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Cohomology Groups on Riemann Surfaces

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Abstract

Amongst all Riemann surfaces the compact ones are especially important. They arise, for example, as those covering surfaces of the Riemann sphere defined by algebraic functions. As well their function theory is subject to interesting restrictions, like the Riemann–Roch Theorem and Abel's Theorem. More recently the theory of Riemann surfaces has been generalized to an extensive theory for complex manifolds of higher dimension. And the methods developed for this are very well suited to proving the classical theorems. One such method is sheaf cohomology and we give a short introduction to this in the present chapter.

To a large extent Chapter 2 is independent of Chapter 1. Essentially only §1 (the definition of Riemann surfaces), the first half of §6 (the definition of sheaves) and §§9 and 10 (differential forms) will be needed. [1]

Contents

The shortest and easiest path between two truths of the real domain most often passes through the complex domain.

Paul Painlevé

Chapter 1

Cohomology Groups

The goal of this section is to define the cohomology groups $H^1(X, \mathscr{F})$, where $\mathscr{F}$ is a sheaf of abelian groups on a topological space X. In our further study of Riemann surfaces, these cohomology groups play a very decided role.

1.1 Cochains, Cocycles, Coboundaries

definition 1.1 Suppose X is a topological space and $\mathscr{F}$ is a sheaf of abelian groups on X. Also suppose that an open covering of X is given, i.e., a family $\mathscr{U} = (U_i)_{i \in I}$ of open subsets of X such that

$$\bigcup_{i \in I} U_i = X.$$

For $q = 0, 1, 2, \ldots$ define the *qth cochain group* of $\mathscr{F}$, with respect to $\mathscr{U}$, as

$$C^q(\mathscr{U}, \mathscr{F}) := \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

The elements of $C^q(\mathscr{U}, \mathscr{F})$ are called q-cochains. Thus a q-cochain is a family

$$(f_{i_0, \ldots, i_q})_{(i_0, \ldots, i_q) \in I^{q+1}}$$

such that

$$f_{i_0, \ldots, i_q} \in \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

for all $(i_0, \ldots, i_q) \in I^{q+1}$. The addition of two cochains is defined component-wise,i.e. Suppose we have two q-cochains, denoted by $f = (f_{i_0, \ldots, i_q})$ and $g = (g_{i_0, \ldots, i_q})$. Their sum $h = f + g$ is defined component-wise: for every index $(i_0, \ldots, i_q) \in I^{q+1}$, let

$$h_{i_0, \ldots, i_q} := f_{i_0, \ldots, i_q} + g_{i_0, \ldots, i_q},$$

where the $+$ denotes the addition in the abelian group $\mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q})$.

definition 1.2 Now define **coboundary operators**

$$\delta : C^0(\mathscr{U}, \mathscr{F}) \to C^1(\mathscr{U}, \mathscr{F})$$

$$\delta : C^1(\mathscr{U}, \mathscr{F}) \to C^2(\mathscr{U}, \mathscr{F})$$

as follows:

(i) For $(f_i)_{i\in I} \in C^0(\mathscr{U}, \mathscr{F})$ let $\delta((f_i)_{i\in I}) = (g_{ij})_{i,j\in I}$ where

$$g_{ij} := f_j - f_i \in \mathscr{F}(U_i \cap U_j).$$

Here it is understood that one restricts f_i and f_j to the intersection $U_i \cap U_j$ and then takes their difference.

(ii) For $(f_{ij})_{i,j\in I} \in C^1(\mathscr{U}, \mathscr{F})$ let $\delta((f_{ij})) = (g_{ijk})$ where

$$g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

Again the terms on the right are restricted to their common domain of definition $U_i \cap U_j \cap U_k$.

These coboundary operators are group homomorphisms.

If we let i,j,k be 1,2,3, then the negative sign appears when 2 disappears.

definition 1.3 Let

$$Z^1(\mathscr{U}, \mathscr{F}) := \operatorname{Ker}\big(C^1(\mathscr{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathscr{U}, \mathscr{F})\big),$$

$$B^1(\mathscr{U}, \mathscr{F}) := \operatorname{Im}\big(C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathscr{U}, \mathscr{F})\big).$$

The elements of $Z^1(\mathscr{U}, \mathscr{F})$ are called *1-cocycles*. Thus by definition a 1-cochain $(f_{ij}) \in C^1(\mathscr{U}, \mathscr{F})$ is a cocycle precisely if

$$f_{ik} = f_{ij} + f_{jk} \quad \text{on } U_i \cap U_j \cap U_k \tag{*}$$

for all $i, j, k \in I$. One calls $(*)$ the **cocycle relation** and it implies

$$f_{ii} = 0, \quad f_{ij} = -f_{ji}.$$

One obtains these from $(*)$ by letting $i = j = k$ for the first and $i = k$ for the second.

The elements of $B^1(\mathscr{U}, \mathscr{F})$ are called *1-coboundaries*. In particular every coboundary is a cocycle. A coboundary is also called a *splitting cocycle*. Thus a 1-cocycle $(f_{ij}) \in Z^1(\mathscr{U}, \mathscr{F})$ splits if and only if there is a 0-cochain $(g_i) \in C^0(\mathscr{U}, \mathscr{F})$ such that

$$f_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j \quad \text{for every } i, j \in I.$$

The quotient group

$$H^1(\mathscr{U}, \mathscr{F}) := Z^1(\mathscr{U}, \mathscr{F})/B^1(\mathscr{U}, \mathscr{F})$$

is called the *1st cohomology group with coefficients in $\mathscr{F}$ with respect to the covering $\mathscr{U}$*.

n open covering $\mathscr{V} = (V_k)_{k\in K}$ is called *finer* than the covering $\mathscr{U} = (U_i)_{i\in I}$, denoted $\mathscr{V} < \mathscr{U}$, if every V_k is contained in at least one U_i. Thus there is a mapping $\tau : K \to I$ such that

$$V_k \subset U_{\tau(k)} \quad \text{for every } k \in K.$$

By means of the mapping τ we can define a mapping

$$\begin{aligned} t^{\mathfrak{U}}_{\mathfrak{V}} : Z^1(\mathfrak{U}, \mathscr{F}) &\to Z^1(\mathfrak{V}, \mathscr{F}) \\ (f_{ij}) &\mapsto (g_{kl}), \end{aligned}$$

where

$$g_{kl} := f_{\tau(k),\tau(l)}|_{V_k\cap V_l} \quad \text{for every } k, l \in K.$$

Lemma 1.1 The mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is independent of the choice of the refining mapping $\tau : K \to I$.

Proof. Suppose $\tilde{\tau} : K \to I$ is another mapping such that $V_k \subset U_{\tilde{\tau}(k)}$ for every $k \in K$.

Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} := f_{\tau(k),\tau(l)}|_{V_k\cap V_l} \quad \text{and} \quad \tilde{g}_{kl} := f_{\tilde{\tau}(k),\tilde{\tau}(l)}|_{V_k\cap V_l}.$$

We have to show that the cocycles (g_{kl}) and $(\tilde{g}_{kl})$ are cohomologous. Since $V_k \subset U_{\tau(k)} \cap U_{\tilde{\tau}(k)}$, one can define

$$h_k := f_{\tau(k),\tilde{\tau}(k)}|_{V_k} \in \mathscr{F}(V_k).$$

On $V_k \cap V_l$ one has

$$\begin{aligned} g_{kl} - \tilde{g}_{kl} &= f_{\tau(k),\tau(l)} - f_{\tilde{\tau}(k),\tilde{\tau}(l)} \\ &= f_{\tau(k),\tau(l)} + f_{\tau(l),\tilde{\tau}(k)} - f_{\tau(l),\tilde{\tau}(k)} - f_{\tilde{\tau}(k),\tilde{\tau}(l)} \\ &= f_{\tau(k),\tilde{\tau}(k)} - f_{\tau(l),\tilde{\tau}(l)} \\ &= h_k - h_l. \end{aligned}$$

Thus the cocycle $(g_{kl}) - (\tilde{g}_{kl})$ is a coboundary. □

Lemma 1.2 The mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is injective.

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a cocycle whose image in $Z^1(\mathfrak{V}, \mathscr{F})$ splits. One has to show that (f_{ij}) itself splits.

Now suppose $f_{\tau(k),\tau(l)} = g_k - g_l$ on $V_k \cap V_l$, where $g_k \in \mathscr{F}(V_k)$. Then on $U_i \cap V_k \cap V_l$ one has

$$g_k - g_l = f_{\tau(k),\tau(l)} = f_{\tau(k),i} + f_{i,\tau(l)} = f_{i,\tau(l)} - f_{i,\tau(k)},$$

and thus $f_{i,\tau(k)} + g_k = f_{i,\tau(l)} + g_l$. Applying sheaf axiom II to the family of open sets $(U_i \cap V_k)_{k \in K}$, one obtains $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i,\tau(k)} + g_k \quad \text{on } U_i \cap V_k.$$

With the elements h_i found in this way, on $U_i \cap U_j \cap V_k$ one has

$$f_{ij} = f_{i,\tau(k)} + f_{\tau(k),j} = f_{i,\tau(k)} + g_k - f_{j,\tau(k)} - g_k = h_i - h_j.$$

Since k is arbitrary, it follows from sheaf axiom I that this equation is valid over $U_i \cap U_j$, i.e., the cocycle (f_{ij}) splits with respect to the covering $\mathfrak{U}$. □

definition 1.4 (The Definition of $H^1(X, \mathcal{F})$) If one has three open coverings such that $\mathfrak{V} < \mathfrak{W} < \mathfrak{U}$, then

$$t_{\mathfrak{V}}^{\mathfrak{W}} \circ t_{\mathfrak{W}}^{\mathfrak{U}} = t_{\mathfrak{V}}^{\mathfrak{U}}.$$

Thus one can define the following equivalence relation $\sim$ on the disjoint union of the groups $H^1(\mathfrak{U}, \mathscr{F})$, where $\mathfrak{U}$ runs through all open coverings of X. Two cohomology classes $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ and $\eta \in H^1(\mathfrak{U}', \mathscr{F})$ are defined to be equivalent, denoted $\xi \sim \eta$, if there exists an open covering $\mathfrak{V}$ with $\mathfrak{V} < \mathfrak{U}$ and $\mathfrak{V} < \mathfrak{U}'$ such that

$$t_{\mathfrak{V}}^{\mathfrak{U}}(\xi) = t_{\mathfrak{V}}^{\mathfrak{U}'}(\eta).$$

The set of equivalence classes is the so-called **inductive limit** of the cohomology groups $H^1(\mathfrak{U}, \mathscr{F})$ and is called the *1st cohomology group of X with coefficients in the sheaf $\mathscr{F}$*. In symbols,

$$H^1(X, \mathscr{F}) := \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) = \left(\bigsqcup_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) \right) \Big/ \sim .$$

Addition in $H^1(X, \mathscr{F})$ is defined by means of representatives as follows. Suppose the elements $x, y \in H^1(X, \mathscr{F})$ are represented by $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ resp. $\eta \in H^1(\mathfrak{U}', \mathscr{F})$. Let $\mathfrak{V}$ be a common refinement of $\mathfrak{U}$ and $\mathfrak{U}'$. Then $x + y \in H^1(X, \mathscr{F})$ is defined to be the equivalence class of

$$t_{\mathfrak{V}}^{\mathfrak{U}}(\xi) + t_{\mathfrak{V}}^{\mathfrak{U}'}(\eta) \in H^1(\mathfrak{V}, \mathscr{F}).$$

One can easily check that this definition is independent of the various choices made and makes $H^1(X, \mathscr{F})$ into an abelian group. If $\mathscr{F}$ is a sheaf of vector spaces, then in a natural way $H^1(\mathfrak{U}, \mathscr{F})$ and $H^1(X, \mathscr{F})$ are also vector spaces.

From Lemma 1.2 it follows that for any open covering $\mathfrak{U}$ of X the canonical mapping

$$H^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is injective. In particular this implies that $H^1(X, \mathscr{F}) = 0$ if and only if $H^1(\mathfrak{U}, \mathscr{F}) = 0$ for every open covering $\mathfrak{U}$ of X.

Theorem 1.1 Suppose X is a Riemann surface and $\mathcal{E}$ is the sheaf of differentiable functions on X. Then

$$H^1(X, \mathcal{E}) = 0.$$

Proof. We give the proof under the assumption that X has a countable topology. However, we assume the manifold is C2 in other textbooks about smooth manifolds.

Suppose $\mathfrak{U} = (U_i)_{i \in I}$ is an arbitrary open covering of X. Then there is a partition of unity subordinate to $\mathfrak{U}$, i.e. a family $(\psi_i)_{i \in I}$ of functions $\psi_i \in \mathcal{E}(X)$ with the following properties (cf. the Appendix):

(i) $\mathrm{Supp}(\psi_i) \subset U_i$.

(ii) Every point of X has a neighborhood meeting only finitely many of the sets $\mathrm{Supp}(\psi_i)$.

(iii) $\sum_{i \in I} \psi_i = 1$.

We will show that $H^1(\mathfrak{U}, \mathcal{E}) = 0$, i.e., every cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{E})$ splits.

The function $\psi_j f_{ij}$, which is defined on $U_i \cap U_j$, may be differentiably extended to all of U_i by assigning it the value zero outside its support. Thus it may be considered as an element of $\mathcal{E}(U_i)$. Set

$$g_i := \sum_{j \in I} \psi_j f_{ij}.$$

Because of (ii), in a neighborhood of any point in U_i, this sum has only finitely many terms which are not zero and thus defines an element $g_i \in \mathcal{E}(U_i)$. For $i, j \in I$

$$\begin{aligned} g_i - g_j &= \sum_{k \in I} \psi_k f_{ik} - \sum_{k \in I} \psi_k f_{jk} \\ &= \sum_{k \in I} \psi_k (f_{ik} - f_{jk}) \\ &= \sum_{k \in I} \psi_k (f_{ik} + f_{kj}) \\ &= \sum_{k \in I} \psi_k f_{ij} \\ &= f_{ij} \end{aligned}$$

on $U_i \cap U_j$ and thus (f_{ij}) is a coboundary. □

Remark 1.1 In exactly the same way one can show that on a Riemann surface X the 1st cohomology groups with coefficients in the sheaves $\mathcal{E}^{(1)}, \mathcal{E}^{1,0}, \mathcal{E}^{0,1}$ and $\mathcal{E}^{(2)}$ also vanish.

Theorem 1.2 Suppose X is a simply connected Riemann surface. Then

(a) $H^1(X, \mathbb{C}) = 0$,

(b) $H^1(X, \mathbb{Z}) = 0$.

Here $\mathbb{C}$ (resp. $\mathbb{Z}$) denotes the sheaf of locally constant functions with values in the complex numbers (resp. integers).

Proof. (a) Suppose $\mathfrak{U}$ is an open covering of X and $(c_{ij}) \in Z^1(\mathfrak{U}, \mathbb{C})$. Since $Z^1(\mathfrak{U}, \mathbb{C}) \subset Z^1(\mathfrak{U}, \mathcal{E})$ and $H^1(\mathfrak{U}, \mathcal{E}) = 0$, there exists a cochain $(f_i) \in C^0(\mathfrak{U}, \mathcal{E})$ such that

$$c_{ij} = f_i - f_j \quad \text{on } U_i \cap U_j.$$

But $dc_{ij} = 0$ implies $df_i = df_j$ on $U_i \cap U_j$, and thus there exists a global differential form $\omega \in \mathcal{E}^{(1)}(X)$ such that $\omega|_{U_i} = df_i$. Since $ddf_i = 0$, it follows that ω is closed. Because X is simply connected, by (10.7) there exists $f \in \mathcal{E}(X)$ such that $df = \omega$. Set

$$c_i := f_i - f|_{U_i}.$$

Since $dc_i = df_i - df = \omega - \omega = 0$ on U_i, c_i is locally constant, i.e., $(c_i) \in C^0(\mathfrak{U}, \mathbb{C})$. On $U_i \cap U_j$ one has

$$c_{ij} = f_i - f_j = (f_i - f) - (f_j - f) = c_i - c_j,$$

and thus the cocycle (c_{ij}) splits.

(b) Suppose $(a_{jk}) \in Z^1(\mathfrak{U}, \mathbb{Z})$. By (a) there exists a cochain $(c_j) \in C^0(\mathfrak{U}, \mathbb{C})$ such that

$$a_{jk} = c_j - c_k \quad \text{on } U_j \cap U_k.$$

Since $\exp(2\pi i a_{jk}) = 1$, one has $\exp(2\pi i c_j) = \exp(2\pi i c_k)$ on the intersection $U_j \cap U_k$. Since X is connected, there exists a constant $b \in \mathbb{C}^*$ such that

$$b = \exp(2\pi i c_j) \quad \text{for every } j \in I.(b exists by the axiom II of sheaf.)$$

Choose $c \in \mathbb{C}$ such that $\exp(2\pi i c) = b$ and let

$$a_j := c_j - c.$$

Since $\exp(2\pi i a_j) = \exp(2\pi i c_j)\exp(-2\pi i c) = 1$, it follows that a_j is an integer, i.e., $(a_j) \in C^0(\mathfrak{U}, \mathbb{Z})$. Moreover

$$a_{jk} = c_j - c_k = (c_j - c) - (c_k - c) = a_j - a_k,$$

i.e., the cocycle (a_{jk}) lies in $B^1(\mathfrak{U}, \mathbb{Z})$.

□

The next theorem shows that in certain cases one can calculate $H^1(X, \mathcal{F})$ using only a single covering of X.

Theorem 1.3 (Larey) Suppose $\mathscr{F}$ is a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i\in I}$ is an open covering of X such that

$$H^1(U_i, \mathscr{F}|_{U_i}) = 0 \quad \text{for every } i \in I.$$

Then

$$H^1(X, \mathscr{F}) \cong H^1(\mathfrak{U}, \mathscr{F}).$$

Such a covering $\mathfrak{U}$ is called a *Leray covering* (of 1st order) for the sheaf $\mathscr{F}$.

Proof. It suffices to show that, for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha\in A}$ with $\mathfrak{V} < \mathfrak{U}$, the mapping $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is an isomorphism(Since limit is unique). From lemma 1.2 this mapping is injective.

Suppose $\tau : A \to I$ is a refining mapping with $V_\alpha \subset U_{\tau(\alpha)}$ for every $\alpha \in A$. To prove the surjectivity of $t^{\mathfrak{U}}_{\mathfrak{V}}$, we must show that given any cocycle $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$, there exists a cocycle $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ such that the cocycle

$$(F_{\tau(\alpha),\tau(\beta)}) - (f_{\alpha\beta})$$

is cohomologous to zero relative to the covering $\mathfrak{V}$.

Now the family $(U_i \cap V_\alpha)_{\alpha\in A}$ is an open covering of U_i which we denote by $U_i \cap \mathfrak{V}$. By assumption $H^1(U_i \cap \mathfrak{V}, \mathscr{F}|_{U_i}) = 0$, i.e., there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

Now on the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$ one has

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}$$

and thus by sheaf axiom II there exist elements $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

Clearly, (F_{ij}) satisfies the cocycle relation and thus lies in $Z^1(\mathfrak{U}, \mathscr{F})$. Let $h_\alpha := g_{\tau(\alpha),\alpha}|V_\alpha \in \mathscr{F}(V_\alpha)$. Then on $V_\alpha \cap V_\beta$ one has

$$\begin{aligned} F_{\tau(\alpha),\tau(\beta)} - f_{\alpha\beta} &= (g_{\tau(\beta),\alpha} - g_{\tau(\alpha),\alpha}) - (g_{\tau(\beta),\alpha} - g_{\tau(\beta),\beta}) \\ &= g_{\tau(\beta),\beta} - g_{\tau(\alpha),\alpha} \\ &= h_\beta - h_\alpha, \end{aligned}$$

and thus $(F_{\tau(\alpha),\tau(\beta)}) - (f_{\alpha\beta})$ splits. □

definition 1.5 Suppose $\mathcal{F}$ is a sheaf of abelian groups on the topological space X and $\mathcal{U} = (U_i)_{i\in I}$ is an open covering of X. Set

$$Z^0(\mathcal{U}, \mathcal{F}) := \operatorname{Ker}\big(C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F})\big),$$

$$B^0(\mathcal{U}, \mathcal{F}) := 0,$$

$$H^0(\mathcal{U}, \mathcal{F}) := Z^0(\mathcal{U}, \mathcal{F})/B^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}).$$

From the definition of δ it follows that a 0-cochain $(f_i) \in C^0(\mathcal{U}, \mathcal{F})$ belongs to $Z^0(\mathcal{U}, \mathcal{F})$ precisely if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every $i, j \in I$. By sheaf axiom II the elements f_i piece together to give a global element $f \in \mathcal{F}(X)$ and there is a natural isomorphism

$$H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X).$$

Thus the groups $H^0(\mathcal{U}, \mathcal{F})$ are entirely independent of the covering $\mathcal{U}$ and one can define

$$H^0(X, \mathcal{F}) := \mathcal{F}(X).$$

1.A Appendix

definition 1.6 (presheaf) Let X be a topological space. A *presheaf* $\mathcal{F}$ of sets on X consists of the following data:

- For every open set $U \subset X$, a set $\mathcal{F}(U)$ (called the set of sections over U).
- For every inclusion of open sets $V \subset U$, a map

  $$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

  (called the restriction map).

These data are required to satisfy the following conditions:

1. $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ for every open set $U \subset X$.
2. For any open sets $W \subset V \subset U$, the diagram

   $$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_{U,V}} & \mathcal{F}(V) \\ & {\scriptstyle \rho_{U,W}} \searrow & \downarrow {\scriptstyle \rho_{V,W}} \\ & & \mathcal{F}(W) \end{array}$$

   commutes, i.e., $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$.

If the sets $\mathcal{F}(U)$ carry additional algebraic structure (e.g., groups, rings, modules) and the restriction maps are homomorphisms, one speaks of a presheaf of groups, rings, modules, etc.

definition 1.7 (sheaf) A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* if for every open set $U \subset X$ and every open covering $\mathfrak{U} = (U_i)_{i \in I}$ of U, the following two axioms are satisfied:

(S1) Locality. Let $s, t \in \mathcal{F}(U)$ be sections such that

$$s|_{U_i} = t|_{U_i} \quad \text{for all } i \in I.$$

Then $s = t$.

(S2) **Gluing.** Let $(s_i)_{i\in I}$ be a family of sections $s_i \in \mathcal{F}(U_i)$ satisfying the compatibility condition

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \text{for all } i, j \in I.$$

Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Remark 1.2 The locality axiom (S1) guarantees the *uniqueness* of the glued section, while the gluing axiom (S2) guarantees its *existence*. Together they imply that for any open covering $(U_i)_{i\in I}$ of U, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer, where the two parallel arrows are given by the two natural restriction maps.

Chapter 2

Dolbeault's lemma

2.1 Dolbeault's lemma

n this section we solve the inhomogeneous Cauchy–Riemann differential equation $\frac{\partial f}{\partial \bar{z}} = g$, where g is a given differentiable function on the disk X. This is then used to show that the cohomology group $H^1(X, \mathbb{C})$ vanishes.

Lemma 2.1 Suppose $g \in \mathcal{E}(\mathbb{C})$ has compact support. Then there exists a function $f \in \mathcal{E}(\mathbb{C})$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Proof. Define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(\zeta) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z - \zeta} \, dz \wedge d\bar{z}.$$

Since the integrand has a singularity when $z = \zeta$, one has to show that the integral exists and depends differentiably on ζ. The simplest way to do this is to change variables by translation and then introduce polar coordinates r, θ, namely let

$$z = \zeta + re^{i\theta}.$$

With regard to the integration ζ is a constant and one has

$$dz \wedge d\bar{z} = -2i \, dx \wedge dy = -2ir \, dr \wedge d\theta.$$

Thus

$$\begin{aligned} f(\zeta) &= -\frac{1}{\pi} \iint \frac{g(\zeta + re^{i\theta})}{re^{i\theta}} r \, dr \, d\theta \\ &= -\frac{1}{\pi} \iint g(\zeta + re^{i\theta}) e^{-i\theta} \, dr \, d\theta. \end{aligned}$$

Since g has compact support, one has only to integrate over a rectangle $0 \le r \le R$, $0 \le \theta \le 2\pi$, provided R is chosen sufficiently large. One may differentiate under the integral sign, i.e., $f \in \mathcal{E}(\mathbb{C})$ and

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = -\frac{1}{\pi} \iint \frac{\partial g(\zeta + re^{i\theta})}{\partial \bar{\zeta}} e^{-i\theta}\, dr\, d\theta.$$

Changing back to the original coordinates, one has

$$\frac{\partial f}{\partial \zeta}(\zeta) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \iint_{B_\epsilon} \frac{\partial g(\zeta + z)}{\partial \zeta} \frac{1}{z}\, dz \wedge d\bar{z},$$

where $B_\epsilon := \{z \in \mathbb{C} : \epsilon \le |z| \le R\}$. Since

$$\frac{\partial g(\zeta + z)}{\partial \zeta} \frac{1}{z} = \frac{\partial g(\zeta + z)}{\partial \bar{z}} \frac{1}{z} = \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta + z)}{z} \right)$$

for $z \ne 0$, one has

$$\frac{\partial f}{\partial \zeta}(\zeta) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \iint_{B_\epsilon} \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta + z)}{z} \right) dz \wedge d\bar{z} = -\lim_{\epsilon \to 0} \iint_{B_\epsilon} d\omega,$$

where the differential form ω is given by

$$\omega(z) = \frac{1}{2\pi i} \frac{g(\zeta + z)}{z}\, dz$$

(here one considers z as a variable and ζ as a constant). By Stokes' Theorem

$$\frac{\partial f}{\partial \zeta}(\zeta) = -\lim_{\epsilon \to 0} \iint_{B_\epsilon} d\omega = -\lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \omega = \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \omega.$$

Parametrizing the circle $|z| = \epsilon$ by $z = \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$, one gets

$$\frac{\partial f}{\partial \zeta}(\zeta) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} g(\zeta + \epsilon e^{i\theta})\, d\theta.$$

Now the integral gives the average value of the function g over the circle $\zeta + \epsilon e^{i\theta}$ for $0 \le \theta \le 2\pi$. Since g is continuous, this converges to $g(\zeta)$ as $\epsilon \to 0$, i.e.,

$$\frac{\partial f}{\partial \zeta}(\zeta) = g(\zeta).$$

□

Theorem 2.1 Suppose $X := \{z \in \mathbb{C} : |z| < R\}$, $0 < R \le \infty$, and $g \in \mathcal{E}(X)$. Then there exists $f \in \mathcal{E}(X)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

This theorem is a special case of the so-called Dolbeault Lemma in several complex variables.

Proof. In this case a solution cannot simply be given as an integral as in lemma 2.1, for the integral will not converge in general. For this reason we use an exhaustion process which allows lemma 2.1 to be applied in the present setting.

Suppose $0 < R_0 < R_1 < \cdots < R_n < \cdots$ is a sequence of radii such that

$$\lim_{n\to\infty} R_n = R$$

and set

$$X_n := \{z \in \mathbb{C} : |z| < R_n\}.$$

There exist functions $\psi_n \in \mathcal{E}(X)$ with compact supports $\mathrm{Supp}(\psi_n) \subset X_{n+1}$ and $\psi_n|_{X_n} = 1$. The functions $\psi_n g$ vanish outside X_{n+1} and thus if one extends them by zero, they become functions on $\mathbb{C}$. By (13.1) there exist functions $f_n \in \mathcal{E}(X)$ such that

$$\bar\partial f_n = \psi_n g \quad \text{on } X.$$

Here and in the following we use the abbreviation $\bar\partial := \partial/\partial\bar z$.

By induction we alter the sequence (f_n) to another sequence $(\bar f_n)$, which for all $n \geq 1$ satisfies

(i) $\bar\partial \bar f_n = g$ on X_n,

(ii) $\|\bar f_{n+1} - \bar f_n\|_{X_{n-1}} \leq 2^{-n}$.

(As usual let $\|f\|_K := \sup_{x\in K} |f(x)|$ denote the supremum norm.) Set $\bar f_1 := f_1$. Suppose $\bar f_1, \ldots, \bar f_n$ are already constructed. Then

$$\bar\partial(f_{n+1} - \bar f_n) = 0 \quad \text{on } X_n,$$

and thus $f_{n+1} - \bar f_n$ is holomorphic on X_n. Hence there exists a polynomial P (e.g., a finite number of terms of the Taylor series of $f_{n+1} - \bar f_n$) such that

$$\|f_{n+1} - \bar f_n - P\|_{X_{n-1}} \leq 2^{-n}.$$

If we set $\bar f_{n+1} := f_{n+1} - P$, then (ii) is satisfied. Moreover, on X_{n+1} one has

$$\bar\partial \bar f_{n+1} = \bar\partial f_{n+1} - \bar\partial P = \bar\partial f_{n+1} = \psi_{n+1} g = g,$$

i.e., (i) also holds. Since every point $z \in X$ is contained in almost all X_n, the limit

$$f(z) := \lim_{n\to\infty} \bar f_n(z)$$

exists. On X_n one may write

$$f = \bar f_n + \sum_{k=n}^{\infty} (\bar f_{k+1} - \bar f_k).$$

For $k \geq n$, the functions $\bar f_{k+1} - \bar f_k$ are holomorphic on X_n, since

$$\bar\partial(\bar f_{k+1} - \bar f_k) = 0.$$

Because of (ii), the series

$$F_n := \sum_{k=n}^{\infty} (\bar{f}_{k+1} - \bar{f}_k)$$

converges uniformly on X_n and is thus holomorphic there. Hence $f = \bar{f}_n + F_n$ is infinitely differentiable on X_n for every n and thus $f \in \mathcal{E}(X)$. As well

$$\bar{\partial} f = \bar{\partial} \bar{f}_n = g \quad \text{on } X_n$$

for every n and thus $\bar{\partial} f = g$ on all of X. □

Remark 2.1 Naturally the solution of the equation $\bar{\partial} f = g$ is not uniquely determined, only up to the addition of an arbitrary holomorphic function.

corollary 2.1 Suppose $X := \{z \in \mathbb{C} : |z| < R\}, 0 < R \leq \infty$. Then given any $g \in \mathcal{E}(X)$, there exists $f \in \mathcal{E}(X)$ such that $\Delta f = g$.

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

is the Laplace operator.

Proof. Choose $f_1 \in \mathcal{E}(X)$ such that $\bar{\partial} f_1 = g$ and $f_2 \in \mathcal{E}(X)$ such that $\bar{\partial} f_2 = f_1$. Then $f := \frac{1}{4} f_2$ satisfies $\Delta f = g$, for

$$\Delta f = 4 \frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f_2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z} \left(\frac{\partial f_2}{\partial \bar{z}} \right) = \frac{\partial f_1}{\partial z} = g.$$

□

Theorem 2.2 Suppose $X := \{z \in \mathbb{C} : |z| < R\}, 0 < R \leq \infty$. Then $H^1(X, \mathcal{O}) = 0$.

Proof. Suppose $\mathfrak{U} = (U_i)_{i \in I}$ is an open covering of X and $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ is a cocycle. Since $Z^1(\mathfrak{U}, \mathcal{O}) \subset Z^1(\mathfrak{U}, \mathcal{E})$ and $H^1(X, \mathcal{E}) = 0$, there exists a cochain $(g_i) \in C^0(\mathfrak{U}, \mathcal{E})$ such that

$$f_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j.$$

Since $\bar{\partial} f_{ij} = 0$, one has $\bar{\partial} g_i = \bar{\partial} g_j$ on $U_i \cap U_j$ and thus there exists a global function $h \in \mathcal{E}(X)$ with $h|_{U_i} = \bar{\partial} g_i$. By (13.2) we can find a function $g \in \mathcal{E}(X)$ such that $\bar{\partial} g = h$. Define

$$f_i := g_i - g.$$

Now f_i is holomorphic, since $\bar{\partial} f_i = \bar{\partial} g_i - \bar{\partial} g = 0$, and thus $(f_i) \in C^0(\mathfrak{U}, \mathcal{O})$. As well on $U_i \cap U_j$ one has

$$f_i - f_j = g_i - g_j = f_{ij},$$

i.e., the cocycle (f_{ij}) splits. □

Theorem 2.3 For the Riemann sphere $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Proof. Set $U_1 := \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 := \mathbb{P}^1 \setminus \{0\}$. Since $U_1 = \mathbb{C}$ and U_2 is biholomorphic to $\mathbb{C}$, it follows from (13.4) that $H^1(U_i, \mathcal{O}) = 0$ for $i = 1, 2$. Thus $\mathfrak{U} = (U_1, U_2)$ is a Leray covering of $\mathbb{P}^1$ and $H^1(\mathbb{P}^1, \mathcal{O}) = H^1(\mathfrak{U}, \mathcal{O})$ by (12.8). Thus the proof is complete once one shows that every cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ splits. In order to do this, it is clearly enough to find functions $f_i \in \mathcal{O}(U_i)$ such that

$$f_{12} = f_1 - f_2 \quad \text{on } U_1 \cap U_2 = \mathbb{C}^*.$$

Let

$$f_{12}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent expansion of f_{12} on $\mathbb{C}^*$. Set

$$f_1(z) := \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad f_2(z) := -\sum_{n=-\infty}^{-1} c_n z^n.$$

Then $f_i \in \mathcal{O}(U_i)$ and $f_1 - f_2 = f_{12}$. □

Reference

[1] Otto Forster. *Lectures on Riemann Surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981. pages ↑2