# 1. Cohomology.

**Def** "Dual" of homology.

$$\text{homology: } \cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots$$

$$\text{cohomology: } \cdots \leftarrow C_n^*(X) \xleftarrow{\partial^* = \delta} C_{n-1}^*(X) \xleftarrow{\partial^* = \delta} C_{n-2}^*(X) \xleftarrow{\delta} \cdots$$

$C_n^*(X) = C^n(X)$, $C_{n-1}^*(X) = C^{n-1}(X)$, $C_{n-2}^*(X) = C^{n-2}(X)$

$$\varphi \circ \partial \longleftarrow\!\shortmid \varphi$$

$$H^n(X) \triangleq H_n(C^\bullet(X)).$$

In general, $H^n(X;G) = H_n(C^\bullet(X;G);G)$, where $C^\bullet(X;G) = C_\bullet^*(X;G)$

- relative cohomology.

$$C^n(X,A;R) = \{\varphi \in C^n(X;R) \mid \varphi(\sigma)=0,\ \forall \sigma: \Delta^n \to A\}.$$

- Cohomoly still satisfies some of Eilenberg–Steenrod Axioms.

In particular, <u>excision</u>. $H^n(X,A) \cong H^n(X \setminus B, A \setminus B)$. $(\overline{B} \subset \mathring{A})$

<u>homotopy invariance</u>. $f \simeq g \Rightarrow f^* = g^*: H^n(Y,B) \to H^n(X,A)$.

<u>induced homomorphisms</u> $f: X \to Y \Longrightarrow f_\#: C_n(X) \to C_n(Y)$, $f^\#: C^n(Y) \to C^n(X)$

- Compare $H^k(X)$ and $H_k(X)^* = \mathrm{Hom}_G(H_k(X), G)$.

$$\begin{array}{ccccccccc}
0 & \to & Z_{n+1} & \xrightarrow{i} & C_{n+1} & \xrightarrow{\partial} & B_n & \to & 0 \\
 & & \downarrow \partial=0 & & \downarrow \partial & & \downarrow 0 & & \\
0 & \to & Z_n & \xrightarrow{i} & C_n & \xrightarrow{\partial} & B_{n-1} & \to & 0
\end{array}$$

$\xrightarrow{\text{dualize}}$

$$\begin{array}{ccccccccc}
0 & \leftarrow & Z_{n+1}^* & \xleftarrow{i^*} & C_{n+1}^* & \leftarrow & B_n^* & \leftarrow & 0 \\
 & & \uparrow 0 & & \uparrow \delta & & \uparrow 0 & & \\
0 & \leftarrow & Z_n^* & \leftarrow & C_n^* & \leftarrow & B_{n-1}^* & \leftarrow & 0
\end{array}$$

homology $\rightsquigarrow$ $B_n^* \xleftarrow{i_n^*} Z_n^* \longleftarrow H^n \longleftarrow B_{n-1}^* \xleftarrow{i_{n-1}} Z_{n-1}^*$

break into SES $\rightsquigarrow$ $0 \leftarrow \ker i_n^* \longleftarrow H^n \longleftarrow \operatorname{coker} i_{n-1}^* \longleftarrow 0$

$$\ker i_n^* = \{\alpha \in Z_n^* \mid \alpha|_{B_n} = 0\} \cong (H_n)^*$$

$$\operatorname{coker} i_{n-1}^* \cong \operatorname{Ext}(H_{n-1}, G)$$

<u>Thm 3.2</u> (UCT for cohomology).

There is splitting SES.

$$0 \longrightarrow \operatorname{Ext}(H_{k-1}(X), G) \longrightarrow H^k(X) \longrightarrow \operatorname{Hom}(H_k(X), G).$$

$$\Rightarrow H^k(X) \cong H_k(X)^* \oplus \operatorname{Ext}(H_{k-1}(X), G)$$

## 2. Cup products

**Def** $\smile$ : $\forall\, \varphi \in C^k(X)$, $\psi \in C^\ell(X)$.

multi in ring $R$.

Define $\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_1, \cdots, v_k]) \cdot \psi(\sigma|[v_{k+1}, \cdots, v_{k+\ell}])$.

**Lemma 3.6** $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$

[proof] Calculation.

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{k+1}])\psi(\sigma|[v_{k+1}, \cdots, v_{k+\ell+1}])$$

$$(-1)^k(\varphi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, \hat{v}_i, \cdots, v_{k+\ell+1}])$$

**Cor** $\smile$ induces $H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$

$(\varphi, \psi) \longmapsto \varphi \smile \psi$.

- Also, $H^k(X) \times H^\ell(X, A) \longrightarrow H^{k+\ell}(X, A)$.

  $H^k(X, A) \times H^\ell(X, A) \longrightarrow H^{k+\ell}(X, A)$.

**prop 3.10** $f^*: H^\cdot(Y) \longrightarrow H^\cdot(X)$

Then $f^*(\alpha \smile \beta) = (f^*\alpha) \smile (f^*\beta)$.

**proof**: Calculation.

$$\begin{aligned}(f^\sharp\varphi \smile f^\sharp\psi)(\sigma) &= f^\sharp\varphi(\sigma|[v_0, \cdots, v_k]) f^\sharp\psi(\sigma|[v_k, \cdots, v_{k+\ell}]) \\ &= \varphi(f\sigma|[v_0, \cdots, v_k])\psi(f\sigma|[v_k, \cdots, v_{k+\ell}]) \\ &= (\varphi \smile \psi)(f\sigma) = f^\sharp(\varphi \smile \psi)(\sigma)\end{aligned}$$

$A \subset X$, $x \in X$.

Set $H^*(X|_x) = H^*(X, X \setminus \{x\})$

$H^*(X|_A) = H^*(X, X \setminus A)$.

$\tilde{M} = \{\mu_x \mid \mu_x$ is an local orientation at $x \in X\}$.

Then $\begin{array}{rcl} \tilde{M} & \longrightarrow & M \\ \mu_x & \longmapsto & x \end{array}$ is a covering map.

<u>Thm 3.26</u> $M$: closed connected $n$-manifold.

Then

[M] is called the fundamental class of M.

$$R[M] \longmapsto R\alpha$$

(a) $M$: $R$-orientible, then $H_n(M;R) \xrightarrow{\simeq} H_n(M|_x;R) \simeq R$.

(b) $M$ not $R$-orientible. then $H_n(M;R) \hookrightarrow H_n(M|_x;R) \simeq R$.

(c) $\forall i > n$, $H^*(M;R) = 0$.

## 3. Cap product

$\frown$ : $k \geq \ell$ $\quad C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$

a singular in $C_\ell(X)$

by $(\sigma \frown \varphi)[v_{\ell+1}, \cdots, v_k] = \varphi(\sigma|[v_1, \cdots, v_\ell]) \cdot \sigma|[v_{\ell+1}, \cdots, v_k]$.

One can check $\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$.

$\Rightarrow \frown : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)$.

$$\partial\sigma \frown \varphi = \sum_{i=0}^{\ell} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots, v_k]$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots, \hat{v}_i, \cdots, v_k]$$
$$\sigma \frown \delta\varphi = \sum_{i=0}^{\ell+1} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots, v_k]$$
$$\partial(\sigma \frown \varphi) = \sum_{i=\ell}^{k} (-1)^{i-\ell} \varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots, \hat{v}_i, \cdots, v_k]$$

### * Some properties

$H_k(X, A) \times H^\ell(X) \xrightarrow{\frown} H_{k-\ell}(X, A)$

$H_k(X, A) \times H^\ell(X, A) \xrightarrow{\frown} H_{k-\ell}(X, A)$

$H_k(X, A \cup B) \times H^\ell(X, A) \longrightarrow H_{k-\ell}(X, B)$. $\quad A, B$ open.

$$\begin{array}{ccccc}
H_k(X) \times H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) & & f_*(\alpha \frown f^*\varphi). \\
\downarrow f_* \quad\quad \uparrow f^* & & \downarrow f_* & & \parallel \\
H_k(Y) \times H^\ell(Y) & \xrightarrow{\frown} & H_{k-\ell}(Y) & & f_*(\alpha) \frown \varphi
\end{array}$$

**Thm 3.26** $M$: closed connected $n$-manifold.

(a) $M$ is $R$-orientable. Then $H_n(M;R) \xrightarrow{\simeq} H_n(M|x;R) \simeq R$

(b) not . Then $\hookrightarrow$ with image $\{r \in R \mid 2r = 0\}$.

(c) $H_i(M) = 0$. $\forall i > n$.

**Lemma 3.27** $M$: $n$-manifold. $A \underset{\text{cpt}}{\subset} M$. Then

(a) If $x \mapsto \alpha_x$ is a section of cover $M_R \twoheadrightarrow M$.

$$M_R = \{(x, \mu_x) \mid x \in M, \mu_x \in \underset{\substack{\text{is} \\ R}}{H_n(M|_x)}\}.$$

then $\exists!\ \alpha_A \in H_n(X|_A)$ s.t. $\begin{array}{ccc} H_n(X|_A) & \hookrightarrow & H_n(X|_x) \\ \alpha_A & \longmapsto & \alpha_x \end{array}$.

(b) $H_k(X|_A) = 0$. $\forall k > n$.

[proof of Lemma].

Step 1 Assume proved for cpt $A$, $B$ and $A \cap B$. Then prove also true for $A \cup B$.

Then

$$\begin{array}{ccccccc} \underset{H_{n+1}(M|_{A\cap B})}{\overset{\|}{0}} \to & H_n(M|_{A\cup B}) & \xrightarrow{\Phi} & H_n(M|_A) \oplus H_n(M|_B) & \xrightarrow{\Psi} & H_n(M|_{A\cap B}) \\ & \alpha & \longmapsto & (\alpha, -\alpha) & & \\ & & & (\alpha, \beta) & \longmapsto & \alpha + \beta \end{array}$$

$\Rightarrow H_{n+1}(M|_{A\cup B}) = 0 \Rightarrow$ (b)

Let $x \mapsto \alpha_x$. section

Then $\exists\ \alpha_A \in H_n(M|_A)$. $\alpha_B \in H_n(M|_B)$.

$\alpha_{A\cap B} \in H_n(M|_{A\cap B})$. $\Psi(\alpha_A, 0) = \Psi(0, \alpha_B) = \alpha_{A\cap B}$.

$\Rightarrow (\alpha_A, -\alpha_B) \in \ker\Psi = \operatorname{im}\Phi \Rightarrow \exists\ \alpha_{A\cup B}$. $\boxed{\text{existence}}$

Uniqueness: Let $\alpha \in H_n(M|_{A\cup B})$ s.t. image 0 at all $x$

Then same for image in $H_n(M|_A)$ and $H_n(M|_B)$. Then by $\Phi$ is injective.

**Step2** Reduce to $M = \mathbb{R}^n$. $A$ cpt $\Rightarrow A = \bigcup_{i=1}^{m} A_i$, $A_i \subset U_i \subset M$
($A_i$ cpt., $U_i$ open.)

Then by induction.

it suffices to show for $m=1$. $\Rightarrow H_i(M|_{A_j}) \overset{\text{excision.}}{\cong} H_i(\mathbb{R}^n|_{A_j})$.

**Step3** Assume $M = \mathbb{R}^n$ and $A_i$ are convex.

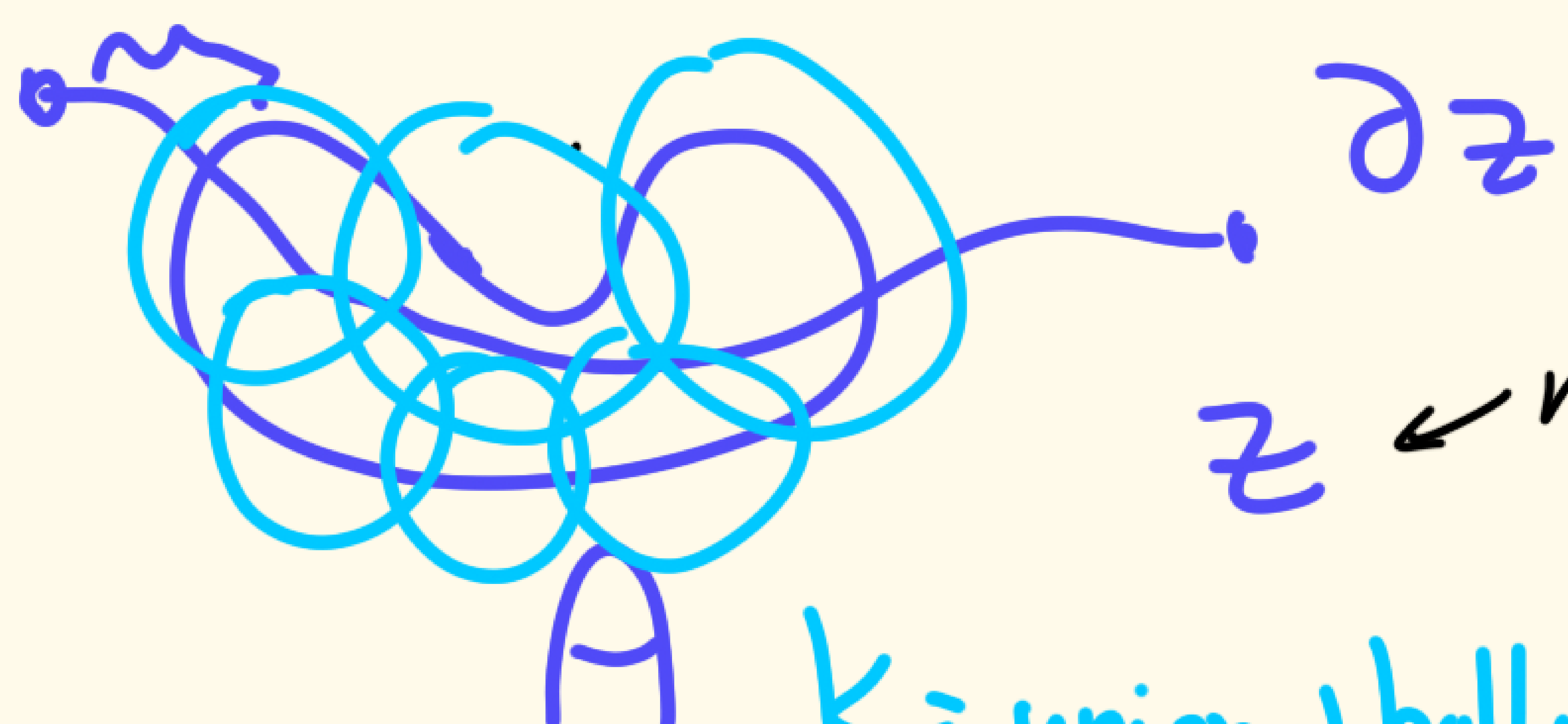
Then $H_i(\mathbb{R}^n|_A) \cong H_i(\mathbb{R}^n|_x)$

(convex $\Rightarrow$ contractable.)

**Step4** Any $A$. Fix $\alpha \in H_i(\mathbb{R}^n|_A)$. represented by $z$ (relative cycle.)

$C$ = union of all singulars in $\partial z$. (finitely many singulars closed in $\mathbb{R}^n$ $\Rightarrow$ $C$ is cpt. $\Rightarrow$ $\delta > 0$.)

$\delta := \mathrm{dist}(C, A) > 0$




$K$ = union of balls of radius $\delta$.

by (3). (convex $\Rightarrow$ by step 2, 3; (finitely many balls))

$\Rightarrow K \cap C = \phi$.

$z$ defines $\alpha_K \in H_i(\mathbb{R}^n|_K)$, mapping to given $\alpha \in H_i(\mathbb{R}^n|_A)$.

If $i > n$, then $H_i(\mathbb{R}^n|_K) = 0 \Rightarrow \alpha_K = 0$.

Uniqueness: If $\alpha_x = 0$ in $H_n(\mathbb{R}^n|_x)$, $\forall x \in A$. Then $\forall x \in K$.

($\alpha_x$ means image of $\alpha_K$ in $H_n(\mathbb{R}^n|_x)$ and $i = n$.)

Since $H_n(\mathbb{R}^n|_B) \cong H_n(\mathbb{R}^n|_x)$ (any ball $B$, $\ni x$)

$\Rightarrow \alpha_K = 0. \Rightarrow \alpha_A = 0$.

existence: Let $\alpha_A$ be image of $\alpha_B$. $\forall B \subset A$ (ball)

[proof of Thm 3.26]

Let $A = M$. (i) is done.

$\Gamma_R(M)$ = set of sections $M_R \to M$.

Then it is an $R$-mod.

$\exists$ homo $H_n(M;R) \longrightarrow \Gamma_R(M)$

$$\alpha \longmapsto (x \longmapsto \alpha_x).$$

where $\alpha_x$ is the image of $x$ under $H_n(M) \longrightarrow H_n(M|x)$.

Then Lemma (a) $\Rightarrow$ is an iso.

# 3. Poincaré Duality.

$$\begin{tikzcd} G_i \arrow[r] \arrow[d] & G_j \\ L & \end{tikzcd}$$

## · Statement

cpt version: $M$: closed $R$-orientible $n$-manifold.

$[M]$ be the fundamental class of $M$.

Then $D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$

$\alpha \longmapsto [M]\frown\alpha$

is an isomorphism.

<u>prop 3.33</u> $X_\alpha$: directed set $X = \bigcup_\alpha X_\alpha$.

And $\forall$ cpt $K \subset X$, $\exists \alpha$ s.t. $K \subset X_\alpha$.

Then $\varinjlim H_*(X_\alpha) \cong H_*(X)$.

$\qquad\qquad\quad \overset{\shortparallel}{\bigcup_\alpha H_*(X_\alpha)}$

<u>Def of $H_c^*$</u>.

w.r.t. rela of inclusion.

$\{K \subset X \mid K \text{ cpt}\}$ forms a directed set. And $K \subset L \Rightarrow H^*(X|_K) \hookrightarrow H^*(X|_L)$.

Define $H_c^*(X) \triangleq \varinjlim H^*(X|_K)$.

Note that $X$ cpt $\Rightarrow H_c^*(X) = H^*(X)$

$x \in \mathrm{supp}\,\varphi \Leftrightarrow \exists \sigma \ni x$ s.t. $\varphi(\sigma) \neq 0$.

$H_c^*(X)$ consists of $\varphi \in H^*(X)$ with cpt supp.

<u>Thm 3.35</u> $M$: $R$-oriented $n$-manifold.

Then $D_M$. $H_c^k(M;R) \xrightarrow{\sim} H_{n-k}(M;R)$

$\alpha \longmapsto [M]\frown\alpha$

<u>Lemma 3.36.</u> $M = U \cup V$. $U, V$ open. Then there is diag. of M-V sequences that commutate (up to sign).

$$\begin{tikzcd} \cdots \arrow[r] & H_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H_c^k(M) \arrow[r, "\delta"] \arrow[d, "D_M"] & H_c^{k+1}(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\ \cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r, "\partial"] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots \end{tikzcd}$$

[Proof of Lemma].

First prove for cpt $K \subset U$, $L \subset V$. Then take direct limit.

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M|_{K\cap L}) \arrow[r] \arrow[d, "\cong\ \text{excision}"] & H^k(M|_K) \oplus H^k(M|_L) \arrow[r] \arrow[d, "\cong"] & H^k(M|_{K\cup L}) \arrow[r] \arrow[dd, "\mu_{K\cup L}"] & H^{k+1}(M|_{K\cap L}) \arrow[r] & \cdots \\
H^k(U\cap V|K\cap L) \arrow[d, "\mu_{K\cap L}\frown"] \arrow[r, phantom, "①"] & & H^k(U|_K) \oplus H^k(V|_L) \arrow[d, "\mu_K \oplus -\mu_L\frown"] \arrow[r, phantom, "②"] & & \\
\cdots\ H_{n-k}(U\cap V) \arrow[r] \arrow[u, phantom, "③"] & & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & \cdots
\end{tikzcd}$$

① and ② are clear.

Now prove ①.

$$\begin{tikzcd}
H^k(M|_{K\cup L}) \arrow[r, "\delta"] \arrow[d, "\mu_{K\cup L}\frown"] & H^{k+1}(M|_{K\cap L}) \arrow[r, "\cong"] & H^{k+1}(U\cap V|K\cap L) \arrow[d, "\mu_{K\cap L}\frown"] \\
H_{n-k}(M) \arrow[rr] & & H_{n-k-1}(U\cap V)
\end{tikzcd} \qquad (*)$$

$$\underbrace{C^*(M,A)}\qquad \underbrace{C^*(M,B)}$$

$$\varphi = \underbrace{(\varphi - \varphi|_A)}_{=\varphi_A} + \underbrace{(\varphi|_A - \overset{0}{\overset{\shortparallel}{\varphi|_{A\cap B}'}})}_{=-\varphi_B}$$

<u>Analize the map $\delta$:</u>

$\boxed{\text{Let } A = M\setminus K,\ B = M\setminus L.}$

$$x \longmapsto (x,-x) \qquad (x,y) \longmapsto x+y.$$

Consider $0 \to C^*(M, A+B) \to C^*(M,A)\oplus C^*(M,B) \to C^*(M, A\cap B) \to 0.$

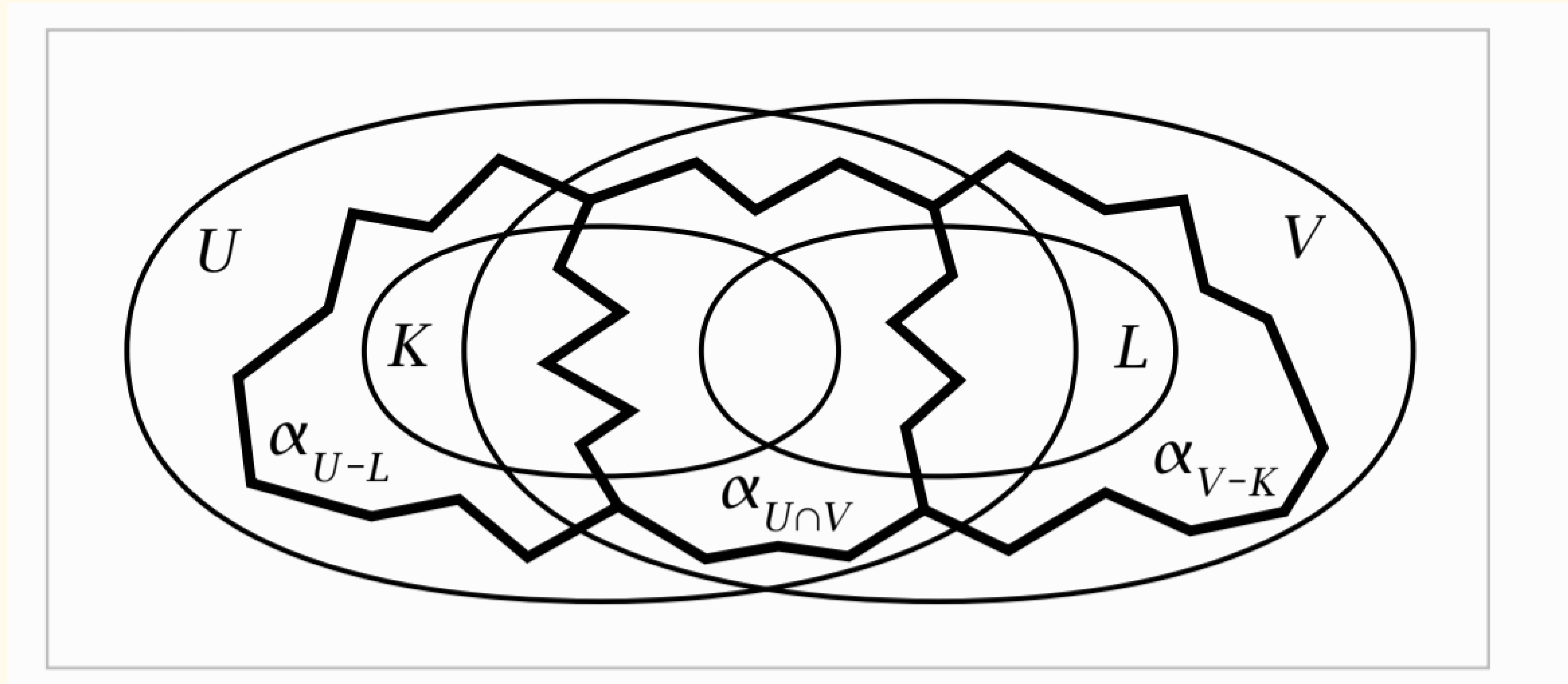
$$\delta\varphi_A \qquad\qquad \varphi = \varphi_A + \varphi_B$$

- $\delta(\varphi_A - \varphi_B) = 0 \Rightarrow$ well-defined.

Similarly, $\partial[z] = \partial[z_U]$. $(z = \underset{\in C_k(U)}{z_U} + \underset{\in C_k(V)}{z_V})$.

Next, $\alpha$ represents $\mu_M$.

$$\alpha = \alpha_{U\setminus L} + \alpha_{V\setminus K} + \alpha_{U\cap V}$$




$$\begin{tikzcd}
\varphi \arrow[r, mapsto] & \delta\varphi_A & \\
H^k(M|_{K\cup L}) \arrow[r, "\delta"] \arrow[d, "\mu_{K\cup L}\frown"] & H^{k+1}(M|_{K\cap L}) \arrow[r, "\cong"] & H^{k+1}(U\cap V|K\cap L) \arrow[d, "\mu_{K\cap L}\frown"] \\
H_{n-k}(M) \arrow[rr, "\partial"] & & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

$\varphi \mapsto \alpha_{U\cup V}\frown\varphi$ $\qquad$ $\delta\varphi_A \mapsto \boxed{\alpha_{K\cap L}\frown\delta\varphi_A}$

$$\alpha_{U\cup V}\frown\varphi = \alpha_{U\setminus L}\frown\varphi + \alpha_{U\cap V}\frown\varphi + \alpha_{V\setminus K}\frown\varphi$$

$\alpha_{U\cap V}\frown\varphi$: $\in\ker\partial$; $\alpha_{V\setminus K}\frown\varphi$: "same"

$$\longmapsto \partial(\alpha_{U\setminus L}\frown\varphi) = (-1)^k(\partial\alpha_{U\setminus L})\frown\varphi = (-1)^k(\partial\alpha_{U\setminus L}\frown\varphi_A) = \boxed{(-1)^{k+1}(\partial\alpha_{U\cap V})\frown\varphi_A}$$

$\delta\varphi = 0$

$(B = M\setminus L)$

$$\varphi_B\big|_{U\setminus L} = \varphi_{M\setminus L}\big|_{U\setminus L} = 0 \longrightarrow \partial\alpha_{U\setminus L}\frown\varphi_B = 0$$

represents $\mu_K$

$$\partial\left(\boxed{\alpha_{U\setminus L} + \alpha_{U\cap V}}\right)\frown\varphi_A = 0, \qquad A = M\setminus K$$

Thus (*) commutes up to a sign depends only on $k$.

[proof of Thm 3.35].

Step 1 Assume $M = U \cup V$. $U, V$ open.

$D_U, D_V, D_{U\cap V}$ are isomorphism.

Then by Five-lemma and commutative diag. in Lemma 3.36,

$D_M$ is an iso.

Step 2 Assume $M = \overset{\infty}{\underset{i=1}{\cup}} U_i$ with $U_i \subset U_{i+1}$. ($U_i$ open)

And $D_{U_i} \left( : H_c^k(U_i) \longrightarrow H_{n-k}(U_i) \right)$ is iso.

Then by prop 3.33, $H_{n-k}(M) \cong \varinjlim H_{n-k}(U_i)$

$\Rightarrow D_M$ is an iso.

Step 3 Assume $M = \mathbb{R}^n \xrightarrow{\cong} \mathring{\Delta}^n$.

Then $D_M : \underbrace{H^k(\Delta^n, \partial\Delta^n)}_{\cong H_c^k(\mathring{\Delta}^n)} \longrightarrow H_{n-k}(\Delta^n) \cong H_{n-k}(pt) \cong \begin{cases} R, & k=n \\ 0, & \text{else} \end{cases}$

$\begin{cases} R, & k=n \\ 0, & \text{else} \end{cases} \cong \mathrm{Hom}(\underbrace{H_k(\Delta^n, \partial\Delta^n)}_{\cong H_k(S^n, pt) = \begin{cases} 0, & k \neq n \\ R, & k=n \end{cases}}, R) \oplus \underbrace{\mathrm{Ext}(\overbrace{H_{k-1}(\Delta^n, \partial\Delta^n)}^{=0}; R)}_{=0}$

In particular, $D_M$ sends generator to generator

take value 1 on $\Delta^n$.

Step 4 Assume $M \cong$ an open set in $\mathbb{R}^n$.

Then write $M = \overset{\infty}{\underset{i=1}{\cup}} U_i$ (open. convex). $V_j = \underset{i<j}{\cup} U_i$.

Then $U_i \cong \mathring{\Delta}^n \Rightarrow$ by step 3, $D_{U_i}$ is isomorphism.

$\Rightarrow$ by step 1, $D_{U_i \cup V_i}$ is iso

$\Rightarrow$ by step 2, $D_M$ is iso.

Step 5 replece "convex open sets" by "open sets" in Step 4.

Then prove for $M \cong \bigcup_{i=1}^{\infty} U_i$ ← open in $\mathbb{R}^n$ (not nece convex).

Step 6 General $M$. Use Zorn's lemma.

$\mathcal{C} = \{ U \underset{\text{open}}{\subset} M \mid D_U \text{ is iso} \}$ with "$\leq$" be inclusion.

by step 5, each chain contains a maximal ele

$\Rightarrow \mathcal{C}$ contains a maximal ele. say $M_0$

Sps $M_0 \subsetneq M$. Take $x \in M \setminus M_0$. $V$: open neigh. of $x$.

Then $M_0 \cup M \in \mathcal{C}$. ↯.

Thus $M_0 = M$. $\Rightarrow D_M$ is an iso