Analytic Continuation

Def 7.1: X is R.S., $u: [0,1] \longrightarrow X$ is path in X

$a = u(0)$, $b = u(1)$. $\psi \in \mathcal{O}_b$ is said to result from the analytic continuation along the curve u of $\varphi \in \mathcal{O}_a$, if

$\exists \{\varphi_t \in \mathcal{O}_{u(t)} \mid t \in [0,1]\}$ s.t.

(i) $\varphi_0 = \varphi$, $\varphi_1 = \psi$

(ii) $\forall \tau \in [0,1]$, $\exists$ neighborhood $T \subseteq [0,1]$ of τ

s.t. $f_{u(t)} = \varphi_t$ for all $t \in T$

Equivalent condition: by figure 5

Lemma 7.2 Condition in 7.1. $\psi \in \mathcal{O}_b$ is a.c. along u of $\varphi \in \mathcal{O}_a$

$\Longleftrightarrow \exists !$ lifting

$$\begin{tikzcd} & {|\mathcal{O}|} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{u}"] \arrow[r, "u"'] & X \end{tikzcd}$$

s.t. $\hat{u}(0) = \varphi$, $\hat{u}(1) = \psi$.

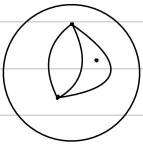
Pf: $\Rightarrow$ $\tilde{u}(t) = \varphi_{u(t)}$ direct from the definition

$\Leftarrow$ $\varphi_{u(t)} = \hat{u}(t)$

Uniqueness of lifting is direct from theorem 4.8

And it implies that give a path u and a germ on the beginning point then the analytic continuation along the curve is determined

Corollary 7.3: X is R.S. u_0, u_1 are path-homotopic. $u_0 \overset{H}{\simeq} u_1$



then $\varphi \in \mathcal{O}_a$ along the pathes in the homotopic class, yields the same function germ $\psi \in \mathcal{O}_b$

Pf: Lifting of Homotopy, theorem 4.10

note that $|\mathcal{O}|$ is Hausdorff

Corollary 7.4. X is a simply connected R.S. $a \in X$, $\varphi \in \mathcal{O}_a$ which admits an analytic continuation along every curve. Then $\exists$ a globally defined holo. $f \in \mathcal{O}(X)$ s.t. $f_a = \varphi$

Pf: $\psi_x \in \mathcal{O}_x$ is the function germ which results from the analytical continuation along any curve from a to x

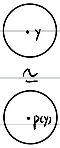
$f(x) \triangleq \psi_x(x)$ holo. $f_a = \varphi$.

f is unique by identity theorem.

•

Y and X are R.S. and $\mathcal{O}_X$ and $\mathcal{O}_Y$

$p: Y \longrightarrow X$ is unbranched holo. $\Rightarrow$ p is locally biholo.



$p^*: \mathcal{O}_{X,p(y)} \longrightarrow \mathcal{O}_{Y,y}$

$p_* \triangleq (p^*)^{-1}$

Def: 7.6 X is R.S. $a \in X$ and $\varphi \in \mathcal{O}_a$. (Y, p, f, b) is called an analytic continuation of φ.

(i) Y is R.S. and $p: Y \longrightarrow X$ unbranched holo.

(ii) $f \in \mathcal{O}(Y)$

(iii) $b \in Y$ s.t. $p(b) = a$ and $p_*(f_b) = \varphi$

Def: maximal analytic continuation of φ (Y, p, f, b)

universal property, if (Z, q, g, c) is any other analytic continuation of φ, then $\exists$ holo. map

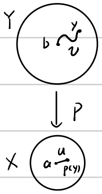
$$\begin{tikzcd} Z \arrow[rr, "F"] \arrow[dr, "q"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$F(c) = b$ and $F^*(f) = g$

Rmk: maximal analytic continuation is up to isomorphism

Lemma 7.7: $a \in X$, $\varphi \in \mathcal{O}_a$ and (Y, p, f, b) is a.c. of φ

Then v is curve in Y s.t. $v(0) = b$, $v(1) \triangleq y$, then $\psi \triangleq p_*(f_y) \in \mathcal{O}_{p(y)}$



is an a.c. of φ along $u = p \circ v$

pf: $\varphi_t \triangleq p_*(f_{v(t)}) \in \mathcal{O}_{p(v(t))} = \mathcal{O}_{u(t)}$.

check a.c.:

then $\varphi_0 = \varphi$ and $\varphi_1 = \psi$

Suppose $t_0 \in [0,1]$.

since p is local homeomorphism, pick open neighborhoods V of $v(t_0)$ and $u(t_0)$ s.t. $p|V \longrightarrow U$ biholomorphic.

then for any $\eta \in V$

$p_*(f_\eta) = [f \circ (p|_V)^{-1}]_{p(\eta)}$ by definition

so $\exists$ open neighborhood T of t_0 s.t. $v(T) \subseteq V \Rightarrow u(T) \subseteq U$

$p_*(f_{v(t)}) = [f \circ (p|_V)^{-1}]_{u(t)}$, $f \circ (p|_V)^{-1} \in \mathcal{O}(U)$

Theorem 7.8. X is R.S. $a \in X$, $\varphi \in \mathcal{O}_a$. Then $\exists$ maximal analytic continuation (Y, p, f, b) of φ.

pf: Y is connected component of $|\mathcal{O}|$ containing φ.

p is the restriction of projection on Y, $p: Y \longrightarrow X$ local homeomorphism.

Theorem 4.6 tells us Y can have complex structure, hence a Riemann surface and p is holo.

define $f: Y \longrightarrow \mathbb{C}$ as follows

$f(\eta) = \eta(p(\eta))$ (evaluation)

f is holo. and $p_*(f_\eta) = (f \circ (p|_V)^{-1})_{p(\eta)} = \eta$

set $b \triangleq \varphi$, then (Y, p, f, b) is analytical continuation of φ

check it is maximal:

suppose (Z, q, g, c) is another analytic continuation of φ.

define $F: Z \longrightarrow Y$, $\xi \in Z$, $x = q(\xi)$

by the lemma 7.7 $q_*(g_\xi) \in \mathcal{O}_x$ arises from a.c. along a curve from a to x of germ φ

By lemma 7.2, $\exists!\ \eta \in Y$ s.t. $q_*(g_\xi) = \eta$

Let $F(\xi) = \eta$, trivially $F: Z \to Y$ is fiber-preserving holo. map s.t. $F(c) = b$, $F^*(f) = g$

Algebraic Functions

- The elementary symmetric functions

Case 1: $\pi : Y \longrightarrow X$ is an n-sheeted unbranch holomorphic covering map and f is meromorphic function on Y.

For each point $x \in X$, U is its elementary neighborhood

$\pi^{-1}(U) = \dot{\bigcup}_{i=1}^{n} V_i$ and $\pi|_{V_i} : V_i \longrightarrow U$ is biholomorphic, $\tau_i \triangleq (\pi|_{V_i})^{-1}$

$$\begin{tikzcd}
Y \arrow[d, "\pi"'] & V_i \arrow[r, "f"] & \mathbb{P}^1 \\
X & U \arrow[u, "\tau_i"'] \arrow[ur, "\circlearrowright"] &
\end{tikzcd}$$

$\tau_i^*(f) = f \circ \tau_i$, denote as f_i

consider $\prod_{i=1}^{n} (T - f_i) = T^n + c_1 T^{n-1} + \cdots +$

$c_i = (-1)^i s_i(f_1, \ldots, f_n)$

Give global meromorphic functions by gluing each open piece

so $c_1, \ldots, c_n \in M(X)$

Case 2 (Thm 8.2): $\pi : Y \to X$ is $\underbrace{\text{n-sheeted}}_{\text{needs proper}}$ branched holomorphic map.

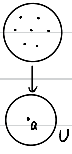
$A \subseteq X$, set of critical value. $B = \pi^{-1}(A)$

$f \in O(Y \setminus B)$ $(M(Y \setminus B))$ and $c_1 \cdots c_n \in O(X \setminus A)$ $(M(X \setminus A))$

Then f may be continued holomorphically (mero) to Y

$\Leftrightarrow$ All c_i can be continued holomorphically (mero). to X

Pf:



$V = \pi^{-1}(U)$

$(U_a, \psi) \xrightarrow[\text{behavior}]{\text{local}} \psi(a) = 0$

$V^* = V \setminus \pi^{-1}(a)$, $U^* = U \setminus \{a\}$

$\prod (f - f_i \circ \pi) \equiv 0$ on V^* $(*)$

(proof of $(*)$: if $y \in V^*$, for $\pi(y)$,

pick elementary neighborhood W, $p^{-1}(W) = \dot{\bigcup}_{i=1}^{n} V_i$, pick V_j s.t. $y \in V_j$,

then $(f - f_j \circ \pi) = (f - f \circ (\tau_j \circ \pi)) = (f - f) \equiv 0$ on V^*)

$f \in O(V^*)$, $f_1, \ldots, f_n \in O(U^*)$, $c_1, \ldots, c_n \in O(U^*)$

and $\prod_{i=1}^{n} (f - f_i \circ \pi) = f^n + (c_1 \circ \pi) \cdot f^{n-1} + \cdots + c_n \circ \pi = 0$

for $x = \pi(y)$, $f^n(y) + c_1(x) \cdot f^{n-1}(y) + \cdots + c_n(x) = 0$

so $f(y)$ is bounded around each point $\pi^{-1}(a)$

$\Leftrightarrow$ $c_i(x)$ is bounded around a.

$\Rightarrow$ $f \in M(V^*)$

$\varphi \triangleq \pi^*(\psi) = \psi \circ \pi \in O(V)$, vanishes at $\pi^{-1}(a)$

$$\begin{tikzcd}
V \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\
U \arrow[r, "\psi"'] & \mathbb{C}
\end{tikzcd}$$

so $\varphi^k \cdot f : V \longrightarrow \mathbb{P}^1$ holomorphic at b_j, for large k

The elementary function of $\varphi^k \cdot f$

$\prod_{i=1}^{n} (T - \varphi^k \cdot f_i) = T^n + \sum_{i=1}^{n} \psi^{ki} c_i T^{n-i}$

by holo case, $\psi^{ki} c_i$ is continued holo.

$\Rightarrow$ c_i is continued mero.

Rmk: we don't use that Y is connected

- Note: $\pi : Y \longrightarrow X$ is non-constant holomorphic map

the push back $\pi^*(f) = f \circ \pi$ induces monomorphism of fields

$\pi^* : M(X) \longrightarrow M(Y)$

$$\begin{tikzcd}
Y \arrow[d, "\pi"'] \arrow[dr, "f \circ \pi"] & \\
X \arrow[r, "f"'] & \mathbb{P}^1
\end{tikzcd}$$

Theorem: $\pi : Y \longrightarrow X$ is a n-sheeted branched holomorphic

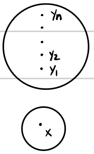
For $f \in M(Y)$, it induces $c_1(x), \ldots, c_n(x) \in M(X)$

then $f^n + (\pi^* c_1) f^{n-1} + \cdots + (\pi^* c_{n-1}) f + \pi^* c_n = 0$ $(*)$

and $\pi^* : M(X) \longrightarrow M(Y)$ is algebraic extension of degree $\leq n$

The equality holds if $\exists f \in M(Y)$ and $\exists x \in X$ with

$\pi^{-1}(x) = \{y_1 \cdots y_n\}$ s.t. $f(y_i)$ are all distinct



$\longrightarrow$ $f(y_1) \ldots f(y_n)$ are roots

so $T^n + \pi^* c_1(x) T^{n-1} + \cdots + \pi^* c_n(x) = 0$

Pf: Let $L \triangleq M(Y)$, $K \triangleq \pi^* M(X) \subseteq L$

So for all $f \in L \Rightarrow f$ is algebraic over K of degree $\leq n$

pick $f_0 \in L$ s.t. its degree n_0 of minimal poly is maximal in L

consider $\forall f \in L$, $\exists g \in L$, $K(f_0, f) = K(g)$ by primitive element theorem ($\operatorname{char} M(X) = 0 \Rightarrow$ perfect $\Rightarrow$ separable)

$n_0 \geq \dim K(g) = \dim K(f_0, f) \geq \dim K(f_0) \geq n_0$

So $K(f_0, f) = K(f_0) \Rightarrow L = K(f_0)$

Equality: f satisfies the condition

suppose $q(T)$ is the minimal poly of f over K, degree $m < n$

$f^m + d_1 \circ \pi f^{m-1} + \cdots + d_m \circ \pi = 0$, over $K = \pi^* M(Y)$

$f(y_1) \cdots f(y_n)$ are distinct roots

- Theorem 8.4: $A \subseteq X$ closed discrete subset and $X' = X \setminus A$.

$\pi': Y' \longrightarrow X'$ is a proper unbranched holomorphic covering map.

Then π' extends to a branched covering of X i.e.

$\exists$ R.S. Y, a proper holomorphic $\pi: Y \longrightarrow X$

and fiber-preserving biholomorphic mapping

$$\begin{tikzcd} Y \setminus \pi^{-1}(A) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & X' & \end{tikzcd}$$

Pf: $\forall a \in A$, choose (U_a, φ_a) on X s.t. $\varphi_a(a) = 0$, $\varphi_a(U_a) = \mathbb{D}$

and $U_a \cap U_{a'} = \phi$ if $a \neq a' \in A$. $U_a^* = U_a \setminus \{a\}$

by properness of π', $\pi'(U_a^*)$ contains finite connected component $\{V_a$

by Massey Lemma 2.1, $\pi'|V_{ai}^* \longrightarrow U_a^*$ is covering map, unbranched

suppose its covering number is k_{ai}

by Thm 5.10, we have ξ_{ai} biholo.

$$\begin{tikzcd} V_{ai}^* \arrow[r, "\xi_{ai}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & D^* \arrow[d, "\pi_{ai}: z \longrightarrow z^{k_{av}}"] \\ U_a^* \arrow[r, "\varphi_a"'] & D^* \end{tikzcd}$$

ideal point p_{ai}, $a \in A$, $i = 1, \ldots, n(a)$ pairwise distinct

$Y \triangleq Y' \cup \{p_{ai} \mid a \in A, i = 1, \ldots, n(a)\}$

Give topolgy: use neighbor basis W of a to define neighbor basis

p_{ai}, $\{p_{ai}\} \cup \{\pi'^{-1}(W^*) \cap V_{ai}^*\}$

Y is Hausdorff

$\pi(y) = \begin{cases} \pi'(y) & y \in Y' \\ a & y = p_{ai} \end{cases}$ π is proper, since the compact subset o

just compact subset of $X' \cup$ finite point

Make Y into Riemann surface

let $\xi_{ai}': V_{ai} \to D$ be the continuation of ξ_{ai}

by commutative diagram, $\xi'_{ai}(p_{ai}) = 0$

holomorphically compatible is trivial

by construction $\varphi: Y \setminus \pi^{-1}(A) \longrightarrow Y'$ identity

1st conclusion:

How elementary symetric function comes from:

covering map: push forward locally to get f_i

proper (n-sheeted): finite product of $(T - f_i)$ to get elementary symetric function

local homeomorphism again: we can glue C_i together

Goal: if $\pi: Y \longrightarrow X$

then $\pi(X) \longrightarrow \pi(Y)$ is finite separable extension

2rd conclusion:

Extension based on closed discrete subset A, along proper unbranched holo. map (sacrifice unbranched)

Extension doesn't enlarge deck group. (isomorphic under extension and restriction)

Theorem 8.5 : X, Y, Z are R.S. and $\pi : Y \longrightarrow X$, $\tau : Z \longrightarrow X$ proper holomorphic map. A is closed discrete subset. $X' = X \setminus A$. $Y' = \pi^{-1}(X')$, $Z' = \tau^{-1}(X')$ Then every F.P.B.M $\sigma' : Y' \longrightarrow Z'$ can extend to FPBM $\sigma : Y \longrightarrow Z$

Pf: $a \in A$. (U, ψ) s.t. $\psi(a) = 0$, $\psi(U) = \mathbb{D}$,

U is small enough that π and τ are unbranched over U^*

$V_1 \cdots V_n$ connected component of $\pi^{-1}(U)$, $W_1 \cdots W_m$ connected comp. of $\tau^{-1}(U)$

$V_i^* = V_i \setminus \pi^{-1}(a)$, Similar construction for $\tau^{-1}(U)$

$\sigma' | \pi^{-1}(U^*) \longrightarrow \tau^{-1}(U^*)$ is biholo. $\Rightarrow n = m$

Since $\pi|_{V_i^*} : V_i^* \longrightarrow U^*$ is finite sheeted unbranched covering

by 5.11

$$\begin{tikzcd}
V_i \arrow[r, "\text{biholo}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{D} \arrow[d, "p_k"] \\
U \arrow[r, "\psi\text{ chart}"'] & \mathbb{D}
\end{tikzcd}$$

$\Rightarrow V_i^* \cap \pi^{-1}(a)$ contains only one (chase by yourself)

Similar, $W_i \cap \tau^{-1}(a)$ contains only one element

So extends $\sigma'|_{\pi^{-1}(U^*)} \longrightarrow \tau^{-1}(U^*)$ to $\sigma|_{\pi^{-1}(U)} \longrightarrow \tau^{-1}(U)$ biholo. by Riemann Removable theorem

Def 8.6 Case above, $Y \rightarrow X$ is called Galois if $Y' \rightarrow X'$ is Galois

- Lemma 8.7: suppose $c_1 \cdots c_n \in \mathcal{O}(\mathbb{D}(R))$, $\mathbb{D}(R)$ radius $< R$

  $w_0 \in \mathbb{C}$ is simple zero of $T^n + c_1(0) T^{n-1} + \cdots + c_n(0) = 0$

  then $\exists\ r \leq R$ and $\varphi \in \mathcal{O}(\mathbb{D}(r))$ s.t. $\varphi(0) = w_0$

  $$\varphi^n + c_1 \varphi^{n-1} + \cdots + c_n = 0 \quad \text{on } \mathbb{D}(r)$$

  Pf: residue theorem

Corollary: 8.8 X is R.S. and $x \in X$

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n T^{n-1} \in \mathcal{O}_x[T]$$

And composite evaluation map:

$$p(T) = T^n + c_1(x) T^{n-1} + \cdots + c_n(x) \in \mathbb{C}[T]$$

has n distinct zeros $w_1 \cdots w_n$. Then $\exists\ \varphi_1 \ldots \varphi_n \in \mathcal{O}_x$ s.t. $\varphi_i(x) = w_i$ and $P(T) = \prod_{i=1}^{n} (T - \varphi_i)$

Theorem 8.9: $P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathcal{M}(X)[T]$ is irreducible polynomial of degree n. Then $\exists\ (Y, \pi, F)$ s.t.

(i) Y is R.S.

(ii) π is n-sheeted branched holo.

(iii) $F \in \mathcal{M}(Y)$, $(\pi^* P)(F) = 0$

(iv) for any (Z, τ, G) satisfies (i), (ii), (iii)

$$\begin{tikzcd}
Z \arrow[rr, "\exists ! \sigma"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\circlearrowright"] & Y \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

s.t. $\sigma^*(F) = G$

(Y, π, F) is called algebraic function defined by $P(T)$

Step 1: Theorem 8.4 allows us to delete some bad points: throw them into A.

(i) collect the point if some $c_1 \ldots c_n$ are not holomorphic on it

(ii) collect the point if $\Delta(x) = 0$ on it, $\Delta \in \mathcal{M}(X)$ is the discriminant of $P(T)$

($\Delta \in \mathbb{C}[c_1(x), \ldots, c_n(x)]$, by the Fundamental theorem on symetric function)

in other words, $X' \triangleq X \setminus A$, $c_1 \cdots c_n$ are holomorphic and $\Delta(x) \neq 0$

Goal: $P_x(T) = T^n + c_1(x) T^n + \cdots + c_n(x) \in \underbrace{\mathbb{C}}_{(i)}[T]$ is $\underbrace{\text{separable}}_{(ii)}$ for any $x \in X'$ and apply corollary 8.8

Step 2: Construct $Y' \longrightarrow X'$ unbranched and apply Thm 8.4

Def: (Topological space associated to a presheaf). X is topo. space $\mathcal{F}$ is its presheaf. $|\mathcal{F}| = \bigcup_{x \in X} \mathcal{F}_x$

U is open in X, $[U, f] \triangleq \{ f_x \mid x \in U \} \subseteq |\mathcal{F}|$, ($\rho_x(f)$ notation)

Facts without proof (Page 43, GTM 81)

1. $[U, f]$ form topo. basis of $|\mathcal{F}|$
2. $p : |\mathcal{F}| \longrightarrow X$ local homeomorphism
3. If X is R.S. with sheaf $\mathcal{O}$ or $\mathcal{M}$, $|\mathcal{O}|$ or $|\mathcal{M}|$ is Hausdorff.

$Y' \triangleq \{ \varphi \in \mathcal{O}_x \mid P(\varphi)=0, \text{for some } x \}$,

consider $\pi' : Y' \longrightarrow X'$ canonical projection

by the corollary, for $\forall\, x \in X'$, then $\exists$ open neighborhood $U \subseteq X'$ and $f_1, \dots, f_n \in \mathcal{O}(U)$ s.t. $P(T) = \prod_{i=1}^{n} (T - f_i)$

then $\pi'^{-1}(U) = \{ \varphi \in \mathcal{O}_x \mid \exists\, W$ open s.t. $x \in W$ and $\prod_{i=1}^{n} (\varphi - f_i) \equiv 0$ on $U \cap W \} = \bigcup_{i=1}^{n} [U, f_i]$

$[U, f_i]$ are disjoint since $\Delta(x) \neq 0$

$\pi'|[U, f_i] \longrightarrow U$ is homeomorphism

so π' is a covering map

The connected components of Y' are Riemann surface

so π' is covering map on each component

and π' is proper

(the key is that π' finite-sheet covering:

K is compact, $\forall\, x \in K$, find elementary open neighborhood U_x

U_x covers K => finite cover $K \subseteq \bigcup_{\text{finite}} U_x$

$\pi'^{-1}(K) \subseteq \bigcup_{\text{finite}} \pi'^{-1}(U_x)$

So $\pi'(K)$ is compact)

Step 3: Construct F by continued meromorphically

Let $f : Y' \longrightarrow \mathbb{C}$ be the evaluation map

use π' pullback the polynomial => $f(y)^n + c_1(\pi'(y)) f(y)^{n-1} + \cdots + c_n(\pi'(y)) = 0$ for any $y \in Y'$.

By theorem 8.4 extend $\pi' : Y' \to X'$ to $\pi : Y \to X$.

it means $c_i \circ \pi'$ can be continued meromorphically, by 8.2

f is continued mero. called $F \in M(Y)$

$(\pi^* P)(F) = 0$

Step 4: Y' is connected

(Note Y' is not connected, then Y may not be connected)

suppose Y has finitely many connected components $Y_1 \dots Y_k$

n_i its covering number, then $n_1 + \cdots + n_k = n$

Use the elementary symmetric on $F|_{Y_i}$, $P_i(T)$ coefficients in $M(X)$

$P_i(T) \mid P(T)$ since $(\pi^* P)(F_i) = 0$

so $P(T) = P_1(T) \cdots P_k(T)$ but it is irreducible

- Group homomorphism $\mathrm{Deck}(Y/X) \longrightarrow \mathrm{Aut}(M(Y))$

$$\sigma \longmapsto \sigma : f \longmapsto f \circ \sigma^{-1}$$

$\mathrm{Deck}(Y/X)$ fixes $\pi^* M(X)$

if $f \in \pi^* M(X)$, $f = g \circ \pi$, $\sigma f = g \circ \underbrace{\pi \circ \sigma^{-1}}_{\pi} = g \circ \pi = f$

by fiber preserving

So the image of σ is in $\mathrm{Aut}(M(Y)/\pi^* M(Y))$

Theorem: X is R.S. $K \triangleq M(X)$ and $P(T) \in K[T]$ is monic and irreducible of degree n. Let (Y, π, F) be the algebraic function by $P(T)$ and $L = M(Y)$. Then L/K is field extension of degree n and $L \simeq K[T]/P(T)$. And $\mathrm{Deck}(Y/X) \longrightarrow \mathrm{Aut}(L/K)$ isomorphism

The covering is Galois $\Longleftrightarrow$ L/K is Galois

pf: $P(F) = 0$ so we have eval homo. $K[T]/(P(T)) \longrightarrow L$

by 8.3 => both field are of degree n over K => isomorphism

injective: if $\sigma F = F$ for any $F \in M(Y)$

if $F \circ \sigma^{-1} = F$ => $\sigma = id$ by Rigity

surjective: $\forall\, \alpha \in \mathrm{Aut}(L/K)$, then $(Y, \pi, \alpha F)$ is still algebraic function by $P(T)$, since αF is also the root $\pi^* P$

by the uniqueness, $\exists\, \tau \in \mathrm{Deck}(Y/X)$ s.t. $\alpha F = F \circ \tau^{-1} = \tau \cdot F$

$L \cong K[T]/(P(T))$ is generated by F

Last statement: $|\mathrm{Aut}(L/K)| = [L:K]$

since Galois covering => acts transitively on $\pi^{-1}(x)$

$|\mathrm{Deck}(Y \backslash X)| = n$ => $|\mathrm{Aut}(L/K)| = n = [L:K]$

the converse is similar

Example 8.10 : $f(z)=(z-a_1)\cdots(z-a_n)$ distinct roots, $\in \mathcal{M}(\mathbb{P}^1)$

$P(T)=T^2-f$

$A=\{a_1,\dots,a_n\}\cup\{\infty\}$

$X'=\mathbb{P}^1\setminus A$, $Y'=\pi^{-1}(X')$

$\pi: Y'\longrightarrow X'$ is 2-sheeted unbranched holo.

Therefore every function germ s.t. $\varphi^2=f$ can be analytically continued along every curve lying in X'

consider exceptional points

(i) choose $r_j>0$ small enough s.t. no other points of A lies in the disk

$U_j \triangleq \{z\in\mathbb{C} \mid |z-a_j|<r_j\}$

$g(z)=\prod_{k\neq j}(z-a_k)$ non-vanishing on U_j

so $\exists$ holomorphic $h: U_j\longrightarrow\mathbb{C}$ s.t. $h^2=g$

$f(z)=(z-a_j)\cdot h^2$

consider $0<\rho<r_i$, $\xi=a_j+\rho\cdot e^{i\theta}$

$\exists$ a germ $\varphi_\xi\in\mathcal{O}_\xi$ s.t. $\varphi_\xi^2=f$

$\varphi_\xi(\xi)=\sqrt{f(\xi)}=\sqrt{\rho}\,e^{i\frac{\theta}{2}}h(\xi)$

this function germ along the curve $\xi=a_j+\rho e^{i\theta}$ $0\le\theta<2\pi$, then we obtain negative germ

$U_j^*=U_j\setminus\{a_j\}$, $V_j^*=\pi^{-1}(U_j^*)$

(I) V_j^* is connected => 2-sheeted covering map

$$\begin{tikzcd}
V_j^* \arrow[rr, "\text{biholo.}"] \arrow[dr, "\pi"'] & & \mathbb{D}^* \arrow[dl, "z\mapsto z^2"] \\
& \mathbb{D}^* &
\end{tikzcd}$$

(II) V_j^* is not connected, split into two single-sheeted covering

Contradiction

V_j^* is always connected => $\pi^{-1}(a_j)$ contains only one point

(ii) suppose $r>\max\{|a_1|,\dots,|a_n|\}$

$U^*\triangleq\{z\in\mathbb{C}\mid |z|>r\}$, $U=U^*\cup\{\infty\}\simeq\mathbb{D}$

write $f=z^nF$, F is holo and non-vanishing

(I) n is odd, $f(z)=z\cdot h^2(z)$

(II) n is even, $f(z)=h^2(z)$

for case (I), one point in Y over ∞

case (II), two points in Y over ∞

- Puiseux Expansion:

$\mathbb{C}\{\{z\}\}$ is field of all Laurent series with finite principal part, $\varphi(z)=\sum_{i=k}^{\infty}c_iz^i$, $k\in\mathbb{Z}$, $c_i\in\mathbb{C}$

converging in $\{0<|z|<r\}$, r depend on φ.

$\mathbb{C}\{\{z\}\}\simeq\mathcal{M}_0$

Theorem: $F(z,w)=w^n+a_1(z)w^{n-1}+\cdots+a_n(z)\in\mathbb{C}\{\{z\}\}[w]$ irreducible. Then $\exists$ Laurent series $\varphi(\xi)=\sum_{i=k}^{\infty}c_i\xi^i$ s.t. $F(\xi^n,\varphi(\xi))=0$

Pf: choose r small enough, $a_i(z)\in\mathcal{M}(D(r))$

$F(z,w)\in\mathcal{M}(D(r))[w]$, choose r small enough s.t.

$F(z,w)\in\mathbb{C}[w]$ has no multiple roots for each z in $D(r)\setminus 0$

(Y,π,f) is algebraic function defined by $F(z,w)$

π is n-sheeted proper. holo.

$$\begin{tikzcd}
Y \arrow[dr, "\pi"'] & & D(\rho) \arrow[ll, "\alpha"'] \arrow[dl, "z\mapsto z^n"] \\
& D(r) &
\end{tikzcd}$$

and $\rho=\sqrt[n]{r}$

So $\pi\circ\alpha(\xi)=\xi^n$

and $F(\pi,f)=0$

$\Rightarrow F(\pi\circ\alpha,f\circ\alpha)=0 \Rightarrow F(\xi^n,\varphi(\xi^n))=0$ for $\varphi=f\circ\alpha$