Differential Form and integration

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April 2, 2026

Preface

This note is written for the seminar **Riemann Surface** held by Prof. Yifei Zhu in Southern University of Science and Technology 2026 Spring semester.

1 Differential Forms in real manifold

Definition 1.1 (Vector Bundle)**.** A vector bundle on M of rank k is a smooth manifold E and a projection map $\pi : E \to M$ such that:

- $\forall x \in M, E_x = \pi^{-1}(x)$ is a k dimensional vector space.
- $\forall x \in M$, there exists an open neighborhood U and a diffeomorphism.

  $$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

  and the following diagram commutes:

  $$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{R}^k \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \pi_1} & \\ U & & \end{array}$$

  Or one can express it in this way:

  $$\forall x \in U, E_x = \Phi^{-1}(\{x\} \times \mathbb{R}^k)$$

- The diffeomorphism restricted on E_x is a linear isomorphism.

Generally speaking, it is a way of attaching a 'smooth' rank k vector space to all points on manifold.

Definition 1.2 (Smooth section on vector bundle)**.** Let $U \subset M$ be an open subset of M. A smooth section of E on U is a smooth mapping $s : U \to E$ such that: $\pi \circ s(x) = x, \forall x \in M$, that is $s(x) \in E_x$.

We use the notation of $\Gamma(U, E)$ to denote all the smooth section of E on U.

1.1 Tangent Bundle

The most common and useful example of the vector bundle is the tangent bundle.

Definition 1.3 (Tangent Bundle)**.** The tangent bundle TM is defined as the disjoint union of all tangent space over M.

$$TM := \coprod_{p \in M} T_p M$$

One can check that TM is a $2n-$dimensional manifold, and the projection map is just: $\pi : (p, v_p) \to p$. So tangent bundle is a special case for vector bundle.

Moreover, every section of tangent bundle is just the vector field.

Similarly, since $T_p M$ is a finite dimensional vector space, from the basic linear algebra knowledge, we can define the dual space of it and glue them together:

Definition 1.4 (Cotangent Bundle)**.** The cotangent bundle TM is defined as the disjoint union of all cotangent space over M.

$$T^*M := \coprod_{p \in M} T_p^* M$$

All this definitions above have a representation using the local coordinates. Let p be a point of M and (U, ϕ) be a local chart near p, then the tangent space $T_p M$ can be locally expressed as:

$$T_p M := span_{\mathbb{R}}\{\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p, \cdots, \frac{\partial}{\partial x_n}|_p\}$$

also, for cotangent space at p:

$$T_p^* M := span_{\mathbb{R}}\{\mathrm{d}x_1|_p, \mathrm{d}x_2|_p, \cdots, \mathrm{d}x_n|_p\}$$

1.2 Tensor field

The above construction of vector bundle can be more generalised into a tensor field.

Definition 1.5. Let M be a smooth manifold and $p \in M$, define:

$$T_l^k (T_p M) := \underbrace{T_p M \otimes \cdots \otimes T_p M}_{l \text{ times}} \otimes \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{k \text{ times}}.$$

Be a local tensor operation on $T_p M$ as a $n-$dimensional vector space.

Definition 1.6. A bundle of (k, l)-tensors on M is defined as as their disjoint union, $T_l^k M := \bigsqcup_{p \in M} T_l^k (T_p M)$

Remark 1.7. $T_l^k M \cong (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$

This is just the change of notations to make things simpler.

Recall what we have learned at advanced linear algebra course. Instead of using k-tensor to generate new bundle. We can also use *Sym* and *Alt* to generate.

1.3 Differential forms on manifolds

Recall that T^kT^*M is the bundle of covariant k-tensors on M. The subset of T^kT^*M consisting of alternating tensors is denoted by Λ^kT^*M :

$$\Lambda^kT^*M = \coprod_{p\in M} \Lambda^k\left(T_p^*M\right).$$

To make this precise, for each $p \in M$ and $\alpha \in T^k(T_p^*M)$, define

$$\operatorname{Alt}(\alpha)(v_1,\ldots,v_k) := \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\,\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)}).$$

A tensor α is called alternating if exchanging any two arguments changes the sign; equivalently, $\alpha(v_1,\ldots,v_k)=0$ whenever two arguments are equal.

For $\alpha \in \Lambda^k(T_p^*M)$ and $\beta \in \Lambda^l(T_p^*M)$, the wedge product is defined by

$$\alpha\wedge\beta := \frac{(k+l)!}{k!\,l!}\operatorname{Alt}(\alpha\otimes\beta) \in \Lambda^{k+l}(T_p^*M).$$

It satisfies bilinearity and graded-commutativity:

$$\alpha\wedge\beta = (-1)^{kl}\,\beta\wedge\alpha.$$

Any smooth section of Λ^kT^*M is called a differentiable $k-$form, or simply $k-$form. We use $\Omega^k(M)$ to denote the vector space of smooth $k-$form.

$$\Omega^k(M) := \Gamma(M, \Lambda^kT^*M)$$

One can also define wedge product over $\Omega^k(M)$ pointwise, $(\omega\wedge\eta)_p = \omega_p\wedge\eta_p$

By the basic knowledge of tensor or wedge product. any $k-$form can be expressed by linear combination of the $k-$wedge product.

$$\omega = \sum_I f_I dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \sum_I f_I dx^I$$

Example 1.8. Consider the case when the manifold is just $\mathbb{R}^3$ space.

- A $0-$form is just a continuous real-valued function
- A $1-$form is a covector field.
- Some examples of the $2-$form are:

$$\eta_1 = dx\wedge dy + dy\wedge dz + dz\wedge dx \quad \eta_2 = \sin xy dy\wedge dz$$

- Every $3-$form is a continuous real-valued function times $dx\wedge dy\wedge dz$

One can think of k-form as a way of integrate on $k-$dimsional object/function.

1.4 Exterior Differentiation

We use one theorem to state what is an exterior differentiation.

Theorem 1.9 (Existence and Uniqueness of Exterior Differentiation)**.** . *Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called exterior differentiation, satisfying the following four properties:*

- *(i) d is linear over $\mathbb{R}$.*
- *(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

- *(iii) $d \circ d \equiv 0$.*
- *(iv) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

The condition (iv) actually tells us that the operator d is just the generalization of the total derivation. We can think of the condition $(i)-(iii)$ gives us a certain kind of reduction and keep the uniqueness up to initial value and the condition (iv) be the actual initial value.

In any smooth coordinate chart, d is given by:

$$d\left(\sum_I f_I \,\mathrm{d}x^I\right) = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial x^j} \,\mathrm{d}x^j \wedge \mathrm{d}x^I.$$

By linearity:

$$d(f_I dx^I) = (df_I) \wedge dx^I$$

We will skipped the proof here.

1.5 De Rham Cohomology

The definition of the $\Omega^k(M)$ and the exterior differentiation d actually give us a chain complex.

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \to 0$$

Similar to the definition of the homology group, we can also define cohomology here.

Definition 1.10 (De Rham Cohomology)**.** Let $\omega \in \Omega^k(M)$.

- ω is called **closed** if $\mathrm{d}\omega = 0$.
- ω is called **exact** if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = \mathrm{d}\eta$.

Denote by

$$Z^k(M) := \ker\left(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)\right), \quad B^k(M) := \operatorname{im}\left(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)\right).$$

By the third properties of the operator d, $Z^k(M) \subset B^k(M)$ always holds. Then the k-th De Rham cohomology group(vector space) is:

$$H^k_{\mathrm{dR}}(M) := Z^k(M)/B^k(M).$$

Example 1.11 (Some examples of the De Rham cohomology)**.** The following are standard computations:

- For the circle S^1: $H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}$, $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$, and $H^k_{\mathrm{dR}}(S^1) = 0$ for $k \geq 2$.
- For the sphere S^n ($n \geq 2$): $H^0_{\mathrm{dR}}(S^n) \cong \mathbb{R}$, $H^n_{\mathrm{dR}}(S^n) \cong \mathbb{R}$, and $H^k_{\mathrm{dR}}(S^n) = 0$ for $0 < k < n$.
- For the torus $T^2 = S^1 \times S^1$: $H^0_{\mathrm{dR}}(T^2) \cong \mathbb{R}$, $H^1_{\mathrm{dR}}(T^2) \cong \mathbb{R}^2$, $H^2_{\mathrm{dR}}(T^2) \cong \mathbb{R}$.
- For the punctured plane $\mathbb{R}^2 \setminus \{0\}$: $H^0_{\mathrm{dR}} \cong \mathbb{R}$ and $H^1_{\mathrm{dR}} \cong \mathbb{R}$ (generated by the class of $\mathrm{d}\theta$), while $H^k_{\mathrm{dR}} = 0$ for $k \geq 2$.

2 Differential forms on Riemann surface

Before talking about that, we recall an interesting fact: In the complex analysis course, we have defined:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

This two operators can be thought to be a linear mapping:

$$\frac{\partial}{\partial z} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U) \quad \frac{\partial}{\partial \bar{z}} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$$

Where the $\mathcal{E}(U)$ is all the real infinitely differentiable function with respect to real coordinates x, y. It is a larger set than $\mathcal{O}(U)$. Moreover, the Cauchy-Riemann Equations tell us that:

$$\mathcal{O}(U) = \ker \frac{\partial}{\partial \bar{z}}$$

Definition 2.1 (Cotangent Space)**.** Denote $\mathfrak{m}(a) \subset \mathcal{E}(a)$ be the vector subspace of all function germs that vanishes at point a, also, we define $\mathfrak{m}^2(a) \subset \mathfrak{m}(a)$ be the vector subspace of those function germs which vanishes to second order.

Which is:

$$\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = 0 \text{ where } z = x + iy \text{ is the local chart}$$

Now we define:

$$T_a^{(1)} := \frac{\mathfrak{m}(a)}{\mathfrak{m}^2(a)}$$

be the cotangent space of X at the point a.

Remark 2.2. For any function $f \in \mathcal{E}(U)$, one can take its differential $\mathrm{d}_a f$ to generates an element in the cotangent space.

$$\mathrm{d}_a f := (f - f(a)) \mod \mathfrak{m}^2(a)$$

In the following theorem, we will show that the definition is actually the same as what is defined in the real sense.

Theorem 2.3. *Suppose X is a Riemann surface, $a \in X$ and $T_a^{(1)}$ be the corresponding cotangent space. Take $(U, z = x + iy)$ be a complex chart near a. Then one can prove that $\mathrm{d}_a z, \mathrm{d}_a \bar{z}$ form a basis of $T_a^{(1)}$, also for $\mathrm{d}_a x, \mathrm{d}_a y$. And if $f \in \mathcal{E}(U)$:*

$$\mathrm{d}_a f = \frac{\partial f}{\partial x}(a)\,\mathrm{d}_a x + \frac{\partial f}{\partial y}(a)\,\mathrm{d}_a y = \frac{\partial f}{\partial z}(a)\,\mathrm{d}_a z + \frac{\partial f}{\partial \bar{z}}(a)\,\mathrm{d}_a \bar{z}$$

Proof. To prove this theorem, we need to prove from 2 sides, i.e. the basis can span the subspace we want and it is linearly independent.

- Show that the basis we choose can span $T_a^{(1)}$. For any $t \in T_a^{(1)}$ and take $\phi \in \mathfrak{m}(a)$ be the representative of t. By the Taylor Expansion at a, one can get:

  $$\phi = c_1(x - x(a)) + c_2(y - y(a)) + \Phi$$

  where $c_1, c_2 \in \mathbb{C}$ and $\Phi \in \mathfrak{m}^2(a)$. Take the equivalent relation moduling $\mathfrak{m}^2(a)$, we have:

$$t = c_1\, \mathrm{d}_a x + c_2\, \mathrm{d}_a y$$

- Show that the basis we choose is linearly independent. Take $t = c_1(x - x(a)) + c_2(y - y(a))$. $t = 0$ is equivalent with $t \in \mathfrak{m}^2(a)$. by taking the partial dirivative with respect to x or y, we get $c_1 = c_2 = 0$.

By above process, we successfully proved that $\mathrm{d}_a x$ and $\mathrm{d}_a y$ is the basis, applying similar result we get $\mathrm{d}_a z$ and $\mathrm{d}_a \bar{z}$ also the basis. $\square$

The next theorem gives the difference between real and complex tangent space.

Theorem 2.4 (Decomposition of contangent space)**.** *The cotangent space has a canonical decomposition:*

$$T_a^{(1)} = T_a^{1,0} \oplus T_a^{0,1}$$

Suppose (U, z) and (U', z') be two complex chart near a. One can prove that:

$$\mathrm{d}_a z = c_1 \mathrm{d}_a z' \quad \mathrm{d}_a \bar{z} = c_2 \mathrm{d}_a \bar{z}' \quad \textit{where } c_1, c_2 \in \mathbb{C}$$

So one can have a decomposition independent of the specific choice of chart. Let:

$$T_a^{(1,0)} = \mathbb{C}\mathrm{d}_a z \quad T_a^{(0,1)} = \mathbb{C}\mathrm{d}_a \bar{z}$$

We can also decomposite the derivative operator d *into two parts*

$$\mathrm{d}_a f = \mathrm{d}'_a f + \mathrm{d}''_a f, \quad \mathrm{d}'_a f \in T_a^{(1,0)}, \quad \mathrm{d}''_a f \in T_a^{(0,1)}$$

Where:

$$\mathrm{d}'_a f = \frac{\partial f}{\partial z}(a)\mathrm{d}_a z, \quad \mathrm{d}''_a f = \frac{\partial f}{\partial \bar{z}}(a)\mathrm{d}_a \bar{z}$$

This is really different from the real case, cause the Cauchy-Riemann equations make the jacobian matirx of the transition map are in simple forms like: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Definition 2.5 (1-form)**.** Suppose Y is an open subset of the Riemann surface X. By a differential form of degree one, or simply a 1-form, on Y we mean a mapping:

$$\omega : Y \longrightarrow \bigcup_{a \in Y} T_a^{(1)}$$

with $\omega(a) \in T_a^{(1)}$ for all $a \in Y$. If $w(a) \in T_a^{(1,0)}$ for every $a \in Y$, then ω is said to be of type (1,0).

Example 2.6. We can go through some examples.

- Suppose $f \in \mathcal{E}(Y)$. Then the mappings $\mathrm{d}f, \mathrm{d}'f, \mathrm{d}''f$, which are defined by:

  $$\mathrm{d}f(a) := \mathrm{d}_a f, \quad \mathrm{d}'f(a) := \mathrm{d}''_a f, \quad \mathrm{d}''f(a) := \mathrm{d}''_a f$$

  for every $a \in Y$, are $1-$forms. Clearly a function f is holomorphic precisely if $\mathrm{d}''f = 0$.

- Suppose ω is an $1-$form on Y and $f : Y \to \mathbb{C}$ is a function. Then the mapping $f\omega$ defined by $(f\omega)(a) := f(a)\omega(a)$ is also a $1-$form on Y.

Remark 2.7. If (U, z) is a complex chart with $z = x + iy$, then every $1-$form on U may be written:

$$\omega = f\mathrm{d}x + g\mathrm{d}y = \varphi\mathrm{d}z + \phi\mathrm{d}\bar{z}$$

But the function above, $f, g, \varphi, \phi : U \to \mathbb{C}$ are not necessarily continuous in general. One need to add a differential or holomorphic condition to restrict it.

Definition 2.8. Suppose Y is an open subset of a Riemann surface X. A 1-form ω on Y is called differentiable (resp. holomorphic) if, with respect to every chart $(U, z), \omega$ may be written

$$\omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z} \text{ on } U \cap Y, \quad \text{where } f, g \in \mathcal{E}(U \cap Y).$$

resp.

$$\omega = f\,\mathrm{d}z \text{ on } U \cap Y, \quad \text{where } f \in \mathcal{O}(U \cap Y).$$

Just the restrction on the 1-form since we only need the 1-form with good property.

Definition 2.9 (The Residue)**.** Suppose Y is an open subset of the Riemann surface, $a \in Y$ and ω is an 1-form on $Y\backslash\{a\}$. Let (U, z) be a coordinate neighborhood of a such that $U \subset Y$ and $z(a) = 0$. Then on $U\backslash\{a\}$ one may write $\omega = f\mathrm{d}z$. Let:

$$f = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent series expansion about a with respect to the coordinate z.

Similar to the complex analysis, we define:

$$c_{-1} = \mathrm{Res}_a(\omega)$$

be the residue of the $1-$ form ω at point a.

Lemma 2.10. *The definition of the residue of an 1-form is independent of the chart z chosen.*

Proof. We use two claims to prove the lemma. Take V be an open neighborhood of a.

Claim 1 : If g is holomorphic on $V\backslash\{a\}$, then the residue of dg at a equals zero and is thus independent of the choice of chart.

Proof. Let (U, z) be any coordiante neighborhood of a with $z(a) = 0$ and suppose g is holomorphic function on $V\backslash\{a\}$ with Laurent series expansion:

$$g = \sum_{-\infty}^{\infty} c_n z^n$$

Then the corresponding differential form dg is:

$$dg = (\sum_{-\infty}^{\infty} nc_n z^{n-1})dz$$

So the coefficient of z^{-1} is zero and is independent of the coordinate chosen. □

Claim 2 : If φ is a holomorphic function on V which has a zero of first order at a, then $\text{Res}_a\left(\varphi^{-1}d\varphi\right) = 1$ and is thus independent of the choice of chart.

Proof. Let (U, z) be any coordiante neighborhood of a with $z(a) = 0$. By the description of φ, it can be written as $\varphi = zh$, so:

$$\frac{d\varphi}{\varphi} = \frac{dz}{z} + \frac{dh}{h}$$
$$\text{Res}_a(\frac{d\varphi}{\varphi}) = \text{Res}_a(\frac{dz}{z}) + \text{Res}_a(\frac{dh}{h})$$

For $\frac{dh}{h}$, since $h(a) \neq 0$, so this form is holomorphic near a, which makes it resdue equals to 0. For $\frac{dz}{z}$, by definition, it has resdue 1. So:

$$\text{Res}_a(\frac{d\varphi}{\varphi}) = \text{Res}_a(\frac{dz}{z}) + \text{Res}_a(\frac{dh}{h}) = 1 + 0 = 1$$

□

This two claim actually is to give independency with respect to two parts of the Laurent series expansion.

Let $\omega = fdz$ be the holomorphic 1-form near a, then we can do a Laurent series expansion for f

$$f = \sum_{-\infty}^{\infty} c_n z^n = g + c_{-1}z^{-1}, \text{ where } g = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=-\infty}^{-2} c_n z^n$$

Then:

$$\omega = fdz = (g + c_{-1}z^{-1})dz = d\tilde{g} + c_{-1}z^{-1}dz$$

Here $\tilde{g} = \sum_{n=-\infty}^{-2} \frac{c_n}{n+1} z^{n+1} + \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$

□

The following is a typical definition of meromorphic stuff.

Definition 2.11 (Meromorphic Differential Forms)**.** A $1-$form ω on an open subset Y of a Riemann surface is said to be a meromorphic differential form on Y if there is an open subset $Y' \subset Y$ such that the following hold:

- ω is a holomorphic 1-form on Y'.
- $Y \backslash Y'$ consists of only isolated points.

- ω has a pole at every point $a \in Y \backslash Y'$

Let $\mathcal{M}^{(1)}(Y)$ denote the set of all meromorphic 1-form on Y. It is easy to check that $\mathcal{M}^{(1)}$ is a sheaf of vector spaces over X.

The meromorphic 1-forms on X are also called *abelian differentials*.

Now we give an example for the indepence of the residue with respect to the coordinates:

Example 2.12. Let $a = 0$ and consider the meromorphic 1-form on a punctured neighborhood of 0:

$$\omega = \frac{\mathrm{d}z}{z}.$$

In the coordinate z, the Laurent expansion is $z^{-1}\,\mathrm{d}z$, so

$$\mathrm{Res}_0(\omega) = 1.$$

Now change coordinate by $w = 2z$. Then $z = w/2$ and $\mathrm{d}z = \mathrm{d}w/2$, hence

$$\omega = \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}w/2}{w/2} = \frac{\mathrm{d}w}{w}.$$

So in the coordinate w, the coefficient of $w^{-1}\,\mathrm{d}w$ is still 1, and therefore

$$\mathrm{Res}_0(\omega) = 1.$$

This gives a direct computation showing that the residue does not depend on the chosen chart.

Theorem 2.13 (**The Residue Theorem.**)**.** *Suppose X is a compact Riemann surface and $a_1, \dots, a_n$ are distinct points in X. Let $X' := X \backslash \{a_1, \dots, a_n\}$. Then for every holomorphic 1-form $\omega \in \Omega(X')$, one has*

$$\sum_{k=1}^{n} \mathrm{Res}_{a_k}(\omega) = 0.$$

The proof will be skipped, one can check **Theorem 10.21** on the page of 80 of the book GTM81[1]

Similar to the real case, we define 2-form pointwise first.

$$T_a^{(2)} := \Lambda^2 T_a^{(1)}$$

Definition 2.14 (2-form)**.** Suppose Y is an open subset of a Riemann surface X. A 2-form on Y is a map

$$\omega : Y \longrightarrow \bigcup_{a \in Y} T_a^{(2)}$$

such that $\omega(a) \in T_a^{(2)}$ for every $a \in Y$. The form ω is called differentiable on Y if, with respect to every complex chart (U, z) on X, it can be written as

$$\omega = f\,\mathrm{d}z \wedge \mathrm{d}\bar{z}, \quad f \in \mathcal{E}(U \cap Y).$$

Here, $\omega = f\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ means

$$\omega(a) = f(a)\,\mathrm{d}_a z \wedge \mathrm{d}_a \bar{z}, \quad \forall a \in U \cap Y.$$

Denote by $\mathcal{E}^{(2)}(Y)$ the vector space of all differentiable 2-forms on Y.

This is a simple corollary of the fact that the bundle of top-degree forms always has rank 1.

Definition 2.15 (Exterior Differentiation)**.** This is the same as in the real manifold. But complex chart and holomorphic gives us more. We can define three operators here: $d, d', d'' : \mathcal{E}^{(1)}(U) \to \mathcal{E}^{(2)}(U)$. take an 1-form $\omega = f_1 dz + f_2 d\bar{z}$, where the f_k and g_k are differentiable functions and z is a local coordinate. Set

$$\begin{aligned} d\omega &:= \sum df_k \wedge dg_k \\ d'\omega &:= \sum d' f_k \wedge dg_k \\ d''\omega &:= \sum d'' f_k \wedge dg_k \end{aligned}$$

Theorem 2.16. *Y is an open subset of a Riemann surface. Then the following hold:*

- *(a) Holomorphic 1-form $\omega \in \Omega(Y)$ is closed.*
- *(b) Every closed 1-form $\omega \in \mathcal{E}^{1,0}(Y)$ is holomorphic.*

Proof. Suppose ω is a differentiable 1 -form of type $(1,0)$. With respect to a coordinate neighborhood (U, z) one may write $\omega = f dz$ for some differentiable function f. Then

$$d\omega = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) \wedge dz = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$$

Thus $d\omega = 0$ is equivalent to $(\partial f / \partial \bar{z}) = 0$ and the results follow. □

3 Integration of one-form

We can define the integration of any $1-$form along an admissble curve.

Theorem 3.1. *Suppose X is a Riemann surface, $c : [0, 1] \to X$ is a piece-wise continuously differentiable curve and $F \in \mathcal{E}(X)$. Then*

$$\int_c dF = F(c(1)) - F(c(0))$$

Proof. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and charts (U_k, z_k) as above. On U_k one has

$$dF = \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y_k} dy_k.$$

Thus

$$\begin{aligned}
\int_c dF &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{\partial F}{\partial x_k}(c(t)) \frac{dx_k(c(t))}{dt} + \frac{\partial F}{\partial y_k}(c(t)) \frac{dy_k(c(t))}{dt} \right) dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{d}{dt} F(c(t)) \right) dt \\
&= \sum_{k=1}^{n} \left(F\left(c\left(t_k\right)\right) - F\left(c\left(t_{k-1}\right)\right)\right) = F(c(1)) - F(c(0))
\end{aligned}$$

□

Definition 3.2. Suppose X is a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$. A function $F \in \mathcal{E}(X)$ is called a **primitive** of ω if $dF = \omega$.

Theorem 3.3. *Suppose X is a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$ is a closed differential form. Then there exist a covering map $p : \hat{X} \to X$ with $\hat{X}$ connected, and a primitive $F \in \mathcal{E}(\hat{X})$ of the differential form $p^*\omega$.*

This is the theorem 10.5 in the textbook page 71[1], we will skip the proof, but use two of its corollary.

Corollary 3.4. *Suppose X is a Riemann surface, $\pi : \tilde{X} \to X$ its universal covering and $\omega \in \mathcal{E}^{(1)}(X)$ a closed differential form. Then there exists a primitive $f \in \mathcal{E}(\tilde{X})$ of $\pi^*\omega$.*

Corollary 3.5. *On a simply connected Riemann surface X every closed differential form $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive $F \in \mathcal{E}(X)$.*

Cause here the universal covering of X is itself and $\pi = id$.

These theorems give us a methodology that is to use the universal covering to deal with the exsitence of primitives.

Lemma 3.6. *Suppose X is a Riemann surface and $p : \widetilde{X} \to X$ is its universal covering. Suppose $\omega \in \mathcal{E}^{(1)}(X)$ is a closed differential form and $F \in \mathcal{E}(\tilde{X})$ is a primitive of $p^*\omega$. If $c : [0, 1] \to X$ is a piece-wise continuously differentiable curve and $\hat{c} : [0, 1] \to \tilde{X}$ is a lifting of c, then*

$$\int_c \omega = F(\hat{c}(1)) - F(\hat{c}(0)).$$

Proof. For every piece-wise continuously differentiable curve $v : [0,1] \to \tilde{X}$ and every differential form $\omega \in \mathcal{E}^{(1)}(X)$ one has

$$\int_v p^*\omega = \int_{p\circ v} \omega$$

This follows directly from the definitions.

3.1 Homopoty invariance and periods homomorphism

Theorem 3.7 (Homotopy invariance). *Suppose X is a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$ is a closed differential form.*

- *(a) If $a, b \in X$ are two points and $u, v : [0,1] \to X$ are two homotopic curves from a to b, then*

$$\int_u \omega = \int_v \omega.$$

- *(b) If $u, v : [0,1] \to X$ are two closed curves which are free homotopic, then*

$$\int_u \omega = \int_v \omega.$$

□

Proof. (a) Let $p : \tilde{X} \to X$ be the universal covering and suppose $\hat{u}, \hat{v} : [0,1] \to \tilde{X}$ are liftings of u and v resp. with the same initial point. By the *homotopy lifting theorem* $\hat{u}$ and $\hat{v}$ also have the same end point. Hence the result follows from Lemma above.

(b) Suppose the curve u has initial and end point x_0 and the curve v has initial and end point x_1. Then there exists a curve w from x_0 to x_1 such that u is homotopic to $w \cdot v \cdot w^-$, cf. (3.13). Hence by (a) one has

$$\int_u \omega = \int_{w\cdot v\cdot w^-} \omega = \int_w \omega + \int_v \omega - \int_w \omega = \int_v \omega.$$

□

Directly speaking, this is saying that any closed form pullbacked to the universal covering must be exact.

This invariance under even free homotopy gives us a well-defineness to construct a morphisim from $\pi_1(X)$.

Definition 3.8 (Periods). Suppose X is a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$ is a closed differential form. Then by Theorem (10.10) one can define the integral

$$a_\sigma := \int_\sigma \omega, \quad \sigma \in \pi_1(X),$$

by choosing any curve representing the homotopy class σ and integrating along that curve. These integrals are called the periods of ω. Clearly

$$\int_{\sigma\cdot\tau} \omega = \int_\sigma \omega + \int_\tau \omega \text{ for } \sigma, \tau \in \pi_1(X).$$

Thus one gets a homomorphism $\pi_1(X) \to \mathbb{C}$ of the fundamental group of X into the additive group $\mathbb{C}$. This homomorphism is called the period homomorphism associated to the closed differential form ω.

Example 3.9. Suppose $X = \mathbb{C}^*$. By common knowledge, $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. A generator of $\pi_1(\mathbb{C}^*)$ is represented by the curve $u : [0,1] \to \mathbb{C}^*, u(t) = e^{2\pi it}$. Let $\omega := (dz/z)$, where z is the canonical coordinate. Then

$$\int_u \omega = \int_u \frac{dz}{z} = 2\pi i$$

Hence the period homomorphism of ω is

$$\mathbb{Z} \to \mathbb{C}, \quad n \mapsto 2\pi in,$$

where we have explicitly realized the isomorphism $\mathbb{Z} \cong \pi_1(\mathbb{C}^*)$ by the correspondence $n \mapsto \mathrm{cl}(u^n)$.

Definition 3.10 (Summands of Automorphy)**.** Let $G := Deck(\tilde{X}\backslash X)$ be the deck tranformation group, by the previous knowlegde, we also know that $G \cong \pi_1(X)$.

Let $\sigma \in G$ and $f : \tilde{X} \to \mathbb{C}$ be a function, then we can define a new function $\sigma f := f \circ \sigma^{-1}$. $\sigma(f+g) = \sigma f + \sigma g$ and $\sigma(fg) = (\sigma f)(\sigma g)$

This kind of like a group action on the $\mathcal{E}^1(\tilde{X})$.

A function $f : \tilde{X} \to \mathbb{C}$ is called **additively automorphic** with constant summands of automorphy of f, if:

$$\exists a_\sigma \in \mathbb{C}, \sigma \in \mathbb{C}, s.t. f - \sigma f = a_\sigma \quad \forall \sigma \in G$$

The constants a_σ, which are uniquely determined by f, are called the summands of automorphy of f. Then $\sigma f - \sigma\tau f = a_\tau$ for any $\sigma, \tau \in G$, since $f - \tau f = a_\tau$. Thus

$$a_{\sigma\tau} = f - \sigma\tau f = (f - \sigma f) + (\sigma f - \sigma\tau f) = a_\sigma + a_\tau.$$

Hence the correspondence $\sigma \mapsto a_\sigma$ is a group homomorphism $\mathrm{Deck}(\tilde{X}/X) \to \mathbb{C}$.

Any function $f : \widetilde{X} \to \mathbb{C}$ which is invariant under covering transformations, i.e., $\sigma f = f$ for every $\sigma \in G$, is an example of an additively automorphic function. In particular its summands of automorphy are all zero. For any such function there exists a function $f_0 : X \to \mathbb{C}$ such that $f = p^* f_0$. If f is differentiable (resp. holomorphic) then f_0 is differentiable (resp. holomorphic) as well.

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