Riemannian surfaces

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1 Covering maps

1.1 fundamental group

Let X be a topological space and let $a, b \in X$. Two curves $u, v : I \to X$ from a to b are called *homotopic*, denoted $u \sim v$, if there exists a continuous mapping $A : I \times I \to X$ satisfying the following conditions:

1. $A(t, 0) = u(t)$ for every $t \in I$,
2. $A(t, 1) = v(t)$ for every $t \in I$,
3. $A(0, s) = a$ and $A(1, s) = b$ for every $s \in I$.

1.2 Branched and Unbranched Coverings

A subset A of a topological space is called *discrete* if every point $a \in A$ has a neighborhood V such that $V \cap A = \{a\}$.

A mapping $p : Y \to X$, between topological spaces X and Y, is said to be *discrete* if the fiber $p^{-1}(x)$ of every point $x \in X$ is a discrete subset of Y.

Given any map $\pi : E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the *fiber at p*. The fiber at p is often written E_p.

For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be *fiber-preserving* if $\phi(E_p) \subset E'_p$ for all $p \in M$.

Equivalently,given two maps $\pi : E \to M$ and $\pi' : E' \to M$, a map $\phi : E \to E'$ is fiber-preserving if and only if the diagram

$$\begin{array}{ccc} E & \stackrel{\phi}{\to} & E' \\ \pi & \searrow \quad \swarrow & \pi' \\ & M & \end{array}$$

commutes.

Theorem 1. *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. Then p is open and discrete.*

Proof. By (2.4) the map p is open. If the fiber of some point $a \in X$ were not discrete, then, by the Identity Theorem (1.11), p would be identically equal to a. □

A holomorphic (resp. meromorphic) function $f : Y \to \mathbb{C}$ (resp. $f : Y \to \mathbb{P}^1$) may also be considered as a multi-valued holomorphic (meromorphic) function on X. If $x \in X$ and $p^{-1}(x) = \{y_j : j \in J\}$, then the $f(y_j), j \in J$, are the different values of this multi-valued function at the point x. Of course it might turn out that $p^{-1}(x)$ is a single point or is empty.

Definition 1. Suppose Y and X are Riemann surfaces and $p : X \to Y$ is a non-constant holomorphic map. A point $y \in Y$ is called a *branch point* or *ramification point* of p, if there is no neighborhood V of y such that $p|_V$ is injective. The map p is called an *unbranched holomorphic map* if it has no branch points.

Theorem 2. *Suppose X and Y are Riemann surfaces. A non-constant holomorphic map $p : Y \to X$ has no branch points if and only if p is a local homeomorphism, i.e., every point $y \in Y$ has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.*

Proof. Suppose $p : Y \to X$ has no branch points and $y \in Y$ is arbitrary. Since y is not a branch point, there exists an open neighborhood V of y such that $p|_V$ is injective. Since p is continuous and open, p maps the set V homeomorphically onto the open set $U := p(V)$.

Conversely, assume $p : Y \to X$ is a local homeomorphism. Then for any $y \in Y$ there exists an open neighborhood V of y which is mapped homeomorphically by p onto an open set in X. In particular, $p|_V$ is injective and y is not a branch point of p. □

Examples

(a) Suppose k is a natural number ≥ 2 and let $p_k : \mathbb{C} \to \mathbb{C}$ be the mapping defined by $p_k(z) := z^k$. Then $0 \in \mathbb{C}$ is a branch point of p_k and the mapping $p_k|_{\mathbb{C}^*} \to \mathbb{C}$ is unbranched.

(b) Suppose $p : Y \to X$ is a non-constant holomorphic map, $y \in Y$ and $x := p(y)$. Then y is a branch point of p precisely if the mapping p takes the value x at the point y with multiplicity ≥ 2, cf. (2.2). By Theorem (2.1) the local behavior of p near y is just the same as the local behavior of the mapping p_k in example (a) near the origin.

(c) The mapping $\exp : \mathbb{C} \to \mathbb{C}^*$ is an unbranched holomorphic map. For exp is injective on every subset $V \subset \mathbb{C}$ which does not contain two points differing by an integral multiple of $2\pi i$.

(d) Suppose $\Gamma \subset \mathbb{C}$ is a lattice and $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is the canonical quotient mapping, cf. (1.5.d). Then π is unbranched.

Theorem 3. *Suppose X is a Riemann surface, Y is a Hausdorff topological space and $p : Y \to X$ is a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic.*

Remark 1. By (2.5) it follows that p is even locally biholomorphic, since a const map isn't a local homeo.

Theorem 4 (Uniqueness of Lifting)**.** *Suppose X and Y are Hausdorff spaces and $p : Y \to X$ is a local homeomorphism. Suppose Z is a connected topological space and $f : Z \to X$ is a continuous mapping. If $g_1, g_2 : Z \to Y$ are two liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$, then $g_1 = g_2$.*

Theorem 5. *Suppose X, Y and Z are Riemann surfaces, $p : Y \to X$ is an unbranched holomorphic map and $f : Z \to X$ is any holomorphic map. Then every lifting $g : Z \to Y$ of f is holomorphic.*

Corollary 1. Suppose X, Y and Z are Riemann surfaces and $p : Y \to X$ and $q : Z \to X$ are unbranched holomorphic maps. Then every continuous fiber-preserving map $f : Y \to Z$ is holomorphic. For f is a lifting of p with respect to q.

Proper Mappings: X,Y are two locally compact Hausdorff spaces, $f : X \to Y$ is proper if preimage of every compact subsets of Y is also compact in X.

Corollary 2. Proper map is closed.

Proof. Let $C \subseteq X$ be a closed set. To show that $f(C)$ is closed in Y, we use the characterization of closed sets in a locally compact Hausdorff space: a subset is closed if and only if its intersection with every compact set is compact.

Let $L \subseteq Y$ be an arbitrary compact set. Since f is proper, the preimage $K := f^{-1}(L)$ is compact in X. Then

$$f(C) \cap L = f(C \cap f^{-1}(L)) = f(C \cap K).$$

Because C is closed and K is compact in a Hausdorff space, the intersection $C \cap K$ is compact. The image of a compact set under a continuous map is compact, hence $f(C \cap K)$ is compact. Therefore $f(C) \cap L$ is compact for every compact $L \subseteq Y$.

By the characterization of closed sets in a locally compact Hausdorff space, it follows that $f(C)$ is closed in Y. Thus f is a closed map. □

Lemma 1. Suppose X and Y are locally compact spaces and $p : Y \to X$ is a proper, discrete map. Then the following hold:

(a) For every point $x \in X$ the set $p^{-1}(x)$ is finite.

(b) If $x \in X$ and V is a neighborhood of $p^{-1}(x)$, then there exists a neighborhood U of x with $p^{-1}(U) \subset V$.

Proof. (a) This follows from the fact that $p^{-1}(x)$ is a compact discrete subset of Y.

(b) We may assume that V is open and thus $Y \setminus V$ is closed. Then $p(Y \setminus V) =: A$ is also closed and $x \notin A$. Thus $U := X \setminus A$ is an open neighborhood of x such that $p^{-1}(U) \subset V$. □

Theorem 6. *Suppose X and Y are locally compact spaces and $p : Y \to X$ is a proper local homeomorphism. Then p is a covering map.*

Proof. Suppose $x \in X$ is arbitrary and let $p^{-1}(x) = \{y_1, \ldots, y_n\}$, where $y_i \neq y_j$ for $i \neq j$. Since p is a local homeomorphism, for every $j = 1, \ldots, n$ there exists an open neighborhood W_j of y_j and an open neighborhood U_j of x, such that $p|_{W_j} \to U_j$ is a homeomorphism. We may assume that the W_j are pairwise disjoint. Now $W_1 \cup \cdots \cup W_n$ is a neighborhood of $p^{-1}(x)$. Thus by (4.21.b) there exists an open neighborhood $U \subset U_1 \cap \cdots \cap U_n$ of x with $p^{-1}(U) \subset W_1 \cup \cdots \cup W_n$. If we let $V_j := W_j \cap p^{-1}(U)$, then the V_j are disjoint open sets with

$$p^{-1}(U) = V_1 \cup \cdots \cup V_n$$

and all the mappings $p|_{V_j} \to U,\ j = 1, \ldots, n$ are homeomorphisms. □

Proper Holomorphic Maps: Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a proper, non-constant, holomorphic mapping. It follows from Theorem (2.1) that the set A of branch points of f is closed and discrete. Since f is proper, $B := f(A)$ is also closed and discrete. One calls B the set of *critical values* of f.

Let $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B) \subset X \setminus A$. Then $f|_{X'} \to Y'$ is a proper unbranched holomorphic covering and by (4.22), (4.16) and (4.21.a) it has a well-defined finite number of sheets n. This means that every value $c \in Y'$ is taken exactly n times. In order to be able to extend this statement to the critical values $b \in B$ as well, we have to consider the multiplicities.

For $x \in X$ denote by $v(f, x)$ the multiplicity, in the sense of (2.2), with which f takes the values $f(x)$ at the point x. Then we will say that f takes the value $c \in Y$, counting multiplicities, m times on X, if

$$m = \sum_{x \in f^{-1}(c)} v(f, x).$$

Theorem 7. *Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic mapping. Suppose $a \in X$ and $b := f(a)$. Then there exists an integer $k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:*

(i) $a \in U$, $\varphi(a) = 0$; $\quad b \in U'$, $\psi(b) = 0$.

(ii) $f(U) \subset U'$.

(iii) The map $F := \psi \circ f \circ \varphi^{-1} : V \to V'$ *is given by*

$$F(z) = z^k \quad \textit{for all } z \in V.$$

Remark 2. The number k in Theorem (2.1) can be characterized in the following way. For every neighborhood U_0 of a there exist neighborhoods $U \subset U_0$ of a and W of $b = f(a)$ such that the set $f^{-1}(y) \cap U$ contains exactly k elements for every point $y \in W$, $y \neq b$. One calls k the *multiplicity* with which the mapping f takes the value b at the point a or one just says that f has multiplicity k at the point a.

Theorem 8. *Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a proper non-constant holomorphic map. Then there exists a natural number n such that f takes every value $c \in Y$, counting multiplicities, n times.*

Proof. Using the same notation as in (4.23) let n be the number of sheets of the unbranched covering $f|X' \to Y'$. Suppose $b \in B$ is a critical value, $p^{-1}(b) = \{x_1, \dots, x_r\}$ and $k_j := v(f, x_j)$. By (2.1) and (2.2) there exist disjoint neighborhoods U_j of x_j and V_j of b such that for every $c \in V_j \setminus \{b\}$ the set $p^{-1}(c) \cap U_j$ consists of exactly k_j points ($j = 1, \dots, r$). By Lemma (4.21.b) we can find a neighborhood $V \subset V_1 \cap \cdots \cap V_r$ of b such that $p^{-1}(V) \subset U_1 \cup \cdots \cup U_r$. Then for every point $c \in V \cap Y'$ we have that $p^{-1}(c)$ consists of $k_1 + \cdots + k_r$ points. On the other hand, for $c \in Y'$ the cardinality of $p^{-1}(c)$ is equal to n. Thus $n = k_1 + \cdots + k_r$. □

Remark 3. A proper non-constant holomorphic map will be called an n-sheeted map, where n is the integer found in the previous Theorem. Note that n-sheeted maps are allowed to have branch points. If we wish to emphasize that there are none, then we will specifically say that the map is unbranched. If we speak of a topological covering map or if there is no complex structure, then we mean a covering map in the sense of (4.11).

Corollary 3. On any compact Riemann surface X every non-constant meromorphic function $f : X \to \mathbb{P}^1$ has as many zeros as poles, where each is counted according to multiplicities.

Proof. The mapping $f : X \to \mathbb{P}^1$ is proper. □

Corollary 4. Any polynomial of n-th degree

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$$

has, counting multiplicities, exactly n zeros.

Proof. By (2.3) we may consider f as a holomorphic mapping $f : \mathbb{P}^1 \to \mathbb{P}^1$ which, counting multiplicities, takes the value ∞ exactly n times, since $f^{(n-1)} = 0, f^{(n)} \neq 0$. □

1.3 The Universal Covering and Covering Transformations

Lemma 2. Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X and $\tilde{x}_i \in \tilde{X}_i$, $i = 1, 2$, points such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Then, there exists a homomorphism φ of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$ if and only if

$$p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subset p_{2*}\pi(\tilde{X}_2, \tilde{x}_2).$$

Let $(\tilde{X}, p)$ be a covering space of X such that $\tilde{X}$ is simply connected. If $(\tilde{X}', p')$ is any other covering space of X, then, by Lemma 6.3, there exists a homomorphism φ of $(\tilde{X}, p)$ onto $(\tilde{X}', p')$, and, by the lemma just proved, $(\tilde{X}, \varphi)$ is a covering space of $\tilde{X}'$; i.e., $\tilde{X}$ can serve as a covering space of any covering space of X. For this reason a simply connected covering space, such as $(\tilde{X}, p)$, is called a **universal covering space**. By Theorem 6.6, any two universal covering spaces of X are isomorphic.

Definition 2. Suppose X and Y are topological spaces and $p : Y \to X$ is a covering map. By a *covering transformation* or *deck transformation* of this covering we mean a fiber-preserving homeomorphism $f : Y \to Y$. With operation the composition of mappings, the set of all covering transformation of $p : Y \to X$ forms a group which we denote by $\mathrm{Deck}(Y/X)$. If there is any chance of confusion, then we will write $\mathrm{Deck}(Y \xrightarrow{p} X)$ instead of $\mathrm{Deck}(Y/X)$.

Definition 3. Suppose X and Y are connected Hausdorff spaces and $p : Y \to X$ is a covering map. The covering is called Galois (the terms normal and regular are also in common usage) if for every pair of points $y_0, y_1 \in Y$ with $p(y_0) = p(y_1)$ there exists a covering transformation $f : Y \to Y$ such that $f(y_0) = y_1$.

Or

Definition 4. Let $p : Y \to X$ be a covering map with X and Y connected and locally path-connected, and let $p(y_0) = x_0$. Then p is called a **Galois covering** (or normal covering) if the induced homomorphism

$$p_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$$

has image $p_*(\pi_1(Y, y_0))$ which is a normal subgroup of $\pi_1(X, x_0)$.

Example 1. The mapping $p : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^k$, is a covering map. It is Galois since for any $z_1, z_2 \in \mathbb{C}^*$ with $p(z_1) = p(z_2)$, one has $z_2 = \omega z_1$ where ω is a k-th root of unity and the mapping $z \mapsto \omega z$ is a covering transformation.

Corollary 5. If $(\tilde{X}, p)$ is a regular covering space of X, then $A(\tilde{X}, p)$ is isomorphic to the quotient group $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$ for any $x \in X$ and any $\tilde{x} \in p^{-1}(x)$.

Theorem 9. *Suppose X is a connected manifold and $p : \tilde{X} \to X$ is its universal covering. Then p is Galois and $\mathrm{Deck}(\tilde{X}/X)$ is isomorphic to the fundamental group $\pi_1(X)$.*

Proof. Only use the above corollary. □

Remark 4. We can replace connected manifold with path connected and locally path connected topological space.

Example 2. (a) $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering of $\mathbb{C}^*$, since $\mathbb{C}$ is simply connected. For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be translation by $2\pi i n$. Then $\exp(\tau_n(z)) = \exp(z + 2\pi i n) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a covering transformation. If σ is any covering transformation, then $\exp(\sigma(0)) = \exp(0) = 1$ and thus there exists $n \in \mathbb{Z}$ such that $\sigma(0) = 2\pi i n$. Since $\tau_n(0) = 2\pi i n$ as well, $\sigma = \tau_n$. Thus

$$\mathrm{Deck}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*) = \{\tau_n : n \in \mathbb{Z}\}.$$

Since the last group is isomorphic to $\mathbb{Z}$,

$$\pi_1(\mathbb{C}^*) \cong \mathbb{Z}.$$

(b) Let

$$H = \{z \in \mathbb{C} : \Re(z) < 0\}$$

be the left half plane and

$$D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Then $\exp : H \to D^*$ is the universal covering of the punctured unit disk. As in Example (a) one can show that the group of covering transformations consists of all translations by integral multiples of $2\pi i$ and that $\pi_1(D^*) \cong \mathbb{Z}$.

Theorem 10. *Suppose X and Y are connected manifolds, $q : Y \to X$ is a covering map and $p : \tilde{X} \to X$ is the universal covering. Let $f : \tilde{X} \to Y$ be a continuous fiber-preserving mapping, which by the definition of the universal covering exists. Then f is a covering map and there exists a subgroup $G \subset \mathrm{Deck}(\tilde{X}/X)$ such that two points $x, x' \in X$ are mapped onto the same point by f precisely if they are equivalent modulo G. Moreover $G \cong \pi_1(Y)$.*

Proof. Let $G := \mathrm{Deck}(\tilde{X}/Y)$. This is a subgroup of $\mathrm{Deck}(\tilde{X}/X)$. Since $\tilde{X}$ is simply connected, $f : \tilde{X} \to Y$ is the universal covering of Y and so is Galois. Hence

$$G \cong \pi_1(Y)$$

and $f(x) = f(x')$ if and only if there exists $\sigma \in G$ such that $\sigma(x) = x'$.

This completes the proof of Theorem. □ □