1 Complex analysis from $\mathbb{C}$ to Riemann surface

We all know that a Riemann surface can be seen as a manifold that locally looks like $\mathbb{C}$, and we have learnt lots of interesting properties on $\mathbb{C}$. The goal of this section is to introduce a similar result on the general Riemann surface.

1.1 What is a Riemann Surface?

Basically, it is the same definition as the surface defined as a two-dimensional smooth manifold, but the base space $\mathbb{R}^2$ is changed into $\mathbb{C}$, smooth is changed into holomorphic, diffeomorphism is replaced by biholomorphism, and the smooth atlas is changed into a complex structure.

Definition 1.1 (Complex Structure).

Let X be a two-dimensional manifold. A complex chart on X is a homeomorphism $\phi : U \longrightarrow V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$.

Two complex charts (U_i, ϕ_i) are called holomorphically compatible if the map:

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$$

is biholomorphic (the same as conformal).

A complex atlas on X is a system $\mathcal{U} = \{(U_i, \phi_i) : \phi_i : U_i \to V_i\}$ which are holomorphically compatible and which covers X, i.e. $\bigcup_{i \in I} U_i = X$.

Two complex atlases $\mathcal{U}$ and $\mathcal{U}'$ are called analytically equicalent if every chart of $\mathcal{U}$ is holomorphically compatible with every chart of $\mathcal{U}'$.

The equivalent class above is called the complex structure on X.

Definition 1.2 (Riemann Surface). A **Riemann surface** is a pair (X, Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X.

Example 1.3.

The complex plane $\mathbb{C}$: Its complex structure is defined by the identity map $id : \mathbb{C} \longrightarrow \mathbb{C}$.

Riemann Sphere $\mathbb{P}^1$: We already learnt the topological structure of $\mathbb{P}^1$ in both complex anaysis and topology course. In a word $\mathbb{P}^\mathbb{K} = \mathbb{C} \cup \{\infty\}$ equipped with the one point compactification topology.

The complex structure can be given by a pair of charts: Let $id : \mathbb{C} \longrightarrow \mathbb{C}$ be the identity map and $f : \mathbb{C}^* \cup \{\infty\} \longrightarrow \mathbb{C}^*$ defined as:

$$f(z) = \begin{cases} 0 & z = \infty \\ \frac{1}{z} & z \in \mathbb{C}^* \end{cases}$$

Then these two charts $(\mathbb{C}, id), (\mathbb{C}^* \cup \{\infty\}, f)$ will be the complex atlas of $\mathbb{P}^\mathbb{K}$

Complex Tori $\mathbb{C}/\Gamma$: Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are two linearly independent complex number over $\mathbb{R}$. Define:

$$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}$$

Γ is called the lattice spanned by ω_1 and ω_2. The **complex tori** can now be defined by the quotient operation $\mathbb{C}/\Gamma$.

Definition 1.4 (Holmorphic function on Riemann surface). Let X be a Riemann surface and $Y \subset X$ be an open subset. A function $f : Y \longrightarrow \mathbb{C}$ is called holomorphic, if for every chart $\phi : U \to V$ on X the function:

$$f \circ \phi^{-1} : \phi(U \cap Y) \longrightarrow \mathbb{C}$$

is holomorphic. We use the notation $\mathcal{O}(Y)$ to denote all the holomorphic functions on Y.

Since all the charts are compatible, meaning the transition mapping is biholomorphic, the requirement in this definition is not particularly strong. One can easily prove that we only need to check that f is holomorphic in any family of charts covering Y.

Remark 1.5. Since the sum and product of holomorphic functions are again holomorphic functions, and also constant functions. Thus $\mathcal{O}(Y)$ is a $\mathbb{C}-$algebra.

Theorem 1.6 (Riemann's removable singularities Theorem). *Let U be an open subset of a Riemann surface and let $a \in U$. Suppose the function $f \in \mathcal{O}(U\backslash\{a\})$ is bounded in some neighborhood of a, then f can be extended to a unique function $f' \in \mathcal{O}(U)$*

This theorem follows directly from Riemann's removable singularities Theorem on $\mathbb{C}$ using the charts covering the point a.

Also, one can consider the mapping between Riemann surfaces.

Definition 1.7 (Holomorphic mapping). Suppose X, Y are two Riemann surfaces. A continuous mapping $f : X \to Y$ is called a holomorphic mapping if for every pair of charts (U_1, ϕ_1) on X and (U_2, ϕ_2) on Y.

Theorem 1.8 (Identity Theorem). *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \longrightarrow Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

Proof. We will prove this theorem using the open-closed argument.

1. First, we define the set G be the sets of all points in X s.t. there exists an open neighborhood W such that $f_1|_W = f_2|_W$. By the direct definition, we know that G is an open set.

$\square$

Definition 1.9 (Meromorphic functions on Riemann surface). Let X be a Riemann surface and Y be an open set of X. By a meromorphic function on Y we mean a holomorphic function $f : Y' \longrightarrow \mathbb{C}$, where $Y' \subset Y$ is an open subset s.t. the following hold:

1. $Y\backslash Y'$ contains only isolated points.

2. For every point $p \in Y\backslash Y'$ one has:

$$\lim_{x \to p} |f(x)| = \infty$$

The points $p \in Y\backslash Y$ are called the **poles** of f.

The set of all meromorphic functions on Y are denoted by $\mathcal{M}(Y)$

The Identity theorem explains why we need restriction 1.

1.2 Properties on Riemann Surface

After defining the Riemann surface and the holomorphic and meromorphic functions on it. We can now formalize the theorems on a general Riemann surface.

Theorem 1.10 (Local behaviour of holomorphic mapping). *Suppose $f : X \to Y$ is the holomorphic mapping. X and Y are the Riemann surfaces. Take $a \in X, b = f(a) \in Y$. Then there exists an integer $k \geq 1$ and two charts $(U_1, \phi_1) \subset X, (U_2, \phi_2) \subset Y$, such that:*

- $\phi_1(a) = \phi_2(b) = 0$. *(Fix the relative position)*

- $f(U_1) \subset U_2$ *(Well-defined)*

- *The map $F = \phi_2 \circ f \circ \phi_1^{-1}$ is given by the power map:*

$$F : \phi_1(U_1) \longrightarrow \phi_2(U_2)$$
$$z \longrightarrow z^k$$

This theorem follows from the Taylor expansion of complex functions.

Main result used here is: For a holomorphic function f with $f(0) = 0$, there exists a unique integer $k \geq 1$ and a non-zero holomorphic function $g(z)$ such that $f(z) = z^k g(z)$.

Proof. First, we randomly pick 2 complex charts satisfying conditions one and two. Then we have:

$$F : \phi_1(U_1) \longrightarrow \phi_2(U_2) \text{ with } F(0) = 0$$

So, by the theorem we have learned in complex analysis, we can get the unique integer k and $G(z)$ holomorphic non-zero in some smaller open subset $V_2 \subset \phi_2(U_2)$ such that $F(z) = z^k G(z)$, which means:

$$F(z) = \phi_2 \circ f \circ \phi_1^{-1} = z^k G(z) \text{ on } V_2$$

Since $G(z)$ non-zero in V_2, so one can take a holomorphic function $g(z)$ in V_2, such that $g^k(z) = G(z)$, now we can define a new holomorphic function $\alpha(z) = zh(z)$ with $\alpha'(0) = h(0) \neq 0$ and by shrinking the domain, we can make α biholomorphic.

Now we want $F(\alpha) = \alpha^k$ to get the final result, so we only need to change our chart slightly by substituting ϕ_2 by $\tilde{\phi}_1(z) = \alpha \circ \phi_1(z)$.

$$\tilde{F}(z) = \phi_2 \circ f \circ \tilde{\phi}_1^{-1} = \phi_2 \circ f \circ \phi_1^{-1} \circ \alpha^{-1}(z) = F(\alpha^{-1}(z)) = z^k$$

Thus, we finish the proof $\hfill\square$

The integer k here is called the multiplicity of the mapping at the point a.

Example 1.11 (Complex Polynomial). Let $f(z) = z^k + c_1 z^{k-1} + \cdot + c_{k-1} z + c_k$ be a polynomial of degree k, then f can be considered as a holomorphic mapping from $\mathbb{P}^1$ to $\mathbb{P}^1$ where $f(\infty) = \infty$. One can prove that ∞ is taken with multiplicity k.

Recall that the chart we have defined is:

$$\phi : U = \mathbb{C}^* \cup \{\infty\} \longrightarrow \mathbb{C}$$

$$z \longrightarrow \begin{cases} \frac{1}{z} & z \in \mathbb{C} \\ 0 & z = \infty \end{cases}$$

Now we consider the function $F = \phi \circ f \circ \phi^{-1}$ defined in an open neighborhood containing $\{\infty\} \in \mathbb{P}^1$.

$$F : 0 \in U \longrightarrow 0 \in V$$

$$z \longrightarrow \begin{cases} 0 & z = 0 \\ \frac{1}{\frac{1}{z}^k + c_1 \frac{1}{z}^{k-1} + \cdots + c_{k-1}\frac{1}{z} + c_k} = \frac{z^k}{c_k z^k + c_{k-1} z^{k-1} + \cdots + c_1 z + 1} & z \in U\backslash\{0\} \end{cases}$$

The rest is to prove that $F^{(n)}(0) = 0, \forall n \in [0, k-1], F^{(k)}(0) \neq 0$

Corollary 1.12. *Let X, Y, f be as described above; then f is an open mapping. i.e., the image of every open set $U \subset X$ is open.*

Consider $f(U) = \phi_2^{-1} \circ F \circ \phi_1(U)$, then obviously open.

Corollary 1.13 (Maximum Principle). *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a non-constant holomorphic function. Then the absolute value of f does not attain its maximum.*

Mainly due to the open mapping property.

Theorem 1.14. *Suppose X, Y are Riemann surfaces. Suppose X is compact and $f : X \to Y$ is a non-constant holomorphic mapping. Then Y is compact and f is surjective.*

Proof. f is an open mapping $\Rightarrow f(X)$ is open.
X compact $\Rightarrow f(X)$ compact $+ Y$ Hausdorff $\Rightarrow f(X) \subset Y$ closed.
Y connected, by open-close argument $\Rightarrow Y = f(X)$.

Corollary 1.15. *Every holomorphic function on a compact Riemann surface is a constant function.*

Since $\mathbb{C}$ is not a compact set.

Remark 1.16. This is a much stronger argument than the real case.

Corollary 1.17. *Every meromorphic function f on $\mathbb{P}^1$ is rational, i.e., can be written as the quotient of two polynomials.*

Proof. Since $\mathbb{P}^1$ is a compact set, a non-constant meromorphic function on $\mathbb{P}^1$ has at most finitely many poles.

Let $a_1, a_2, \cdots, a_n$ be the collection of poles. Take the principal part for each of the poles:

$$h_v(z) = \sum_{j=k_v}^{-1} c_{vj}(z - a_v)^j$$

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Then we eliminate them by subtracting them, and define g to be the holomorphic function on $\mathbb{P}^1$.

$$g := f - (h_1 + h_2 + \cdots + h_n)$$

By the corollary above, f is a constant function, which means:

$$f = \sum_{v=1}^{n} h_v(z) + z, \text{ where } z \in \mathbb{C} \text{ is a complex constant}$$

By simple calculation, one can prove that f is rational. $\qquad\square$

Theorem 1.18 (Liouville's Theorem). *Every bounded holomorphic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is constant.*

  With the tools above, we can provide a new proof.

Proof. By Riemann's Removable Singularities Theorem, f can be analytically continued to a holomorphic mapping $f : \mathbb{P}^1 \longrightarrow \mathbb{C}$.(Since f is bounded).
  By the corollary above, f is constant.

Theorem 1.19 (Fundamental Theorem of Algebra). *For any complex polynomial $f(z) = z^k + c_1 z^{k-1} + \cdots + c_k$, there always exists a root.*

Proof. We view the polynomial as a mapping from $\mathbb{P}^1$ to $\mathbb{P}^1$, this is obvious a holomorphic function. Since $\mathbb{P}^1$ is compact, the rest is to show that f non constant. By taking $0, \infty \in \mathbb{P}^1$, $f(0) = c_k \neq \infty = f(\infty)$, so f non constant, which then shows f is surjective and there exists a root.

2 More examples of Riemann Surfaces

2.1 Complex Projective Line $\mathbb{CP}^1$

Let $\mathbb{CP}^1$ denote the complex projective line, which is all complex 1-dimensional subspaces of $\mathbb{C}^2$.

First, we need to define the complex structure by defining its complex chart. Let $\mathbb{CP}^1 := \{[z, w] \mid (z, w) \in \mathbb{C}^2\}$ where $[z, w] = [\lambda z, \lambda w], \forall \lambda \in \mathbb{C}^*$, so the bracket actually denotes an equivalent class under a multiplication of complex number.

Then we denote two open sets, $U_1 := \{[z, w] \mid z \neq 0\}$ and $U_2 := \{[z, w] \mid w \neq 0\}$, then it is obvious that $U_1 \cup U_2 = \mathbb{CP}^1$. Define:

$$\phi_1 : U_1 \longrightarrow \mathbb{C} \qquad\qquad \phi_2 : U_2 \longrightarrow \mathbb{C}$$
$$[z, w] \longrightarrow \frac{w}{z} \qquad\qquad [z, w] \longrightarrow \frac{z}{w}$$

These two functions are two homeomorphisms and their transition function:

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) = \mathbb{C}^* \longrightarrow \phi_2(U_1 \cap U_2) = \mathbb{C}^*$$
$$z \longrightarrow \frac{1}{z}$$

is a biholomorphic function.

Since U_1, U_2 are both connected open sets and have a non-empty intersection, their union $\mathbb{CP}^1$ is a connected topological space.

Finally, we need to check that the $\mathbb{CP}^1$ is Hausdorff. Randomly pick two points $p, q \in \mathbb{CP}^1$.

- They are in the same chart, then trivial since U_1, U_2 are Hausdorff.

- Not in the same, this forces (p, q) to be $[1, 0], [0, 1]$, then this can be separated by $\phi_1^{-1}(\mathbb{D})$ and $\phi_2^{-1}(\mathbb{D})$

2.2 Projective Curves

The above example, $\mathbb{CP}^1$, is the first in a series of the most important and interesting compact Riemann surfaces.

These are surfaces that are embedded in the Projective Space.

Definition 2.1 (Projective Plane $\mathbb{P}^2$)**.** The projective plane $\mathbb{P}^2$ is the set of $1-$dimensional subspaces of $\mathbb{C}^3$.

If (x, y, z) is a nonzero vector in $\mathbb{C}^3$, its span is denoted by $[x : y : z]$ and is a point in the projective plane. Every point in the projective plane may be written in this way:

$$[x : y : z] = [\lambda x : \lambda y : \lambda z]$$

For a non-zero $\lambda \in \mathbb{C}^*$. This is called the homogeneous coordinates of the corresponding point.

$\mathbb{CP}^2$ can be viewed as the quotient space pf $\mathbb{C}^3 \backslash \{0\}$ by the multiplicative action $\mathbb{C}^*$. In this way, $\mathbb{CP}^2$ inherits the Hausdorff property.

Similarly, one can also define:

$$U_1 = \{[x:y:z] \mid x \neq 0\}, \quad U_2 = \{[x:y:z] \mid y \neq 0\}, \quad U_3 = \{[x:y:z] \mid z \neq 0\}$$

and the three functions

$$\phi_1 : U_1 \longrightarrow \mathbb{C}^2 \quad \phi_2 : U_2 \longrightarrow \mathbb{C}^2 \quad \phi_3 : U_3 \longrightarrow \mathbb{C}^2$$

$$[x:y:z] \longrightarrow (\frac{y}{x}, \frac{z}{x}) \quad [x:y:z] \longrightarrow (\frac{x}{y}, \frac{z}{y}) \quad [x:y:z] \longrightarrow (\frac{x}{z}, \frac{y}{z})$$

Definition 2.2 (Homogeneous Polynomial)**.** A polynomial F is homogeneous if every term has the same degree in the variables; this degree is the degree of the homogeneous polynomial.

For example, $x^2y - 2xyz + 3z^3$ is homogeneous of degree 3 in the variables x, y, z.

Let $F(x, y, z)$ be a homogeneous polynomial of degree d. It does not make sense to evaluate F at a point of the projective plane; if $[x_0 : y_0 : z_0] \in \mathbb{P}^2$, then $F(x_0, y_0, z_0)$ is not well defined, because the homogeneous coordinates x_0, y_0, and z_0 are themselves not well defined. In particular, one sees easily that

$$F(\lambda x_0, \lambda y_0, \lambda z_0) = \lambda^d F(x_0, y_0, z_0)$$

For a normal polynomial with degree $\neq 0$, it can not be a well-defined function on homogeneous coordinates. But the locus of the polynomial can make sense.

$$X = \{[x:y:z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$$

Moreover, it is a closed subset of $\mathbb{P}^2$. The intersection X_i of X with the open sets U_i is exactly an *affine plane curve* when transported to $\mathbb{C}^2$. For example, in U_1 where $x \neq 0$, we have after transporting to $\mathbb{C}^2$ that

$$X_1 = X \cap U_1 \cong \{(a, b) \in \mathbb{C}^2 \mid F(1, a, b) = 0\}$$

which is the affine plane curve described by the polynomial $f(a, b) = 0$, where $f(a, b) = F(1, a, b)$.

Now we want to show that under a nonsingularity assumption on F, the locus X is a Riemann surface. But in either case, X is called the *projective plane curve* defined by F.

Definition 2.3 (Nonsingular condition)**.** A homogeneous polynomial $F(x, y, z)$ is nonsingular if there are no common solutions to the system of equations

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

in the projective plane $\mathbb{P}^2$.

This condition is equivalent to requiring that there be no nonzero solutions to the above system in $\mathbb{C}^3$.

Proposition 2.4. *Let $F(x, y, z)$ be a nonsingular homogeneous polynomial. Then the projective plane curve X, which is its zero locus in $\mathbb{P}^2$, is a compact Riemann surface.*

Proof. Since $X \subset \mathbb{P}^2$ is a closed subset, then it is compact.

The Riemann surface property can be proved by considering X as the union of the X_i defined above and by defining a proper complex chart on each X_i.

2.3   Higher-dimensional Projectice Space

One can also generalise it to a higher dimension $\mathbb{CP}^n$

Definition 2.5 (Complete Intersection). Let $F_1, \ldots, F_{n-1}$ be $n - 1$ homogeneous polynomials in $n + 1$ variables $x_0, \ldots, x_n$. Let X be their common zero locus in $\mathbb{P}^n$. We say X is a smooth complete intersection curve in $\mathbb{P}^n$ if the $(n-1) \times (n+1)$ matrix of partial derivatives $(\partial F_i / \partial x_j)$ has maximal rank $n - 1$ at every point of X.

Proposition 2.6. *A smooth complete intersection curve in $\mathbb{P}^n$ is a compact Riemann surface.*

One can read the specific details in Miranda's book

References

[1] Rick Miranda. *Algebraic Curves and Riemann Surfaces*. Vol. 5. Graduate Studies in Mathematics. American Mathematical Society, 1995.