· Moduli space

Def: a parameter space for classes of geometric object of interest

Ex: moduli space of lines in $\mathbb{R}^2$

if not vertical to x-axis $y=mx+b$ $(m,b) \Rightarrow \mathbb{R}^2$

$ax+by+c=0$ $(a,b,c)\sim(\lambda(a,b,c), \lambda\neq 0$; a,b not all 0 $\Rightarrow \mathbb{P}^1\setminus\{(0,0,1)\}$ Möbius strip

Ex: triangles $\sim$: iso $\{(a,b,c)\mid a+b>c,\ b+c>a,\ a+c>b,\ a,b,c>0\}\subset\mathbb{R}^3$ ① to coarse moduli space that satisfy 1-to-1-corre and universal property

① hope to keep self-auto $\triangle: S_2$ $\triangle: S_3$ $\Leftarrow$ stack

stack: a category $\mathcal{M}$ together with a functor from $\mathcal{M}$ to category Top of topological spaces / Schemes / geometry you're interested in, satisfying some precise extra conditions we won't get into.

· The Moduli Spaces $\mathcal{M}_{g,n}$

Fix $g,n\geq 0$ with $2g-2+n>0 \Rightarrow$ finite self-auto

$M_{g,n}$ denotes the moduli space of genus g, n marked Riemann Surfaces.

write $\mathcal{M}_g=\mathcal{M}_{g,0}$ $\Leftarrow$ a theorem about existence

Ex ($M_{0,3}$) genus 0 $\Rightarrow$ only $\mathbb{P}^1 \cong \mathbb{C}\cup\{\infty\}$ ~~only one~~

$f(z)=\frac{az+b}{cz+d}$ $f(z)=z \Rightarrow (z^2+(d-a)z+b$ has ≥ 3 roots $\therefore c=(d-a)=b=0$ $f(z)=z$

$\therefore$ only 1 self-auto ~~$f(z)=$~~ for (a,b,c) let $f(z)=\frac{(z-a)(b-c)}{(z-c)(b-a)}$ $\therefore (a,b,c)\Rightarrow(0,1,\infty)$

$\therefore \forall\ (x_1,y_1,z_1)\to(x_2,y_2,z_2)$ $\exists!$ iso $\therefore$ a point

Ex ($M_{0,4}$) first three $\to\{0,1,\infty\}$ $cr(a,b,c,d)=\frac{(d-a)(b-c)}{(b-a)(d-c)}=f(d)$

able to get $M_{g,n}$ for $n>3$

Ex ($M_{1,1}$) genus 1 $\Rightarrow \mathbb{C}/\Lambda$ $\Lambda=\{\mathbb{Z}a+\mathbb{Z}b\}$, $\frac{b}{a}\notin\mathbb{R}$ iso $f:\mathbb{C}\to\mathbb{C}$ s.t. $f(z)=mz+t$

s.t. marked point $\leftrightarrows$ $(0,0)$ $(a,b)\to(1,\tau)$ we can exchange (a,b) such that

$\tau\in$ upper half plane H consider self-auto $\Leftrightarrow \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}$ $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$

$\Leftrightarrow \mathbb{C}/(1,\tau)=\mathbb{C}/(c\tau+d,a\tau+b)\cong\mathbb{C}/(1,\frac{a\tau+b}{c\tau+d})$. where $\frac{a\tau+b}{c\tau+d}\in H$ (Möbius transformation)

$M_{1,1}=H/SL_2(\mathbb{Z})$

Ex (M_2): $M_{0,6}/S_6$ genus 2 get from branched cover of $\mathbb{P}^1$ of degree 2, branched exactly over $(2g+2)=6$ points

· The Deligne-Mumford-Knudsen Compactification $\overline{M}_{g,n}$

Other than $M_{0,3}$, space $M_{g,n}$ not compact, $\Rightarrow$ modular compactification

Def 1. A nodal curve of genus g is a proper, connected algebraic curve X over $\mathbb{C}$ with arithmetic genus $h^1(X, \mathcal{O}_X) = g$ whose only singularities, if any, are nodes. A node is a complex point $x \in X$ with analytic-local equation $uv = 0$: two branches meeting transversely

Def 2. A nodal, n-marked curve is a nodal curve X as above with $p_1, \cdots, p_n \in X$ distinct smooth complex points of X. Simply put, you are ~~fro~~ forbidden from marking a node.

Say $(X, p_1, \cdots, p_n)$ stable if its auto-map is finite

Def 3. Fix $g, n \geq 0$, with $2g-2+n > 0$ then $\overline{M}_{g,n}$ denotes the moduli space of n-marked stable curve of genus g.

Again, def $\Rightarrow$ a remarkable $\exists$ theorem

In fact, boundary $\overline{M}_{g,n} \setminus M_{g,n}$ admits a stratification in which the strata are assembled in a combinatorial way from smaller moduli spaces $M_{g',n'}$

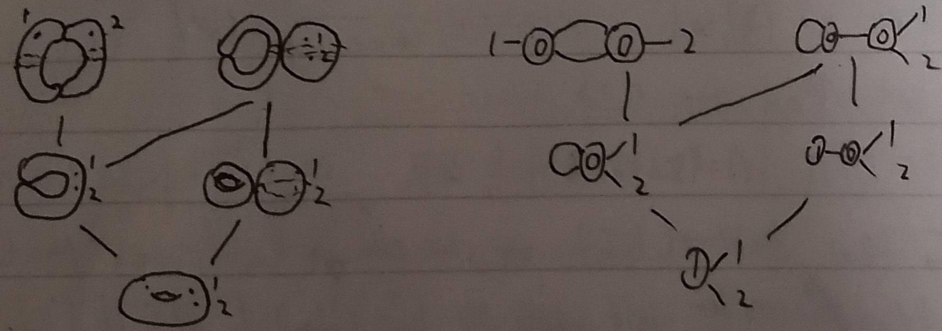
Def: A ~~g~~ dual-graph for stable ~~curve~~ curve $(X, p_1, \cdots, p_n)$ is a triple $\mathbb{G} = (G, m, w)$ that

1. G is a (muti)graph, with a vertex v corresponding to each irreducible component C_v of X, and an edge $v \frown w$ for every node of X on $C_v \cap C_w$

   allow loops, muti-edge?

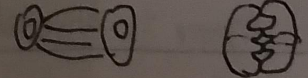
2. marking function $m: \{1, \cdots, n\} \to V(G)$ sends $m(i) = v$ when p_i lies on C_v
3. weight function $w: V(G) \to \mathbb{Z}_{\geq 0}$ $w(v) := $ genus of $\tilde{C_v}$, the normalization of C_v

Ex



stability: $\forall v$, $2w(v) - 2 + n_v = 0$ $\quad$ $n_v :=$ degree of v $\quad$ Fix stable graph G

Let $\widetilde{M}_G = \prod_{v \in V(G)} M_{w(v), n(v)}$ then $M_G = [\widetilde{M}_G / \mathrm{Aut}(G)]$

Ex:  $M_G = [(M_{0,4} \times M_{0,4}) / (S_2 \times S_4)]$

· Tropical Curves

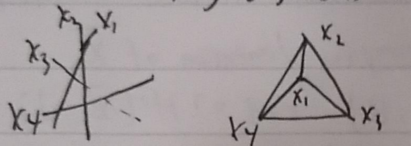
degeneration techniques in algebra geometry

basic idea: get information on the general behavior of a smooth algebraic curve, say, by studying a one-parameter family of curves, which degenerate in the limit to a singular curve, instead.

Ex: family of projective plan quartics C_t : $t(x^4+y^4+z^4)+xyz(x+y+z)=0$

$t=0 \Rightarrow xyz(x+y+z)=0$

slogan Tropical geometry is a very drastic degeneration technique in algebraic geometry in which the limiting object is entirely combinatorial



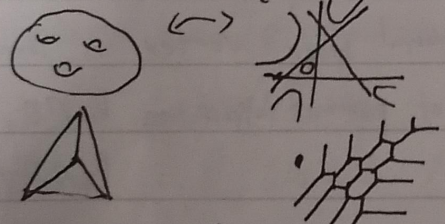
edge length 1

$g(G) = \dim H_1(G,\mathbb{Q}) = |E(G)|-|V(G)|+\#\{\text{connected component of } G\} = 3 =$ genus of smooth curve C_t

Def : A tropical curve is a pair (G, l), where

1. G is a stable graph 2. $l : E(G) \to \mathbb{R}_{>0}$ a function on the edge set of G

abstract | embedded (as if they are real 1-dim curve)



(tropical curves are drawn very differently)

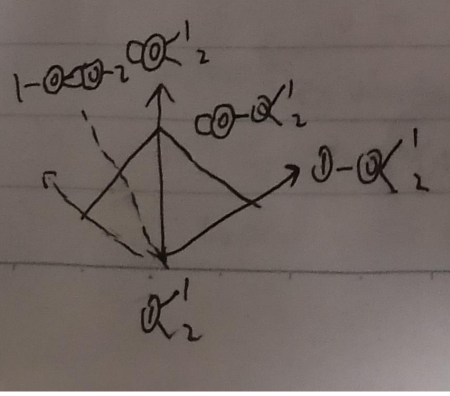
· Moduli space of Tropical Curves

$M^{trop}_{g,n}$ a combinatorial space glued from polyhedral cones

fix a stable graph G $M^{trop}_G = \mathbb{R}^{E(G)}_{>0} / Aut(G)$ 1-⓪-⓪-2

contracting : allow edges of length 0 $\Rightarrow$ loop $0 \to$ ~~...~~ $w(v)$++ ~~...~~

$M^{trop}_{g,n} = \left(\coprod_G \mathbb{R}^{E(G)}_{\geq 0} / Aut(G) \right) / \sim$ $\sim$ is generated by contracting edges



def: link $\Delta_{g,n}$: subspace of $M^{trop}_{g,n}$ where $\sum$ length of edge $=1$

· The weight Filtration

Def. A pure Hodge structure of weight $n\in\mathbb{Z}$ is a finitely generated free abelian group $H_{\mathbb{Z}}$ together with a decomposition of $H_{\mathbb{C}} = H_{\mathbb{Z}}\otimes\mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$ s.t. $H^{q,p} = \overline{H^{p,q}}$

X a complex variety (no need smooth or compact)

Deligne defines a weight filtration ~~on~~ : $W_0 \subset \cdots \subset W_{2i} = H^i(X;\mathbb{Q})$

weight j graded piece $Gr_j^W H^i := W_j/W_{j-1}$ equipped with a pure Hodge structure of weight j.

Def. X smooth. $X\subset\overline{X}$ is a simple normal crossings compactification of X, ~~which~~ it ~~this~~ means is

1. $\overline{X}$: a smooth variety that is complete i.e. it is compact
2. the irreducible components of the boundary $D=\overline{X}\setminus X$ are smooth and intersect transversely

$D_1,\cdots,D_e$: irreducible components of the boundary, d : complex dimension of $\overline{X}$

aim: $Gr^W_{2d-j} H^*(X,\mathbb{Q})$ consider $H^j(D_{i_0}\cap\cdots\cap D_{i_r};\mathbb{Q})$ $r=-1 \Rightarrow H^j(\overline{X};\mathbb{Q})$

natural maps : $H^j(D_{i_0}\cap\cdots\cap\widehat{D_{i_j}}\cap\cdots D_{i_r};\mathbb{Q}) \longrightarrow H^j(D_{i_0}\cap\cdots\cap D_{i_r};\mathbb{Q})$

chain complex $0\longrightarrow H^j(\overline{X};\mathbb{Q}) \xrightarrow{\delta_0} \bigoplus_{i_0} H^j(D_{i_0};\mathbb{Q}) \xrightarrow{\delta_1} \bigoplus_{i_0<i_1} H^j(D_{i_0}\cap D_{i_1};\mathbb{Q}) \xrightarrow{\delta_2} \cdots \to 0$ (3)

$Gr^W_j H_c^{j+i}(X,\mathbb{Q}) \cong \frac{\ker\delta_i}{\operatorname{im}\delta_{i-1}}$ Poincaré duality, $\Rightarrow Gr^W_{2d-j} H^{2d-j-i}(X,\mathbb{Q}) \cong (Gr^W_j H_c^{j+i}(X,\mathbb{Q}))^*$

compactly support cohomology $j=0 \Rightarrow H^0(Y,\mathbb{Q})$ # {component of Y}

Def: dual complex/boundary complex of D $\Delta(X\subset\overline{X})$ combinatorial space : vertex is irreducible component D_{i_0}, an edge for every irr- component of pairwise intersection $D_{i_0}\cap D_{i_1}$

triangle $D_{i_0}\cap D_{i_1}\cap D_{i_2}$

$j=0$, the chain complex (3) is the reduced cochain complex of the boundary complex $\Delta(X\subset\overline{X})$

$Gr^W_{2d} H^{2d-i}(X,\mathbb{Q}) \cong \widetilde{H}_{i-1}(\Delta(X\subset\overline{X});\mathbb{Q})$ fact: cohomology never appears in weight $>2d$

$H^{2d-i}(X,\mathbb{Q}) \twoheadrightarrow \widetilde{H}_{i-1}(\Delta(X\subset\overline{X});\mathbb{Q})$

· The Top-Weight Cohomology of $M_{g,n}$

theorem of Abramovich-Caporaso-Payne gives $\Delta_{g,n} \cong \Delta(M_{g,n}\subset\overline{M}_{g,n})$

$H^{6g-6+2n-i}(M_{g,n};\mathbb{Q}) \twoheadrightarrow \widetilde{H}_{i-1}(\Delta_{g,n};\mathbb{Q})$

~~Graph~~

- Graph Complexes : an umbrella term for a (co)chain complex of vector space with labels of decorations.

$G^{(g,n)}$ $g, n \geq 0,\ 2g-2+n > 0$, generators : triples (G, m, w)

G: connected (multi)graph of first Betti number g $(|E|-|V|+1)$, w is a total order on the edge set of G, $m: \{1,\cdots,n\} \to V(G)$ marking function s.t. $\deg(v) \geq 3$ for $\forall$ vertex v (add marked)

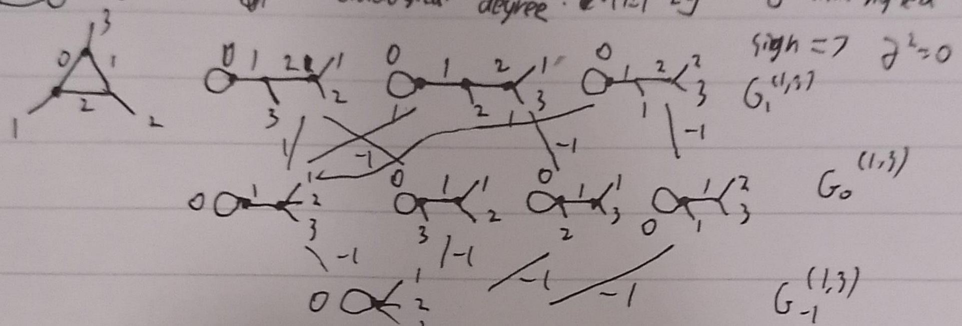
$(G,m,w) = \pm (G',m',w')$ whenever $\exists$ iso of marked graph $(G,m) \to (G',m')$

the sign depend on ~~even/odd~~ even/odd permutation $\therefore$ self-identity odd $\Rightarrow (G,m,w) = 0$

$0\!\!\!-\!\!\!0 = 0$ homological degree : $|E| - 2g$ ∂ : signed sum of 1-edge-contractions.

sign $\Rightarrow \partial^2 = 0$

Ex:



M. Chan, S. Galatius, S. Payne : $\tilde{H}_{k+2g-1}(\Delta_{g,n}; \mathbb{Q}) \cong H_k(G^{(g,n)})$ $2g-1 \Rightarrow 70$ condition

proof: symmetric Δ-complexes : roughly like CW complexes $\Delta_{1,3} \simeq S^2$

~~other~~ cellular homology

The Grothendieck-Teichmüller Lie Algebra $\mathfrak{grt}_1$

$$GC = \prod_{g \geq 2} \mathrm{Hom}(G^{(G)}, \mathbb{Q}) \qquad \text{Willwacher} : H^0(GC) \cong \mathfrak{grt}_1$$

Brown $\mathfrak{f}_{Lie}(\sigma_3, \sigma_5, \sigma_7, \cdots) \hookrightarrow \mathfrak{grt}_1 \Rightarrow$ the dimensions of the graded pieces of RHS grow at least as fast as those of LHS.

$\Rightarrow$ degree g graded pieces grows faster than β^g for $\forall \beta < \beta_0$, where

$\beta_0 \simeq 1.3247\ldots$ (real root of $t^3 - t - 1 = 0$)

Theorem 4 $\dim H^{4g-6}(M_g; \mathbb{Q}) > \beta^g +$ constant

rational cohomology of M_g must vanish in degree above $4g-5$

$$H^{4g-5}(M_g; \mathbb{Q}) = 0$$

problem find elusive, yet abundant, odd degree cohomology groups of M_g

- $H^{15}(M_6; \mathbb{Q})$; $H^{23}(M_8, \mathbb{Q})$ $H^{27}(M_{10}, \mathbb{Q})$ nonzero through translate back through tropical moduli spaces and boundary complexes over to the top weight cohomology of M_g