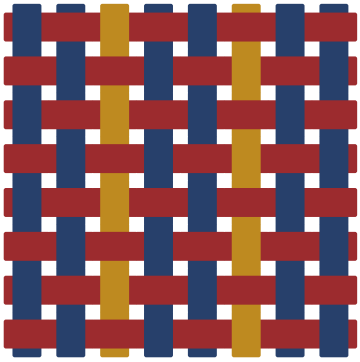


Basic Preparation

category
topological group and completion

From three handwritten pages and Atiyah–Macdonald, Chapter 10

Seminar preparation notes

dyed in indigo, madder & weld; ruled in iron-gall. woven on LOOM.

CONTENTS

category 2

1 Cones and universal properties 2

2 Colimits and basic examples 3

3 Ordered categories and systems 4

4 Equalizers and tensor products 5

5 The bridge: inverse limits as kernels 6

topological group and completion 7

6 Topological groups and separatedness 7

7 Completion by Cauchy sequences 8

8 Completion as an inverse limit 9

9 Inverse limits and exactness 10

10 Quotients, idempotence and adic topology 11

11 Stable filtrations and graded algebra 12

12 Artin–Rees: the central mechanism 13

13 Exact completion and tensoring 14

14 The completed ring and Krull intersection 15

15 Separatedness and localization 16

16 Associated graded rings 17

17 Lifting from graded information 18

18 Why completion is Noetherian 19

19 Worked exercises and pitfalls 20

20 Why finiteness cannot be omitted 21

21 Two further applications 22

22 Chapter 10 concordance 23

category

1 CONES AND UNIVERSAL PROPERTIES

A universal object is not merely an object satisfying equations. It is an object through which every other solution factors **uniquely**.

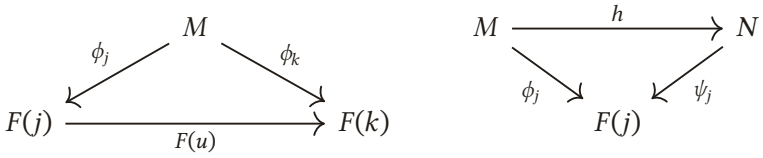
Definition 1.1 *(Initial and terminal objects).* An object T of a category $\mathscr{C}$ is *terminal* if $\mathrm{Hom}_{\mathscr{C}}(X, T)$ has exactly one element for every object X. An object I is *initial* if $\mathrm{Hom}_{\mathscr{C}}(I, X)$ has exactly one element for every X.

For example, a singleton is terminal in **Set**, while the empty set is initial. The zero module is both initial and terminal in R-**Mod**.

Definition 1.2 *(Diagram and cone).* Let $\mathscr{J}$ be a small category and $F : \mathscr{J} \to \mathscr{C}$ a functor. A *cone* to F with vertex M is a family of arrows $\phi_j : M \to F(j)$ satisfying

$$F(u)\phi_j = \phi_k \qquad (u : j \to k \text{ in } \mathscr{J}).$$

A map of cones $(M, \phi_j) \to (N, \psi_j)$ is a map $h : M \to N$ such that $\psi_j h = \phi_j$ for every j.



Cones with these maps form the category $\mathrm{Cone}(F)$. A cone is equivalently a natural transformation $\Delta M \Rightarrow F$, where ΔM is the constant diagram.

Definition 1.3 *(Limit).* A *limit* of F is a terminal object (L, p_j) of $\mathrm{Cone}(F)$. Thus for every cone (M, ϕ_j) there is a unique $h : M \to L$ with $p_j h = \phi_j$ for all j. Write $L = \varprojlim_{\mathscr{J}} F$ when it exists.

2 COLIMITS AND BASIC EXAMPLES

contents ↑

Definition 2.1 *(Cocone and colimit).* A *cocone* from F is an object M with maps $\iota_j : F(j) \to M$ such that $\iota_k F(u) = \iota_j$ for every $u : j \to k$. A map of cocones $(M, \iota_j) \to (N, \kappa_j)$ is $h : M \to N$ with $h\iota_j = \kappa_j$.
A *colimit* is an initial object (C, i_j) of this category. Every compatible family $\iota_j : F(j) \to M$ factors uniquely as $\iota_j = hi_j$ for a map $h : C \to M$. Write $C = \varinjlim_{\mathcal{J}} F$.

Proposition 2.2 *(Uniqueness up to unique compatible isomorphism).* Any two limits of a diagram are uniquely isomorphic by an isomorphism commuting with the projections. The dual statement holds for colimits.

Proof. If (L, p_j) and (L', p'_j) are limits, terminality gives compatible maps $a : L \to L'$ and $b : L' \to L$. Both ba and id_L have composites p_j with all projections, so uniqueness gives $ba = \mathrm{id}_L$. Similarly $ab = \mathrm{id}_{L'}$. The proof for colimits reverses the arrows. ∎

The shape of the diagram decides the construction

Indexing shape	Limit	Colimit
Empty category	Terminal object	Initial object
Discrete family	Product $\prod_j X_j$	Coproduct $\coprod_j X_j$
Two parallel arrows	Equalizer	Coequalizer
$X \to Z \leftarrow Y$	Pullback $X \times_Z Y$	—
$X \leftarrow Z \to Y$	—	Pushout $X \amalg_Z Y$

In R-**Mod** the coproduct is the direct sum, not the direct product for an infinite family.

Example 2.3 *(Products).* A cone to a discrete diagram (X_j) is exactly a family of maps $M \to X_j$. The unique mediating map to $\prod_j X_j$ sends m to the tuple of its images. This proves the product universal property directly.

3 ORDERED CATEGORIES AND SYSTEMS

contents ↑

Definition 3.1 *(Poset as a category).* A partially ordered set $(I, \leq)$ determines a category with objects $i \in I$ and a single arrow $i \to j$ exactly when $i \leq j$. Composition is forced by transitivity. It is *directed* if it is nonempty and every pair has a common upper bound.

Definition 3.2 *(Direct and inverse systems).* A *direct system* over I is a functor $I \to \mathscr{C}$. Its transition maps $f_{ij} : M_i \to M_j$ for $i \leq j$ satisfy

$$f_{ii} = \mathrm{id}_{M_i}, \qquad f_{jk} f_{ij} = f_{ik} \quad (i \leq j \leq k).$$

Its direct limit is its colimit. An *inverse system* is a functor $I^{\mathrm{op}} \to \mathscr{C}$, with maps $q_{ji} : M_j \to M_i$ for $i \leq j$ satisfying the reversed compatibility. Its inverse limit is its limit.

For a sequence, it is enough to give $q_n : M_{n+1} \to M_n$: all longer transition maps are composites.

Proposition 3.3 *(Concrete inverse limits).* In sets, abelian groups, rings or modules,

$$\varprojlim_n M_n = \left\{ (x_n) \in \prod_{n \geq 1} M_n \;:\; q_n(x_{n+1}) = x_n \text{ for all } n \right\}.$$

The projections are the coordinate maps; the algebraic operations are coordinatewise.

Proof. Given a compatible family $u_n : X \to M_n$, the only possible mediating map is $u(x) = (u_n(x))_n$. Compatibility says this tuple lies in the displayed subset. The map has the required algebraic structure because each u_n does, and uniqueness follows coordinatewise. ∎

Example 3.4 *(Directed unions).* For an increasing family of submodules M_i whose index set is directed, $\varinjlim M_i = \bigcup_i M_i$. Directedness makes the union a submodule and ensures any finite calculation takes place in one common stage.

More generally, in modules the direct limit is

$$\left(\bigoplus_i M_i \right) \Big/ \left\langle e_i(x) - e_j(f_{ij} x) \;:\; i \leq j \right\rangle.$$

Here e_i is the inclusion into the direct sum. The imposed relations are exactly the compatibility required of a cocone.

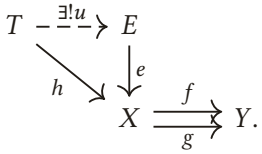
4 EQUALIZERS AND TENSOR PRODUCTS

contents ↑

Definition 4.1 *(Equalizer and coequalizer).* For parallel maps $f, g : X \rightrightarrows Y$, an *equalizer* is $e : E \to X$ with $fe = ge$, universal among maps $h : T \to X$ satisfying $fh = gh$. Thus h factors uniquely through e.
A *coequalizer* is $q : Y \to Q$ with $qf = qg$, universal among maps out of Y with this property. In modules,

$$E = \ker(f - g), \qquad Q = \operatorname{coker}(f - g) = Y / \operatorname{im}(f - g).$$

This immediately identifies equalizers as limits and coequalizers as colimits. The equalizer factorization has the form



The tensor construction in the third photograph

Let R be a commutative ring and M, N be R-modules. Write $F(S)$ for the free R-module on a set S, with basis symbol $[s]$. Put

$$B = F(M \times N), \qquad D = F(M \times M \times N) \oplus F(M \times N \times N)$$

$$\oplus F(R \times M \times N) \oplus F(R \times M \times N).$$

Define $d_0, d_1 : D \rightrightarrows B$ on the four types of generators by

$$\begin{aligned}
[m, m', n] &\mapsto [m + m', n], \quad [m, n] + [m', n], \\
[m, n, n'] &\mapsto [m, n + n'], \quad [m, n] + [m, n'], \\
[r, m, n]_{\mathrm{L}} &\mapsto [rm, n], \quad r[m, n], \\
[r, m, n]_{\mathrm{R}} &\mapsto [m, rn], \quad r[m, n].
\end{aligned}$$

In each row the two outputs specify d_0 and d_1, respectively.

Proposition 4.2 *(Tensor product as a colimit).* $M \otimes_R N \cong \operatorname{coeq}(D \rightrightarrows B)$ in R-**Mod**.

Proof. A linear map $B \to P$ is a set map $M \times N \to P$. It coequalizes d_0, d_1 precisely when that map is additive and R-linear in both arguments. The coequalizer therefore represents bilinear maps, which is the defining universal property of $M \otimes_R N$. ∎

5 THE BRIDGE: INVERSE LIMITS AS KERNELS

contents ↑

For a sequence of abelian groups or modules with maps $q_n : M_{n+1} \to M_n$, define the *difference map*

$$d : \prod_{n\geq 1} M_n \longrightarrow \prod_{n\geq 1} M_n, \qquad d((x_n)) = (x_n - q_n(x_{n+1}))_n.$$

By the coordinate description of inverse limits,

$$\varprojlim_n M_n = \ker d. \tag{1}$$

Equivalently, it is the equalizer of the identity map and the shifted transition map $(x_n) \mapsto (q_n(x_{n+1}))_n$.

> This is the main connection between the two parts. Completion will be a compatible family of finite-level approximations. Its compatibility equations make it a **limit**; expressing that limit as a kernel explains its exactness properties.

Example 5.1 *(Compatible truncated polynomials).* Take $M_n = k[t]/(t^n)$ and reduction maps. A compatible family chooses the same coefficient of t^j at every stage $n > j$. Therefore

$$\varprojlim_n k[t]/(t^n) \cong k[[t]].$$

The right side is a ring of *formal* power series. No analytic convergence or numerical value for t is involved.

Example 5.2 *(Why surjectivity at each stage is subtle).* A compatible family in a quotient system can be lifted individually at every stage, but the chosen lifts may fail to be compatible. Surjectivity on inverse limits requires a way to correct the differences between successive lifts. This correction will occur in the *kernel system*; see the exactness theorem.

topological group and completion

6 TOPOLOGICAL GROUPS AND SEPARATEDNESS

Definition 6.1 *(Topological abelian group).* An abelian group G with a topology is a *topological group* if addition $G\times G \to G$ and inversion $G \to G$ are continuous. Every translation $x \mapsto x+a$ is a homeomorphism. Thus the neighborhoods of 0 determine all neighborhoods.

If U is a neighborhood of 0, continuity gives a symmetric neighborhood V with $V + V \subseteq U$. Symmetric means $-V = V$; replace a smaller neighborhood by its intersection with its negative if needed.

Lemma 6.2 *(The Hausdorff quotient; A–M 10.1).* Let $H = \bigcap_{U \ni 0} U$, the intersection of all neighborhoods of 0. Then H is a subgroup, $H = \overline{\{0\}}$, and G/H is Hausdorff. In particular,

$$G \text{ Hausdorff} \iff H = \{0\}.$$

Proof. For $x, y \in H$ and $U \ni 0$, choose symmetric V with $V + V \subseteq U$. Then $x - y \in V - V \subseteq U$, so $x - y \in H$.

A point x lies in $\overline{\{0\}}$ exactly when $0 \in x + V$ for every symmetric neighborhood V of 0. This is equivalent to $x \in V$ for every such V, hence to $x \in H$.

The quotient map $G \to G/H$ is open and its point preimages are the closed cosets of H. Thus every point of G/H is closed. In any topological group with closed $\{0\}$, the diagonal is closed, since it is the preimage of $\{0\}$ under $(x, y) \mapsto x - y$. A space with closed diagonal is Hausdorff. Applying this to G/H finishes the proof. ∎

Uniform continuity without a metric

The translation-invariant uniformity of an abelian topological group has basic entourages

$$E_U = \{(x, y) : x - y \in U\} \qquad (U \ni 0).$$

A homomorphism $f : G \to K$ is continuous if and only if it is continuous at 0. Given a neighborhood $V \ni 0$ in K, continuity provides $U \ni 0$ with $f(U) \subseteq V$. Consequently

$$x - y \in U \implies f(x) - f(y) = f(x - y) \in V.$$

This is uniform continuity. In particular, continuous homomorphisms preserve Cauchy sequences.

7 COMPLETION BY CAUCHY SEQUENCES

contents ↑

Standing assumption for this section: G has a countable neighborhood basis at 0. With this assumption, sequences suffice; general topological groups require Cauchy filters or nets.

Definition 7.1 *(Cauchy sequences and equivalence).* A sequence (x_r) is Cauchy if for every $U \ni 0$ there is r_0 with $x_r - x_s \in U$ whenever $r, s \geq r_0$. Two Cauchy sequences are equivalent if $x_r - y_r \to 0$.

This is an equivalence relation. For transitivity, choose V with $V+V \subseteq U$ and add the two eventual differences. Coordinatewise addition and negation preserve Cauchy sequences and equivalence. Thus their equivalence classes form an abelian group $\widehat{G}$.

The canonical map is

$$\iota_G : G \to \widehat{G}, \qquad x \mapsto [(x, x, \ldots)].$$

A constant sequence represents zero precisely when its value lies in every neighborhood of 0. Therefore

$$\ker \iota_G = \overline{\{0\}}. \tag{2}$$

Hence a non-Hausdorff group is separated by completion before it can be embedded.

Proposition 7.2 *(Extension property).* Equipped with its completion uniformity, $\widehat{G}$ is complete and Hausdorff, and $\iota_G(G)$ is dense. If K is a complete Hausdorff abelian topological group and $f : G \to K$ is a continuous homomorphism, there is a unique continuous homomorphism $\widetilde{f} : \widehat{G} \to K$ with $\widetilde{f}\iota_G = f$.

Construction and uniqueness. Define $\widetilde{f}([(x_r)]) = \lim_r f(x_r)$. Uniform continuity makes the image Cauchy; completeness supplies its limit; Hausdorffness and equivalence make that limit independent of the representative. Addition commutes with limits, so this is a homomorphism. Continuity follows by choosing a smaller neighborhood whose closure is inside a prescribed neighborhood and using uniform continuity of f. Uniqueness follows from density and Hausdorffness of K.

For the subgroup-basis case used from now on, the topology, density and completeness are proved explicitly in the next section. For a general countable basis one obtains the same construction from the invariant uniformity (or a compatible invariant pseudometric). ∎

This property determines the completion up to a unique isomorphism commuting with ι_G. A continuous map $f : G \to K$ also induces $\widehat{f} : \widehat{G} \to \widehat{K}$ by $[(x_r)] \mapsto [(f(x_r))]$. The construction preserves identity maps and composition.

8 COMPLETION AS AN INVERSE LIMIT

contents ↑

Assume that the neighborhoods of 0 have a basis of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots.$$

Every G_n is open, and its complement is a union of open cosets, so G_n is also closed. Give G/G_n the discrete topology.

Theorem 8.1 *(Coherent approximations).* There is a canonical identification of topological groups

$$\widehat{G} \cong \varprojlim_{n\geq 1} G/G_n. \tag{3}$$

The inverse limit has the subspace topology from $\prod_n G/G_n$. Its basic neighborhoods of zero are $K_n = \ker(\widehat{G} \to G/G_n)$, and $\iota_G(x) = (x + G_n)_n$.

Proof. A Cauchy sequence is eventually constant modulo each G_n, with eventual value ξ_n. Reduction sends ξ_{n+1} to ξ_n. Equivalent sequences have identical eventual values.

Conversely choose a representative $x_n \in G$ of each coherent class ξ_n. For $r, s \geq n$, their reductions modulo G_n coincide, so $x_r - x_s \in G_n$. This is a Cauchy sequence. Different representatives give equivalent sequences. The two constructions are inverse and preserve addition.

Use the displayed inverse-limit neighborhoods to describe the completion topology. They restrict to the original basis because $\iota_G^{-1}(K_n) = G_n$. Every cylinder $\xi + K_n$ meets $\iota_G(G)$: choose a representative in G of the nth coordinate. This proves density.

Finally a Cauchy sequence in the inverse limit is eventually constant in each discrete coordinate. Its eventual coordinate values are compatible, hence define a limit point. Thus the inverse limit is complete; it is Hausdorff as a subspace of a product of discrete Hausdorff groups. ∎

Equivalent filtrations

Two decreasing subgroup bases (G_n) and (G'_n) define the same topology exactly when they are cofinal: for every n some m satisfies $G_m \subseteq G'_n$, and conversely. The continuous identity maps then induce inverse isomorphisms of completions. In particular, passing to a cofinal subsequence does not change completion.

9 INVERSE LIMITS AND EXACTNESS

contents ↑

All systems in this section are sequences of abelian groups (or modules); exactness is meant at every index and the transition diagrams commute.

Theorem 9.1 *(A–M 10.2).* For an exact sequence of inverse systems

$$0 \to (A_n) \to (B_n) \to (C_n) \to 0,$$

the sequence $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$ is exact. If all transition maps $A_{n+1} \to A_n$ are surjective, the final map to $\varprojlim C_n$ is also surjective.

Proof. Injectivity and equality of image and kernel can be checked coordinatewise: a coherent tuple in B_n mapping to zero has unique preimages in A_n, which are compatible by injectivity.

For surjectivity, let (c_n) be coherent. Choose a lift $b_1 \in B_1$. Suppose b_n has been chosen. Choose any lift $t \in B_{n+1}$ of c_{n+1}. The difference $q_n^B(t) - b_n$ lies in A_n. Lift this difference to $a \in A_{n+1}$ by the surjectivity assumption and set $b_{n+1} = t - a$, using the inclusion $A_{n+1} \subset B_{n+1}$. Then $q_n^B(b_{n+1}) = b_n$ and b_{n+1} still lifts c_{n+1}. Induction constructs a coherent lift. ∎

The difference-map proof and the obstruction

Use d_A as in (1), and define d_B, d_C similarly. The snake lemma for the two exact rows of products gives

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \operatorname{coker} d_A \to \cdots.$$

For sequences, $\operatorname{coker} d_A$ is denoted $\varprojlim{}^1 A_n$. If the transitions of A_n are onto, the equations $x_n - q_n(x_{n+1}) = a_n$ can be solved recursively; hence d_A is onto and this obstruction vanishes.

Corollary 9.2 *(Completion in induced topologies; A–M 10.3).* Let $0 \to G' \to G \xrightarrow{p} G'' \to 0$ be exact. Give G its subgroup filtration (G_n), G' the filtration $G' \cap G_n$, and G'' the filtration $p(G_n)$. Then

$$0 \to \widehat{G'} \to \widehat{G} \to \widehat{G''} \to 0$$

is exact.

Proof. At each level use $0 \to G'/(G' \cap G_n) \to G/G_n \to G''/p(G_n) \to 0$. The left system has surjective transitions, so apply the theorem. ∎

10 QUOTIENTS, IDEMPOTENCE AND ADIC TOPOLOGY

contents ↑

Corollary 10.1 *(A–M 10.4–A–M 10.5).* The completion of G_n for its induced topology identifies with $K_n \subset \widehat{G}$, and

$$\widehat{G}/\widehat{G_n} \cong G/G_n, \qquad \widehat{\widehat{G}} \cong \widehat{G}.$$

In particular, completion is idempotent.

Proof. Apply the induced-topology corollary to $0 \to G_n \to G \to G/G_n \to 0$. The last group has the discrete induced topology and is already complete. Thus $\widehat{G_n}$ is the projection kernel K_n. Taking the inverse limit of the quotient isomorphisms proves the second assertion. ∎

Definition 10.2 *(Adic topology).* For an ideal $\mathfrak{a} \subseteq A$, the *$\mathfrak{a}$-adic topology* on A has basis $\mathfrak{a}^n$ at zero. On an A-module M it has basis $\mathfrak{a}^n M$. Write

$$\widehat{A} = \varprojlim_n A/\mathfrak{a}^n, \qquad \widehat{M} = \varprojlim_n M/\mathfrak{a}^n M.$$

The ring operations on $\widehat{A}$ are coordinatewise. Multiplication on A is continuous since the basic neighborhoods are ideals and $\mathfrak{a}^r\mathfrak{a}^s \subseteq \mathfrak{a}^{r+s}$. Similarly $\widehat{M}$ is a topological $\widehat{A}$-module. Every A-linear map is adically continuous, since $f(\mathfrak{a}^n M) \subseteq \mathfrak{a}^n N$.

Example 10.3 *(Two prototypes).* For $A = k[t]$ and $\mathfrak{a} = (t)$, $\widehat{A} = k[[t]]$, as in the truncated-polynomial example. For $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$, $\widehat{A} = \mathbb{Z}_p$, with unique digit expansion

$$x = \sum_{j \geq 0} a_j p^j, \qquad 0 \leq a_j < p.$$

The sum is interpreted by compatibility modulo p^n; carrying is part of addition and multiplication. The field of p-adic numbers is $\mathbb{Q}_p = \operatorname{Frac}(\mathbb{Z}_p)$, not the completion of $\mathbb{Z}$ itself.

No Noetherian hypothesis has been used in defining these completions. That hypothesis becomes essential when comparing a submodule's intrinsic adic topology with its induced topology.

11 STABLE FILTRATIONS AND GRADED ALGEBRA

contents ↑

Definition 11.1 *(Filtration and stability).* A filtration $M = M_0 \supseteq M_1 \supseteq \cdots$ is an $\mathfrak{a}$*-filtration* if $\mathfrak{a}M_n \subseteq M_{n+1}$. It is *stable* if equality holds for all sufficiently large n.

Lemma 11.2 *(Bounded difference; A–M 10.6).* Every stable $\mathfrak{a}$-filtration defines the adic topology. More precisely, some $c \geq 0$ satisfies

$$M_{n+c} \subseteq \mathfrak{a}^n M \subseteq M_n \quad (n \geq 0).$$

Any two stable filtrations consequently have bounded difference.

Proof. The second inclusion follows inductively from $M_0 = M$. Choose c after stability begins; then $M_{n+c} = \mathfrak{a}^n M_c \subseteq \mathfrak{a}^n M$. For two filtrations compare each with $(\mathfrak{a}^n M)$, and enlarge the two constants to their maximum. ∎

Graded objects

A nonnegatively graded ring is $S = \bigoplus_{n\geq 0} S_n$ with $S_r S_s \subseteq S_{r+s}$. A graded module $E = \bigoplus E_n$ satisfies $S_r E_s \subseteq E_{r+s}$. Each element is a *finite* sum of homogeneous elements.

Proposition 11.3 *(A–M 10.7).* A nonnegatively graded ring S is Noetherian if and only if S_0 is Noetherian and S is a finitely generated S_0-algebra.

Proof. If S is Noetherian, so is $S_0 \cong S/S_+$, where $S_+ = \bigoplus_{n>0} S_n$. Choose finitely many homogeneous positive-degree generators x_i of the ideal S_+. Induction on degree shows that they generate S as an S_0-algebra: a homogeneous y is a sum $\sum a_i x_i$ with each nonzero homogeneous coefficient a_i of strictly smaller degree. The converse follows from the Hilbert basis theorem. ∎

The Rees construction

For an ideal $\mathfrak{a}$ and an $\mathfrak{a}$-filtration (M_n), put

$$\mathscr{R}(A,\mathfrak{a}) = \bigoplus_{n\geq 0} \mathfrak{a}^n t^n \subseteq A[t], \qquad \mathscr{R}(M) = \bigoplus_{n\geq 0} M_n t^n.$$

The symbol t records the degree. If $\mathfrak{a} = (a_1, \dots, a_r)$, then $\mathscr{R}(A,\mathfrak{a}) = A[a_1 t, \dots, a_r t]$. Thus it is Noetherian whenever A is Noetherian.

12 ARTIN–REES: THE CENTRAL MECHANISM

contents ↑

Throughout this section, A is Noetherian and M is finitely generated.

Lemma 12.1 *(A–M 10.8).* For an $\mathfrak{a}$-filtration (M_n), the Rees module $\mathscr{R}(M)$ is finitely generated over $\mathscr{R}(A,\mathfrak{a})$ if and only if the filtration is stable.

Proof. If stability begins at c, the finitely generated modules $M_0,\dots,M_c$ supply finitely many homogeneous generators, since $M_{c+r}=\mathfrak{a}^r M_c$.

Conversely take homogeneous generators in degrees at most c. For $n\ge c$, every degree $n+1$ expression has coefficient in $\mathfrak{a}^{n+1-d}=\mathfrak{a}\,\mathfrak{a}^{n-d}$ for a generator of degree $d\le c$. It follows that $M_{n+1}\subseteq \mathfrak{a}M_n$. The reverse inclusion is the filtration property. ∎

Theorem 12.2 *(Artin–Rees; A–M 10.9–A–M 10.10).* If $N\subseteq M$ and (M_n) is stable, then $(N\cap M_n)$ is a stable $\mathfrak{a}$-filtration of N. In particular, for some c,

$$N\cap \mathfrak{a}^n M=\mathfrak{a}^{n-c}(N\cap \mathfrak{a}^c M)\quad (n\ge c). \tag{4}$$

Proof. The graded module $\bigoplus_n (N\cap M_n)t^n$ is a submodule of the finite Rees module $\mathscr{R}(M)$. The Rees ring is Noetherian, so this submodule is finite. Apply the lemma, then iterate its eventual equality for $M_n=\mathfrak{a}^n M$. ∎

Corollary 12.3 *(Intrinsic and induced topologies; A–M 10.11).* The filtrations $(\mathfrak{a}^n N)$ and $(N\cap \mathfrak{a}^n M)$ have bounded difference. Hence the adic topology on N equals the topology induced from M.

Proof. Both filtrations are stable by the theorem. More explicitly, (4) gives

$$\mathfrak{a}^n N\subseteq N\cap \mathfrak{a}^n M,\qquad N\cap \mathfrak{a}^{n+c}M\subseteq \mathfrak{a}^n N.$$

These inclusions are the required comparison. ∎

> In general $\mathfrak{a}^n N$ need not equal $N\cap\mathfrak{a}^n M$. Artin–Rees supplies a uniform shift of the index that makes their **topologies** coincide. This is exactly what completion needs.

13 EXACT COMPLETION AND TENSORING

contents ↑

Theorem 13.1 *(A–M 10.12).* For a Noetherian ring A and an exact sequence of finitely generated A-modules $0 \to N \to M \to P \to 0$, adic completion gives

$$0 \to \widehat{N} \to \widehat{M} \to \widehat{P} \to 0.$$

Proof. The quotient topology on P is its adic topology, because the image of $\mathfrak{a}^n M$ is $\mathfrak{a}^n P$. Artin–Rees identifies the induced topology on N with its own adic topology. Now apply exactness for induced filtrations. ∎

Proposition 13.2 *(Base change; A–M 10.13).* There is a natural map

$$\alpha_M : \widehat{A} \otimes_A M \to \widehat{M}, \qquad a \otimes m \mapsto a\, \iota_M(m).$$

It is surjective when M is finitely generated over any ring A. If A is also Noetherian, it is an isomorphism.

Proof. Choose $A^r \twoheadrightarrow M$. Completion is onto for this surjection: at level n, the kernel is the image of its original kernel in $A^r/\mathfrak{a}^n A^r$, and these images form a surjective inverse system. Thus $\widehat{A}^r \twoheadrightarrow \widehat{M}$, and α_M is onto since its image contains the images of the standard generators with arbitrary $\widehat{A}$ coefficients.

If A is Noetherian, choose a finite presentation $A^s \to A^r \to M \to 0$. Tensoring is right exact, and completion is exact on these finite modules, so both $\widehat{A} \otimes_A M$ and $\widehat{M}$ are the cokernel of the same matrix $\widehat{A}^s \to \widehat{A}^r$. The canonical map is therefore an isomorphism. ∎

Corollary 13.3 *(Flatness; A–M 10.14).* For Noetherian A, its completion $\widehat{A}$ is a flat A-algebra.

Proof. Tensoring with $\widehat{A}$ is exact on finite modules by the two results above. Every ideal of a Noetherian ring is finite, so $I \otimes_A \widehat{A} \to \widehat{A}$ is injective for every ideal I. The ideal criterion for flatness proves the claim. ∎

For arbitrary modules, tensoring with $\widehat{A}$ remains exact by flatness; the identification with completion need not hold. See the counterexample in the exercises.

14 THE COMPLETED RING AND KRULL INTERSECTION

contents ↑

Assume A is Noetherian and put $\hat{\mathfrak{a}} = \mathfrak{a}\widehat{A}$.

Proposition 14.1 *(A–M 10.15–A–M 10.16).* The completed ideal embeds in $\widehat{A}$ and identifies with $\mathfrak{a}\widehat{A}$. Moreover

$$\widehat{\mathfrak{a}^n} = (\mathfrak{a}\widehat{A})^n, \qquad \widehat{A}/(\mathfrak{a}\widehat{A})^n \cong A/\mathfrak{a}^n,$$

$$\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong (\mathfrak{a}\widehat{A})^n/(\mathfrak{a}\widehat{A})^{n+1}, \qquad \mathfrak{a}\widehat{A} \subseteq \operatorname{Jac}(\widehat{A}).$$

If $(A, \mathfrak{m})$ is local and the completion is $\mathfrak{m}$-adic, then $\widehat{A}$ is local with maximal ideal $\mathfrak{m}\widehat{A}$ and the same residue field as A.

Proof. Complete $0 \to \mathfrak{a}^n \to A \to A/\mathfrak{a}^n \to 0$ and apply base change; the image of $\widehat{\mathfrak{a}^n}$ is $\mathfrak{a}^n\widehat{A} = (\mathfrak{a}\widehat{A})^n$. Taking successive quotients proves the graded-piece formula.

If $x \in \mathfrak{a}\widehat{A}$, then $\sum_{j\ge 0} x^j$ converges and is an inverse to $1 - x$. For every $b \in \widehat{A}$, also $bx \in \mathfrak{a}\widehat{A}$, so $1 - bx$ is a unit. This is the Jacobson-radical criterion. In the local case, every maximal ideal therefore contains $\mathfrak{m}\widehat{A}$, whose quotient is the field $A/\mathfrak{m}$; it is the unique maximal ideal. ∎

Theorem 14.2 *(Krull intersection; A–M 10.17).* For a finite A-module M, let $E = \bigcap_{n\ge 1} \mathfrak{a}^n M$. Then

$$E = \{x \in M : (1 - a)x = 0 \text{ for some } a \in \mathfrak{a}\}.$$

In fact a single element $1 - a$ annihilates all of E.

Proof. Artin–Rees applied to $E \subseteq M$ gives $E = \mathfrak{a}E$, since $E \cap \mathfrak{a}^n M = E$ at every stage. The module E is finite. The determinant trick for $E = \mathfrak{a}E$ gives an $a \in \mathfrak{a}$ with $(1 - a)E = 0$. Conversely, $(1 - a)x = 0$ implies $x = a^n x \in \mathfrak{a}^n M$ for every n. ∎

15 SEPARATEDNESS AND LOCALIZATION

contents ↑

Corollary 15.1 *(A–M 10.18–A–M 10.20).* 1. If A is a Noetherian domain and $\mathfrak{a} \neq A$, then $\bigcap_n \mathfrak{a}^n = 0$.

2. If A is Noetherian, $\mathfrak{a} \subseteq \operatorname{Jac}(A)$ and M is finite, then $\bigcap_n \mathfrak{a}^n M = 0$.

3. In particular, finite modules over a Noetherian local ring $(A, \mathfrak{m})$ are Hausdorff in their $\mathfrak{m}$-adic topology.

Proof. In a domain, $1 - a \neq 0$ for $a \in \mathfrak{a} \neq A$, so it annihilates no nonzero element of A. If $\mathfrak{a} \subseteq \operatorname{Jac}(A)$, each $1 - a$ is a unit, so it annihilates no nonzero module element. Apply Krull's theorem in both cases. ∎

Corollary 15.2 *(Primary ideals; A–M 10.21).* For a prime ideal $\mathfrak{p}$ in a Noetherian ring A,

$$\bigcap_{\substack{Q \subset A \\ Q \text{ is } \mathfrak{p}\text{-primary}}} Q = \ker(A \to A_{\mathfrak{p}}).$$

Proof. In the local ring $A_{\mathfrak{p}}$, every power of its maximal ideal is primary, and each primary ideal with that radical contains a power. Their intersection is therefore zero by the preceding corollary. Extension and contraction identify these primary ideals with the $\mathfrak{p}$-primary ideals of A. Contracting the zero intersection gives the kernel of localization. ∎

Localization compared with completion

Set $S = 1 + \mathfrak{a}$. The kernel of $M \to S^{-1}M$ consists of elements annihilated by some $s \in S$. Krull's theorem therefore says, for finite M,

$$\ker(M \to \widehat{M}) = \ker(M \to S^{-1}M).$$

Every $s \in S$ becomes a unit in $\widehat{A}$ by the geometric-series argument, giving a canonical map $S^{-1}A \to \widehat{A}$. Its kernel is zero: if the image of a/s is zero, then a lies in the completion kernel and is annihilated by an element of S.

Example 15.3 *(Completion can discard a component).* Let $A = k \times k$ and $\mathfrak{a} = k \times 0$. Since $\mathfrak{a}^n = \mathfrak{a}$ for $n \geq 1$, the completion is $A/\mathfrak{a} \cong k$, and the canonical map kills the first factor. The adic topology on A is not Hausdorff. The Noetherian hypothesis by itself does not imply separatedness for every ideal.

16 ASSOCIATED GRADED RINGS

contents ↑

Definition 16.1 *(Associated graded objects).* For an ideal $\mathfrak{a} \subseteq A$ and an $\mathfrak{a}$-filtration (M_n), set

$$\operatorname{gr}_{\mathfrak{a}}(A) = \bigoplus_{n\geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1}, \qquad \operatorname{gr}(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}.$$

Multiplication is induced by representatives: $[a]_r[b]_s = [ab]_{r+s}$. If a representative changes by an element of $\mathfrak{a}^{r+1}$, its product changes by an element of $\mathfrak{a}^{r+s+1}$, so multiplication is well-defined. The analogous rule makes $\operatorname{gr}(M)$ a graded $\operatorname{gr}_{\mathfrak{a}}(A)$-module.

Proposition 16.2 *(A–M 10.22).* If A is Noetherian, then $\operatorname{gr}_{\mathfrak{a}}(A)$ is Noetherian and

$$\operatorname{gr}_{\mathfrak{a}}(A) \cong \operatorname{gr}_{\mathfrak{a}\widehat{A}}(\widehat{A}).$$

If M is finite and its filtration is stable, $\operatorname{gr}(M)$ is a finite graded $\operatorname{gr}_{\mathfrak{a}}(A)$-module.

Proof. Choose generators $a_1, \dots, a_r$ of $\mathfrak{a}$. Their degree-one classes generate the graded ring over $A/\mathfrak{a}$. Thus there is a surjection $(A/\mathfrak{a})[T_1, \dots, T_r] \twoheadrightarrow \operatorname{gr}_{\mathfrak{a}}(A)$, proving Noetherianity. The isomorphism follows degree by degree from the completed-quotient formulas.

If stability begins at c, then $M_{c+r} = \mathfrak{a}^r M_c$. The finite collection of finite $A/\mathfrak{a}$-modules M_n/M_{n+1} for $0 \leq n \leq c$ generates the whole graded module. ∎

Example 16.3 *(Power series and leading terms).* For $A = k[[t]]$ with $\mathfrak{a} = (t)$, the associated graded ring is $k[T]$, where T has degree one. For a field k and $A = k[x_1, \dots, x_r]$ with $\mathfrak{a} = (x_1, \dots, x_r)$, the associated graded ring is again the polynomial ring with its usual total-degree grading. Completion retains precisely these finite-order layers.

The Rees module stores the whole filtration with its inclusions. The associated graded module stores the successive layers. They serve different purposes: Rees proves Artin–Rees; the associated graded construction transfers finite-generation information back to complete objects.

17 LIFTING FROM GRADED INFORMATION

contents ↑

Lemma 17.1 *(A–M 10.23).* Let $f : G \to H$ preserve decreasing filtrations with $G_0 = G$, $H_0 = H$. If $\mathrm{gr}(f)$ is injective, then $\widehat{f}$ is injective. If $\mathrm{gr}(f)$ is surjective, then $\widehat{f}$ is surjective.

Proof. Write $f_n : G/G_n \to H/H_n$. Use the short exact sequences

$$0 \to G_n/G_{n+1} \to G/G_{n+1} \to G/G_n \to 0$$

and their counterparts for H. Starting at the zero quotients for $n = 0$, the snake lemma gives, by induction, injectivity of all f_n when every graded map is injective. This implies injectivity on inverse limits.

If every graded map is onto, the same diagram gives that every f_n is onto and $\ker f_{n+1} \to \ker f_n$ is onto. Apply inverse-limit exactness to $0 \to \ker f_n \to G/G_n \to H/H_n \to 0$ to obtain surjectivity of $\widehat{f}$. ∎

Proposition 17.2 *(A–M 10.24).* Suppose A is complete for its adic topology, (M_n) is an $\mathfrak{a}$-filtration, M is Hausdorff for that filtration, and $\mathrm{gr}(M)$ is a finite $\mathrm{gr}_{\mathfrak{a}}(A)$-module. Then M is a finite A-module.

Proof. Choose homogeneous generators of $\mathrm{gr}(M)$, represented by $x_i \in M_{d_i}$, $1 \le i \le r$. Let $F = \bigoplus_i Ae_i$ with shifted filtration

$$F_n = \bigoplus_i \mathfrak{a}^{\max(n-d_i,0)} e_i.$$

The map $f : F \to M$, $e_i \mapsto x_i$, is filtered and its graded map is onto. By the lemma, $\widehat{F} \to \widehat{M}$ is onto. Each shifted filtration has the adic topology, so completeness of A gives $F \cong \widehat{F}$.

For any $m \in M$, lift its image in $\widehat{M}$ to an element $z \in F$. The images of $f(z)$ and m in $\widehat{M}$ coincide. Since M is Hausdorff, $M \to \widehat{M}$ is injective, and $f(z) = m$. Hence f is onto and the x_i generate M. ∎

Notice that completeness is assumed for A, while only separatedness is initially assumed for M. Replacing these hypotheses casually would break the argument.

18 WHY COMPLETION IS NOETHERIAN

contents ↑

Corollary 18.1 *(A–M 10.25).* Under the hypotheses of the finite-generation proposition, if $\mathrm{gr}(M)$ is Noetherian as a $\mathrm{gr}_{\mathfrak{a}}(A)$-module, then M is Noetherian as an A-module.

Proof. For every submodule $N \subseteq M$, use the induced filtration $N_n = N \cap M_n$. It is an $\mathfrak{a}$-filtration, is Hausdorff, and its associated graded module embeds in $\mathrm{gr}(M)$. Thus $\mathrm{gr}(N)$ is finite by the Noetherian hypothesis. The proposition shows N is finite. Since every submodule is finite, M is Noetherian. ∎

Theorem 18.2 *(A–M 10.26).* The adic completion of a Noetherian ring is Noetherian.

Proof. The ring $\widehat{A}$ is complete and Hausdorff for the $\mathfrak{a}\widehat{A}$-adic topology, by the quotient formulas and idempotence of completion. Its associated graded ring is isomorphic to $\mathrm{gr}_{\mathfrak{a}}(A)$, which is Noetherian by the graded finite-generation proposition. Apply the preceding corollary with the complete base ring $\widehat{A}$ and its module $\widehat{A}$. ∎

Corollary 18.3 *(Formal power series; A–M 10.27).* If A is Noetherian, then $A[[x_1, \dots, x_r]]$ is Noetherian.

Proof. The polynomial ring $A[x_1, \dots, x_r]$ is Noetherian by the Hilbert basis theorem. Its $(x_1, \dots, x_r)$-adic completion is $A[[x_1, \dots, x_r]]$: coherent truncations specify exactly one coefficient for each monomial. Apply the theorem. ∎

The two proof chains to remember

$$\text{Rees Noetherianity} \implies \text{Artin–Rees} \implies \text{induced = intrinsic topology}$$
$$\implies \text{exact completion} \implies \widehat{A} \otimes_A M \cong \widehat{M} \implies \widehat{A} \text{ flat.}$$

The other chain uses successive layers:

$$\mathrm{gr}_{\mathfrak{a}}(A) \cong \mathrm{gr}_{\mathfrak{a}\widehat{A}}(\widehat{A}) \text{ Noetherian} \quad + \quad \widehat{A} \text{ complete} \quad \implies \quad \widehat{A} \text{ Noetherian.}$$

19 WORKED EXERCISES AND PITFALLS

contents ↑

1. An inverse limit that is not right exact

Fix a prime p. Consider the exact sequences

$$0 \to p^n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0,$$

with inclusion on the left, identity in the middle and reduction on the right. Show that taking inverse limits is not right exact.

Solution. The left inverse limit is $\bigcap_n p^n\mathbb{Z} = 0$, the middle one is $\mathbb{Z}$, and the right one is $\mathbb{Z}_p$. The resulting map $\mathbb{Z} \to \mathbb{Z}_p$ is not onto. For example, the p-adic expansion whose digit is 1 exactly at positive square positions is not an ordinary integer. A nonnegative integer has eventually zero digits; a negative integer has eventually $p-1$ digits. This expansion has neither form. The missing hypothesis in the exactness theorem is surjectivity of the transitions $p^{n+1}\mathbb{Z} \to p^n\mathbb{Z}$. ∎

2. Equivalent filtrations need not agree term by term

Let $A = k[t]$, $\mathfrak{a} = (t)$, $M = A$ and $N = t^s A$ with $s \geq 1$. Compare the intrinsic and induced filtrations on N.

Solution. They are

$$\mathfrak{a}^n N = t^{n+s}A, \qquad N \cap \mathfrak{a}^n M = t^{\max(n,s)}A.$$

For $n \geq s$, the Artin–Rees formula holds with $c = s$:

$$N \cap t^n A = t^{n-s}(N \cap t^s A) = t^n A.$$

The terms differ, but shifting the index by s gives cofinal filtrations. Both completions of N identify with $t^s k[[t]]$ inside $\widehat{M} = k[[t]]$. ∎

3. A nonzero module killed by completion

Compute the (t)-adic completion of $M = k[t]/(t-1)$.

Solution. Multiplication by t is the identity on M, so $t^n M = M$ for every n. All quotient modules are zero and $\widehat{M} = 0$, although $M \neq 0$. The element $1 - t \in 1 + (t)$ kills M, as predicted by Krull's theorem. The ideal (t) is not contained in $\mathrm{Jac}(k[t])$. ∎

20 WHY FINITENESS CANNOT BE OMITTED

contents ↑

This example explains the mechanism of Chapter 10, Exercise 1. Put

$$U = \bigoplus_{j\geq 1} \mathbb{Z}/p\mathbb{Z}, \qquad B = \bigoplus_{j\geq 1} \mathbb{Z}/p^j\mathbb{Z},$$

and embed U in B by sending the generator of its jth summand to p^{j-1} in the jth summand of B.

Intrinsic versus induced completion

Intrinsically $pU = 0$, so the p-adic topology on U is discrete and $\widehat{U} = U$. Its induced neighborhoods from B are

$$U \cap p^r B = \bigoplus_{j\geq r+1} \mathbb{Z}/p\mathbb{Z}.$$

Indeed, p^{j-1} is divisible by p^r in $\mathbb{Z}/p^j\mathbb{Z}$ exactly when $j - 1 \geq r$. The quotients are the first r summands, so completion in the induced topology gives

$$\varprojlim_r U/(U \cap p^r B) = \prod_{j\geq 1} \mathbb{Z}/p\mathbb{Z}.$$

The intrinsic and induced topologies differ; the finite-module conclusion of Artin–Rees fails here.

Failure of right exactness of completion

Let $C = B/U$. Define $b^{(r)} \in B$ by putting p^{j-1} in coordinates $j \leq r$ and zero elsewhere. Then $b^{(r+1)} - b^{(r)} \in p^r B$, so $(b^{(r)})$ is Cauchy and determines $b \in \widehat{B}$. Every $b^{(r)}$ belongs to U, so b maps to zero in $\widehat{C}$.

But b is not in the image of $\widehat{U} = U$: its projection to every summand of B is nonzero, while each element of U has finite support. Therefore

$$\operatorname{im}(\widehat{U} \to \widehat{B}) \neq \ker(\widehat{B} \to \widehat{C}).$$

Completion is not right exact on all $\mathbb{Z}$-modules, even though $\mathbb{Z}$ is Noetherian and $\mathbb{Z}_p$ is flat. Tensoring with $\mathbb{Z}_p$ is exact; it need not coincide with completion on infinite modules.

> The obstruction is visible before taking the limit: the intrinsic topology on the submodule differs from the inherited topology. Finite generation is what Artin–Rees uses to eliminate this discrepancy.

21 TWO FURTHER APPLICATIONS

contents ↑

4. Faithful flatness

For a Noetherian ring A, prove that $\widehat{A}$ is faithfully flat over A if and only if $\mathfrak{a} \subseteq \operatorname{Jac}(A)$.

Solution. Flatness is already known. A flat algebra B is faithfully flat if and only if $B/\mathfrak{m}B \neq 0$ for every maximal ideal $\mathfrak{m}$ of A.

If $\mathfrak{a} \subseteq \operatorname{Jac}(A)$, then $\mathfrak{a} \subseteq \mathfrak{m}$ for every maximal ideal. The module $A/\mathfrak{m}$ has discrete adic topology, so

$$\widehat{A}/\mathfrak{m}\widehat{A} \cong \widehat{A} \otimes_A A/\mathfrak{m} \cong \widehat{A/\mathfrak{m}} \cong A/\mathfrak{m} \neq 0.$$

Conversely, if $\mathfrak{a} \not\subseteq \mathfrak{m}$ for some maximal ideal, then $\mathfrak{a}(A/\mathfrak{m}) = A/\mathfrak{m}$. Its completion is zero, so base change gives $\widehat{A}/\mathfrak{m}\widehat{A} = 0$, violating faithful flatness. ∎

5. A simple-root Hensel lifting example

Construct a square root of 2 in $\mathbb{Z}_7$, starting from $3 \pmod 7$.

Solution. Suppose $a_n^2 \equiv 2 \pmod{7^n}$ and seek $a_{n+1} = a_n + c_n 7^n$, $0 \leq c_n < 7$. The equation to solve is

$$\frac{a_n^2 - 2}{7^n} + 2a_n c_n \equiv 0 \pmod 7.$$

Since $a_n \equiv 3 \pmod 7$, the coefficient $2a_n \equiv 6$ is invertible modulo 7. Thus c_n is unique. Starting with $a_1 = 3$ gives $c_1 = 1$, $a_2 = 10$; next $c_2 = 2$, $a_3 = 108$. Indeed $108^2 - 2 = 11662 = 34 \cdot 343$.

The coherent sequence $(a_n \bmod 7^n)$ defines $a \in \mathbb{Z}_7$ with $a^2 = 2$. This illustrates the simple-root case of Hensel lifting (Chapter 10, Exercises 9–10): an invertible derivative corrects one more digit at each stage. ∎

A quick distinction

Flatness preserves injectivity of multiplication by a non-zero-divisor of A. It does not by itself show the completed ring is a domain: it controls images of elements of A, not every new element of $\widehat{A}$.

22 CHAPTER 10 CONCORDANCE

contents ↑

Numbers refer to the original textbook. Each topic links to its discussion here; the last column gives printed textbook pages.

A–M	Topic in these notes	p.
10.1	Closure of zero and the Hausdorff quotient	101–102
10.2	Inverse-limit exactness and surjective kernel systems	104
10.3	Completion in induced topologies	104–105
10.4	Unchanged discrete quotients	105
10.5	Idempotence of completion	105
10.6	Stable filtrations have bounded difference	106
10.7	Noetherian graded rings	106
10.8	Finite Rees modules and stable filtrations	107
10.9	Artin–Rees for stable filtrations	107
10.10	Artin–Rees equality for ideal powers	107
10.11	Intrinsic and induced adic topologies	107
10.12	Exactness on finite modules	108
10.13	Completion compared with tensoring	108
10.14	Flatness of the completed ring	109
10.15	Completed ideals and the Jacobson radical	109
10.16	Local completion and residue field	109
10.17	Krull's kernel theorem	110
10.18	Separatedness for domains	110
10.19	Ideals in the Jacobson radical	110–111
10.20	The Noetherian local case	111
10.21	Intersection of primary ideals with a fixed radical	111
10.22	Associated graded rings and modules	111–112
10.23	Passing graded injectivity/surjectivity to completions	112
10.24	Lifting finite generation over a complete base	112–113
10.25	Lifting Noetherianity	113
10.26	Noetherianity of completion	113
10.27	Noetherianity of formal power series rings	113