In this section we consider.
① sheaf homomorphism,
② exact sequence of sheaves
③ LES
④ cohomology groups.

Def. $\mathcal{F}$ $\mathcal{G}$ sheaves of abelian groups on topological space X.

A sheaf homomorphism $\alpha: \mathcal{F} \to \mathcal{G}$ is a family of group homomorphism
$\alpha_U: \mathcal{F}(U) \to \mathcal{G}(U)$ U open in X
which are compatible with the restriction.
i.e $V \subset U \subset X$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha} & \mathcal{G}(U) \\ \text{restr.} \downarrow & & \downarrow \text{restr.} \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(U) \end{array}$$

is commute.

α is called isomorphism if all the α_U are isomorphisms,

Similarly, def. homomorphism of sheaves of vector spaces

Remark ① In this def, axiom of uniqueness and extencess compatible.

② In category thm, It's Natural transformation between 2 Funtor

15.2 Examples (Riemann surface

(a) $\mathcal{E} \xrightarrow{d} \mathcal{E}^{(1)}$ $d: \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ $f \to \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ $w = f dx + g dy$
$dw = d(f dx + g dy)$
$= df \wedge dx + dg \wedge dy$

(b) inclusion $\mathcal{O} \to \mathcal{E}$ $\mathbb{C} \to \mathcal{E}$ $\mathbb{Z} \to \mathcal{E}$

(c) ex: $\mathcal{O} \to \mathcal{O}^*$
$f \mapsto \exp(2\pi i f)$

Def: kernel of a sheaf homo
$\alpha: \mathcal{F} \to \mathcal{G}$ on X.

For U open X.

$\mathcal{K}(U) \triangleq \mathrm{Ker}(\mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U))$

It is a sheaf

Denote it as $\mathcal{K} = \ker\alpha$

Pf: $s \in \ker\alpha_U$ $\alpha_U(s) = 0$

$s_V \in \ker\alpha_V$ $\alpha_V(s_V) = (\alpha_U(s))|_V = 0$

$\ker\alpha_U$ is subset of $\mathcal{F}(U)$ U_i cover of U

So if $s|_{U_i} = 0$ for all U_i $s = 0$

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ in $\ker\alpha_U$

So $\exists s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$

Then, $\alpha(s)|_{U_i} = \alpha_{U_i}(s|_{U_i}) = 0$

So $\alpha(s)$ is 0 for all U_i

$\alpha(s) \in \mathcal{G}$ $\mathcal{G}$ is a sheaf:

so $\alpha(s) = 0$

Example $\mathcal{O} = \mathrm{Ker}(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1})$ $d'' = \bar\partial$

$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$

$\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ $d = \partial + \bar\partial$

$\Omega = \mathrm{Ker}(\mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)})$

$\mathcal{E}^{(0,1)} = \{f(z,\bar z)d\bar z\}$

$\mathbb{Z} = \mathrm{Ker}(\mathcal{O} \xrightarrow{ex} \mathcal{O}^*)$

only $2\pi i z = 0$
$z \in \mathbb{C}$

$\mathcal{E}^{(1,0)} = \{f(z,\bar z)\, dz\}$

$d(f\, dz) = (\partial f + \bar\partial f) \wedge dz = \bar\partial f \wedge dz = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz$

Def Image presheaf
$\alpha: \mathcal{F} \to \mathcal{G}$

$\mathcal{B}(U) = \mathrm{Im}(\mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U))$

Remark Counter example. Def of Log in Complex analysis

$ex: \mathcal{O} \to \mathcal{O}^*$

$f \to \exp(2\pi i f)$.

$U_1 = \mathbb{C}^* \setminus \mathbb{R}_-$ $U_2 = \mathbb{C}^* \setminus \mathbb{R}_+$

$\exp(2\pi i f) = z$.

Then on $\mathbb{C}^*$

$\exp(2\pi i F) = z$ impossible.

Def Exact sequences

$\alpha: \mathcal{F} \to \mathcal{G}$ induced

$\alpha_x: \mathcal{F}_x \to \mathcal{G}_x$ for each $x \in X$ stalks.

A sequence of sheaf homo $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{K}$ is called exact

if for each $x \in X$, the sequence

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{K} \text{ is exact.}$$

i.e. $\operatorname{Ker} \beta_x = \operatorname{Im} \alpha_x$.

A sheaf homo is a monomorphism if $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact

is an epimorphism if $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0$ is exact.

An exact sequence of then form $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$ is called a short exact sequence.

Lemma Suppose $\alpha: \mathcal{F} \to \mathcal{G}$ is a sheaf monomorphism

Then for every open subset $U \subset X$ the mapping $\alpha_U: \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

Pf: For $f \in \mathcal{F} \to \mathcal{G}$ is a monomorphism

α_x is injective so there is an open neighborhood $V_x \subset U$ such that $f|_{V_x} = 0$.

By sheaf axiom, $f = 0$.

Remark: If $\alpha: \mathcal{F} \to \mathcal{G}$ is sheaf epimorphism.
It's not necessarily true that for every open set U the mapping $\alpha_U: \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

e.g. $\mathcal{O} \to \mathcal{O}^*$ ex: $f \to \exp(2\pi i f)$

ex. $\mathcal{O}_X \to \mathcal{O}_X^*$ is surjective since $\exp(2\pi i f) = \ldots$ loc.

But $\mathcal{O}_X(\mathbb{C}^*) \to \mathcal{O}_X^*(\mathbb{C}^*)$ is not surjective.

II. inverse of the 2 Lemma are true
all α_U injective (surjective) $\Rightarrow$ α mono (epi)

Lemma: Suppose $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{K}$ is an exact sequece of sheaves.
Then for every open set $U \subset X$ the sequence

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U) \xrightarrow{\beta} \mathcal{K}(U).$$

is exact.

Pf. (a) $\operatorname{Ker} \alpha_U = 0$ is proved.
(b) $\operatorname{Ker} \beta = \operatorname{Im} \alpha$.

$\operatorname{Im} \alpha_U \subset \operatorname{Ker} \beta_U$

suppose $f \in \mathcal{F}(U)$ $\alpha(f) \in \mathcal{G}(U)$.

$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{K}_x$ is exact for every $x \in U$.

So for every $x \in U$, $\exists V_x \ni x$ s.t. $\beta(\alpha(f))|_{V_x} = 0$.

By axiom $\beta(\alpha) f = 0$.

$\operatorname{Im} \alpha_U \supset \operatorname{Ker} \beta_U$

Let $g \in \mathcal{G}(U)$. $\beta_U(g) = 0$. for every $x \in U$. $\operatorname{Ker} \beta_x = \operatorname{Im} \alpha_x$
There is an open covering $(V_i)_{i \in I}$ of U and elements $f_i \in \mathcal{F}(V_i)$ such that $\alpha(f_i) = g|_{V_i}$
for every $i \in I$. (Because of def of stalk
And $\alpha(f_i - f_j) = 0$, so $f_i = f_j$ By Lemma.
So there exist an $f \in \mathcal{F}(U)$ st $f|_{V_i} = f_i$ for every i
It follows that $\alpha(f)|_{V_i} = \alpha(f|_{V_i}) = g|_{V_i}$. so $\alpha(f) = g$.

Example (short exact sequence on Riemann surface.
(a) $0 \to \mathcal{O} \xrightarrow{i} \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$

(b) Let $\mathcal{Z} := \mathrm{Ker}(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)})$

be the sheaf of closed differential forms

$0 \to \mathbb{C} \xhookrightarrow{i} \mathcal{E} \xrightarrow{d} \mathcal{Z} \to 0$

Pf ① $\mathrm{Ker}\, i = 0$

② $\ker d = \mathrm{Im}\, i$

$\ker(d) \subset \mathrm{Im}(i)$. $\quad$ $\mathrm{Im}(i) \subset \ker(d)$

If $df = 0$ $f \in \mathbb{C}$ $\quad$ If. $dc = 0$.

③ $d\, df = 0$

④ Poincaré Lemma. Local existence of Primitives. 10.4

For open neighborhood. V

$w \in \mathcal{Z}(U)$. $x \in U$, $\exists V$ s.t $f \in \mathcal{E}(V)$ $df = w|_V$

(c) $0 \to \mathbb{C} \xhookrightarrow{i} \mathcal{O} \xrightarrow{d} \Omega \to 0$

Since $\Omega = \mathcal{O} dx$ (holomorphic version of b)

(d) $\Omega = \mathrm{Ker}(\mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)})$

$0 \to \Omega \xrightarrow{i} \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$.

Pf. d is epim.

for all $x \in X$, surjective (U_z)

$d(f dz) = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz$

Find $\frac{\partial f}{\partial \bar{z}} = g$.

Dolbeault Lemma.

On disk for any smooth function g $\frac{\partial f}{\partial \bar{z}} = g$ $\exists f$.

so for any $x \in X$ surjective.

(e) $0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{ex} \mathcal{O}^* \to 0$

proved

15.10 Any homomorphism $\alpha: \mathcal{F} \to \mathcal{G}$ of sheaves on topological space X induces homomorphisms

$$\alpha^0: H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G})$$
$$\alpha^1: H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$$

α^0 is $\alpha_X\ \mathcal{F}(X) \to \mathcal{G}(X)$.

α^1 is constructed as follows $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X.

$\alpha_{\mathfrak{U}}: C^1(\mathfrak{U}, \mathcal{F}) \longrightarrow C^1(\mathfrak{U}, \mathcal{G})$

which assigns to each cochain $\xi = (f_{ij}) \in C^1(\mathfrak{U}, \mathcal{F})$ the cochain

$$\alpha_{\mathfrak{U}}(\xi) = (\alpha(f_{ij})) \in C^1(\mathfrak{U}, \mathcal{G})$$

Then the mapping induce the to cohomology of open covering

$$\bar{\alpha}_{\mathfrak{U}}: H^1(\mathfrak{U}, \mathcal{F}) \to H^1(\mathfrak{U}, \mathcal{G})$$

Let $\mathfrak{U}$ runs over covering of X. it induces homomorphism α^1.

Def Connecting Homomorphism

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{K} \to 0$$

SES.

A connecting homomorphism" $\delta^* H^0(X, \mathcal{K}) \to H^1(X, \mathcal{F})$

Suppose $h \in H^0(X, \mathcal{K}) = \mathcal{K}(X)$.

Since all the homomorphism $\beta_X: \mathcal{G}_x \to \mathcal{K}_x$ are surjective $\exists$ an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of X and a cohain $(g_i) \in C^0(\mathfrak{U}, \mathcal{G})$ such that $\beta(g_i) = h|U_i$ for every $i \in I$.

Hence $\beta(g_j - g_i) = 0$ on $U_i \cap U_j$ $g_i - g_j \in \mathcal{G}$

$\exists f_{ij} \in \mathcal{F}(U_i \cap U_j)$ s.t.

$$\alpha(f_{ij}) = g_j - g_i$$

$$\alpha(f_{ij} + f_{jk} - f_{ik}) = g_j - g_i + g_k - g_j - (g_k - g_i) = 0.$$

α mono $\Rightarrow f_{ij} + f_{jk} = f_{ik}$

So $f_{ij} \in Z^1(\mathfrak{U}, \mathcal{F})$

Let $\delta^* h \in H^1(X, \mathcal{F})$ $\delta^* h = f_{ij} + B^1(X, \mathcal{F})$.

Thm. X space.

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{K} \to 0 \quad \text{SES}$$

induced sequence of cohomology groups is exact

$$0 \to H^0(X,\mathscr{H}) \xrightarrow{\alpha^0} H^0(X,\mathscr{G}) \xrightarrow{\beta^0} H^0(X,\mathscr{K}) \xrightarrow{\delta^*}$$
$$\xrightarrow{\delta^*} H^1(X,\mathscr{H}) \xrightarrow{\alpha^1} H^1(X,\mathscr{G}) \xrightarrow{\beta^1} H^1(X,\mathscr{K})$$

Pf (a) $0 \to H^0(X,\mathscr{H}) \xrightarrow{\alpha^0} H^0(X,\mathscr{G}) \xrightarrow{\beta^0} H^0(X,\mathscr{K})$ follows from Lemma.

(b) $\mathrm{Im}\,\beta^0 \subset \mathrm{Ker}\,\delta^*$

$g \in H^0(X,\mathscr{G})$ $h = \beta^0(g)$.

$\delta^*(h) = f_{ij} + B^1(X,\mathscr{H})$ Let $g_i = g|U_i$ In def.

So $\alpha(f_{ij}) = 0 \Rightarrow f_{ij} = 0$

$\delta^*(h) = 0$

(c) $\mathrm{Ker}\,\delta^* \subset \mathrm{Im}\,\beta^0$

$h \in \mathrm{Ker}\,\delta^*$ $f_{ij} \in Z^1(\mathfrak{U},\mathscr{H})$ $\delta^* h = 0$.

So $f_{ij} \in B^1(\mathfrak{U},\mathscr{H})$

$f_{ij} = f_i - f_j$ on $U_i \cap U_j$

Set $\hat{g_i} := g_i - \alpha(f_i)$ $\hat{g_j} - \hat{g_i} = g_j - g_i - \alpha(f_j - f_i) = \alpha(f_{ij}) - \alpha(f_{ij}) = 0$

So $\hat{g_i}$ glue together get g

and $\beta(\hat{g_i}) = \beta(g_i - \alpha(f_i)) = \beta(g_i) = h$

$h \in \mathrm{Im}\,\beta^0$

(d) $\mathrm{Im}\,\delta^* \subset \mathrm{Ker}\,\alpha^1$

$\alpha(f_{ij}) = g_i - g_j$ $g_i - g_j \in B^1(X,\mathscr{H})$

(e) $\mathrm{Ker}\,\alpha^1 \subset \mathrm{Im}\,\delta^*$

Assume $\xi \in \mathrm{Ker}\,\alpha^1$ $f_{ij} \in Z^1(\mathfrak{U},\mathscr{H})$ $\xi = (f_{ij})$

$\alpha^1(\xi) = 0$. $\alpha^1(\xi) \in B^1(\mathfrak{U},\mathscr{G})$

$\alpha^1(f_{ij}) = g_i - g_j$ on $U_i - U_j$

$0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i)$ on $U_i \cap U_j$

$\exists\, h \in \mathscr{K}(X)$ such. $h|U_i = \beta(g_i)$.

So $\delta^* h = \xi$.

(f) $\mathrm{Im}\,\alpha^1 \subset \mathrm{Ker}\,\beta^1$

$\mathscr{H}(U_i \cap U_j) \xrightarrow{\alpha} \mathscr{G}(U_i \cap U_j) \xrightarrow{\beta} \mathscr{K}(U_i \cap U_j)$. iseaxct

(g) $\operatorname{Ker}\beta' \subset \operatorname{Im}\alpha'$ suppose $\eta \in \operatorname{Ker}\beta'$ $\eta = (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{G})$
$\mathfrak{U} = (U_i)_{i\in I}$. Then there is a cochain $(h_i) \in C^0(\mathfrak{U}, \mathcal{K})$ such that $\beta(g_{ij}) = h_j - h_i$

λ

$x \in X$, , choose τx near by such that $x \in U_{\tau x}$.
$\beta_x : \mathcal{G}_x \to \mathcal{K}_x$ is surjective, then there is an open neighborhood $V_x \subset U_{\tau x}$ of x.
and there is an element $g_x \in \mathcal{G}(V_x)$ such that $\beta(g_x) = h_{\tau x}/V_x$. Let $\mathfrak{V} = (V_x)_{x\in X}$
and $\hat{g}_{xy} = g_{\tau x, \tau y} / V_x \cap V_y$. Then $(\hat{g}_{xy}) \in Z^1(\mathfrak{V}, \mathcal{G})$.

def : $\psi_{xy} = \hat{g}_{xy} - g_y + g_x$
So $\psi_{xy} \equiv \hat{g}_{xy}$ in $H^1(X, \mathcal{G})$
$\beta(\psi_{xy}) = 0$. by injective.
$\exists f_{xy}$ s.t $\alpha(f_{xy}) = \psi_{xy}$

Thm. $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$ ES on X $H^1(X, \mathcal{G}) = 0$
Then

$$H^1(X, \mathcal{F}) \cong \mathcal{K}(X) / \beta\mathcal{G}(X)$$

Pf: Since $H^1(X, \mathcal{G}) = 0$.

By $\mathcal{G}(X) \xrightarrow{\beta} \mathcal{K}(X) \xrightarrow{\delta^*} H^1(X, \mathcal{F}) \to 0$
It is clear.

construct the iso.
Assume $\mathcal{F} = \operatorname{Ker}\beta$ $\alpha : \mathcal{F} \to \mathcal{G}$ inclusion.
Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$
$f_{ij} \in Z^1(\mathfrak{U}, \mathcal{G}) = 0$
$f_{ij} = g_j - g_i$ on $U_i \cap U_j$
$\beta(f_{ij}) = 0$,
So $\beta(g_j) = \beta(g_i)$
So $\exists h \in \mathcal{K}$ s.t $h|_{U_i} = \beta(g_i)$
Then $h = \Phi(\xi)$.

Dolbeault's Theorem.

$$H^1(X,\mathcal{O}) \cong \mathcal{E}^{0,1}(X) / d'' \mathcal{E}(X)$$

$$H^1(X,\Omega) \cong \mathcal{E}^{(2)}(X) / d\, \mathcal{E}^{1,0}(X)$$

Since $H^1(X,\mathcal{E}) = H^1(X,\mathcal{E}^{1,0}) = 0$

de Rham Group

$$Rh^1(X) = \frac{\operatorname{Ker}(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X))}{\operatorname{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X))}$$

Thm. $H^1(X,\mathbb{C}) \cong Rh^1(X)$

by 15.13 15.9 b

$Rh^1(X) = 0$ if every closed 1-form has a primitive.

if X simply connected, $Rh^1(X) = 0$