Defining Group Homology and Cohomology

Last Update: November 4, 2026

Author: HUANG Qinpei

Southern University of Science and Technology

Seminar in Geometry and Topology[1], Fall 2026

Meetings 2–4: October 9, 16, and 23

Advisor: ZHU Yifei

Abstract

These notes cover the 2–4 meetings of the Fall 2026 Geometry and Topology Seminar. Starting from modules and resolutions, we construct left and right derived functors and prove independence from the chosen resolutions. Tensor and Hom functors yield Tor and Ext functors, and hence group homology and cohomology through coinvariants and invariants. The bar resolution provides general formulas.

Contents

1 Left Modules, Presentations, and Free Resolutions **2**

1.1 Rings and left modules . . . 2

1.2 Free left modules and presentations . . . 3

1.3 Exact sequences and free resolutions . . . 4

2 From Projective Resolutions to Group Homology **6**

2.1 Projective modules . . . 6

2.2 Projective resolutions . . . 6

2.3 Tensor functors and left derived functors . . . 13

2.4 Tor functors . . . 20

2.5 Group homology . . . 21

3 From Injective Resolutions to Group Cohomology **25**

3.1 Injective modules and resolutions . . . 25

3.2 Right derived functors . . . 27

3.3 Hom and Ext functors . . . 29

3.4 Group cohomology . . . 30

4 The Bar Resolution **33**

4.1 Left and right bar resolutions . . . 33

4.2 Explicit definitions of group homology and cohomology . . . 36

Summary **38**

References **39**

[1]Webpage: `https://sustech-topology.github.io/undergrad/`

1 Left Modules, Presentations, and Free Resolutions

1.1 Rings and left modules

Ring. A *unital ring* is an abelian group $(R, +, 0)$ with a multiplication and an element 1_R such that, for all $r, s, t \in R$,

$$(rs)t = r(st), \qquad r(s+t) = rs + rt, \qquad (r+s)t = rt + st, \qquad 1_R r = r = r 1_R.$$

All rings discussed below are assumed unital, but not necessarily commutative; $1_R = 0$ is allowed.

Left module. A *left R-module* is an abelian group $(M, +, 0)$ with a left R-action $R \times M \to M$, $(r, m) \mapsto rm$, satisfying

$$r(m+m') = rm + rm', \quad (r+r')m = rm + r'm,$$
$$(rr')m = r(r'm), \qquad 1_R m = m$$

for all $r, r' \in R$ and $m, m' \in M$. We write R-**Mod** for (the category of) left R-modules.

Submodule. A *R-submodule* K of M is an additive subgroup of $(M, +)$ satisfying $rk \in K$ for all $r \in R$ and $k \in K$.

Quotient module. For a R-submodule $K \subseteq M$, the set of cosets $M/K = \{m + K : m \in M\}$ is a left R-module with

$$(m+K) + (m'+K) = (m+m') + K, \qquad r(m+K) = rm + K.$$

This left R-module is called a *R-quotient module* of M. The quotient map $q : M \to M/K$; $q(m) = m + K$, is surjective.

Left module homomorphism. A map $f : M \to N$ is left *R-linear* or a left module homomorphism between R-modules if

$$f(m+m') = f(m) + f(m'), \qquad f(rm) = rf(m).$$

An R-linear map is a *monomorphism* if it is injective, an *epimorphism* if it is surjective, and an *isomorphism* if it is bijective.

Kernel and image. For an R-linear map $f : M \to N$, its *kernel* and *image* are defined as

$$\ker f = \{m \in M : f(m) = 0\}, \qquad \operatorname{im} f = \{f(m) : m \in M\}$$

They are submodules of M and N, respectively.

Isomorphism theorem. Every R-linear map $f : M \to N$ induces an isomorphism

$$M/\ker f \xrightarrow{\cong} \operatorname{im} f, \qquad m + \ker f \longmapsto f(m).$$

Proof. Since $f(m) = f(m')$ holds exactly when $m + \ker f = m' + \ker f$, the displayed map is well defined and injective. It is R-linear by linearity of f and surjective by the definition of $\operatorname{im} f$, hence an isomorphism. □

1.2 Free left modules and presentations

Definition by the universal property. A *free left R-module on a set S* is a pair (F, ι), where F is a left R-module and $\iota : S \to F$ is a map, such that for every left R-module N and every map $a : S \to N$, there is a unique R-linear map $\widetilde{a} : F \to N$ with

$$\widetilde{a} \circ \iota = a.$$

A left R-module is *free* if it is free on some set.

Existence. Set

$$F_R(S) = \bigoplus_{s \in S} Re_s.$$

Here each Re_s is a copy of the left module R, labelled by s, and the *direct sum* means

$$\bigoplus_{s \in S} Re_s = \{(r_s)_{s \in S} : r_s \in R,\ r_s = 0 \text{ for all but finitely many } s\},$$

with coordinatewise addition and left scalar multiplication. The vector e_s has entry 1_R at s and 0 elsewhere, and we may identify it with $(\cdots, 1_R, \cdots)$. Thus every element has a unique expression $\sum_{s \in S} r_s e_s$ with finitely many nonzero coefficients; $(e_s)_{s \in S}$ is its *basis*.

Put $\iota(s) = e_s$. For a function $a : S \to N$, the formula

$$\widetilde{a}\left(\sum_s r_s e_s\right) = \sum_s r_s a(s)$$

defines an R-linear map extending a. Every such extension must satisfy this formula, proving the universal property. □

Uniqueness. If (F, ι) and (F', ι') are free on the same set S, their universal properties give unique R-linear maps

$$u : F \to F', \quad u\iota = \iota', \qquad v : F' \to F, \quad v\iota' = \iota.$$

Uniqueness gives $vu = \mathrm{id}_F$ and $uv = \mathrm{id}_{F'}$. Hence, there is a unique isomorphism $u : F \to F'$ such that $u\iota = \iota'$. □

Generators and relations. A subset $S \subseteq M$ *generates* M if every element of M is a finite sum of elements rs with $r \in R$ and $s \in S$. Chosen such an S (taking $S = M$ is always possible), the universal property gives a surjection

$$\varepsilon : F_0 = F_R(S) \longrightarrow M, \qquad \sum_s r_s e_s \longmapsto \sum_s r_s s.$$

with $K_0 = \ker \varepsilon$. The isomorphism theorem gives

$$M \cong F_0/K_0, \quad \text{where} \quad K_0 = \left\{\sum_s r_s e_s : \sum_s r_s s = 0 \text{ in } M\right\}.$$

The basis elements e_s are *generators*; the elements of K_0 are *relations* among their images in M. This is called a *presentation* of M.

1.3 Exact sequences and free resolutions

From a quotient to a short exact sequence. Write $M \cong F_0/K_0$ from Section 1.2 as

$$0 \longrightarrow K_0 \xrightarrow{i_0} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where i_0 is inclusion, ε is surjective, and $\operatorname{im} i_0 = K_0 = \ker \varepsilon$. This is a *short exact sequence.*

In general, a sequence of left R-modules and left R-linear maps

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

is *exact at* A_n if $\operatorname{im} f_{n+1} = \ker f_n$, and *exact* if this holds at every term having an incoming and an outgoing map. In particular,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact precisely when f is injective, $\operatorname{im} f = \ker g$, and g is surjective.

Relations among relations. Choose a generating set $T \subseteq K_0$. The presentation free left R-module $F_1 = F_R(T)$ maps onto K_0 by $p_1(e_t) = t$. Put $K_1 = \ker p_1$ and let $i_1 : K_1 \hookrightarrow F_1$ be inclusion. This gives a second short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_1 \xrightarrow{p_1} K_0 \longrightarrow 0.$$

Define $d_1 = i_0 p_1 : F_1 \to F_0$. Splicing the two sequences gives

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

which is exact since $\ker d_1 = K_1$ and $\operatorname{im} d_1 = K_0$.

The elements $t \in T$ are chosen relations in F_0, and $\sum_t a_t e_t \in F_1$ belongs to K_1 exactly when

$$\sum_t a_t t = 0 \quad \text{in } F_0.$$

Thus K_1 consists of the *relations among the chosen relations.*

Iterating to get a free resolution. For each $n \geq 1$, choose a free left R-module F_n and a surjection $p_n : F_n \to K_{n-1}$, put $K_n = \ker p_n$, and let $i_n : K_n \hookrightarrow F_n$ be inclusion. Set $d_n = i_{n-1} p_n$. Then

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact: $\operatorname{im} d_1 = \ker \varepsilon$ and $\operatorname{im} d_{n+1} = K_n = \ker d_n$ for every $n \geq 1$.

An exact sequence of this form with every F_n free is called a *free resolution* of M; ε is its *augmentation.* The construction proves that every left R-module has a free resolution. □

Higher free left R-modules encode successive relations among relations. A free resolution thus unravels a R-module, layer by layer: generators, relations, relations among relations, and so on.

Removing the augmentation to get a chain complex. Delete M and ε from the resolution, leaving

$$F_\bullet : \qquad \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0.$$

Its consecutive maps compose to zero.

In general, a *chain complex* of left R-modules is a family $(C_n)_{n\in\mathbb{Z}}$ with left R-linear maps $d_n : C_n \to C_{n-1}$, called *differentials*, such that

$$d_n d_{n+1} = 0 \qquad \text{for every } n.$$

Equivalently, $\operatorname{im} d_{n+1} \subseteq \ker d_n$; equality is not required.

The *nth homology* of $C_\bullet$ is the quotient left R-module

$$H_n(C_\bullet) = \ker d_n / \operatorname{im} d_{n+1}.$$

Its scalar action is $r[x] = [rx]$. Elements of $\ker d_n$ are *n-cycles* or *closed n-chains*; elements of $\operatorname{im} d_{n+1}$ are *n-boundaries* or *exact n-chains*. Thus $H_n(C_\bullet) = 0$ precisely when the complex is exact at C_n.

The complex $F_\bullet$ has $F_n = 0$ for $n < 0$ and $d_0 = 0$. In terms of homology, the free resolution gives

$$H_0(F_\bullet) = F_0 / \operatorname{im} d_1 \cong M, \qquad H_n(F_\bullet) = 0 \quad (n \neq 0).$$

Example 1: one generator and one relation. Let $n \geq 2$ and regard $M = \mathbb{Z}/n\mathbb{Z}$ as a left $\mathbb{Z}$-module. It is generated by $\overline{1}$ with the relation $n\overline{1} = 0$. The standard free resolution is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0, \qquad \varepsilon(a) = \overline{a}.$$

Multiplication by n is injective and its image is $n\mathbb{Z} = \ker \varepsilon$, so the sequence is exact.

Example 2: a redundant relation. Keep the same generator, but choose both relations $n\overline{1} = 0$ and $2n\overline{1} = 0$. Thus $F_0 = \mathbb{Z}$, $F_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 = \mathbb{Z}^2$, and

$$d_1 : F_1 \longrightarrow F_0, \qquad d_1(a, b) = na + 2nb = n(a + 2b).$$

The second relation is redundant: $2n = 2 \cdot n$ in F_0. We have

$$\ker d_1 = \{(a, b) \in \mathbb{Z}^2 : a + 2b = 0\} = \mathbb{Z}(-2, 1).$$

The nonzero element $-2e_1 + e_2 \in F_1$ records the identity $-2n + 2n = 0$ in F_0, a relation among the chosen relations. The map

$$d_2 : \mathbb{Z} \longrightarrow \mathbb{Z}^2, \qquad t \longmapsto (-2t, t)$$

is injective with image $\ker d_1$. Since $\operatorname{im} d_1 = n\mathbb{Z} = \ker \varepsilon$, we obtain the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

The additional term records the redundancy of the chosen relations.

2 From Projective Resolutions to Group Homology

2.1 Projective modules

Definition by the lifting property. A left R-module P is *projective* if, for every epimorphism $q : A \to B$ of left R-modules and every R-linear map $f : P \to B$, there exists an R-linear map $\widetilde{f} : P \to A$ such that

$$q \circ \widetilde{f} = f.$$

The map $\widetilde{f}$ is a *lift* of f through q. The lift need not be unique.

Proposition. Every free left R-module is projective.

Proof. Let F be free with basis $(e_s)_{s \in S}$. Given an epimorphism $q : A \to B$ and an R-linear map $f : F \to B$, choose $a_s \in A$ with $q(a_s) = f(e_s)$ for each $s \in S$. By the universal property of F, the assignment $e_s \mapsto a_s$ extends uniquely to an R-linear map $\widetilde{f} : F \to A$. The maps $q \circ \widetilde{f}$ and f agree on every basis element, so $q \circ \widetilde{f} = f$. This proves projectivity. The extension is unique for the chosen a_s, but the choices themselves need not be unique. □

Since every left R-module is a quotient of a free left R-module, every left R-module admits an epimorphism from a projective left R-module. Thus there are *enough projective left modules.*

Remark. Freeness is strictly stronger than projectivity in general: every free left R-module is projective, but a projective left R-module need not be free.

We introduce projective left R-modules because their lifting property is useful in later proofs and dualizes to the extension property of injective left R-modules. For most explicit computations, we will use free left or right R-modules, as appropriate, and free resolutions: they provide concrete generators and relations, whereas identifying suitable projective or injective left R-modules, and assembling them into resolutions, can be more difficult.

2.2 Projective resolutions

Definition and notation. A *projective resolution* of a left R-module M, written $P_\bullet \xrightarrow{\varepsilon} M$, is an exact sequence of left R-modules and R-linear maps

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

in which every P_n is projective. The map $\varepsilon : P_0 \to M$ is the *augmentation.*

The notation $P_\bullet$ alone denotes the underlying chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0,$$

with $P_n = 0$ for $n < 0$ and $d_0 = 0$. It does not include M, but the augmentation identifies $H_0(P_\bullet)$ with M; $H_n(P_\bullet) = 0$ for $n > 0$.

Existence. Every left R-module has a free resolution. Since every free left R-module is projective, this is also a projective resolution. □

Towards uniqueness. The two resolutions in Section 1.3 have different lengths and are not isomorphic degree by degree. We will establish uniqueness up to *chain homotopy equivalence.*

The preparation parallels topology: continuous maps between spaces, homotopies between maps, and homotopy equivalences between spaces.

Chain maps. Let $C_\bullet$ and $D_\bullet$ be chain complexes of left R-modules. A *chain map* $f : C_\bullet \to D_\bullet$ is a family of R-linear maps $f_n : C_n \to D_n$ such that

$$d_n^D \circ f_n = f_{n-1} \circ d_n^C \qquad \text{for every } n.$$

Equivalently, the following squares commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \xrightarrow[d_{n+1}^D]{} & D_n & \xrightarrow[d_n^D]{} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

Composition and identity maps are defined degree by degree.

Proposition. Every chain map $f : C_\bullet \to D_\bullet$ canonically induces an R-linear map

$$H_n(f) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet), \qquad [x] \longmapsto [f_n(x)]$$

for every n. These maps satisfy $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for composable chain maps, and $H_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{H_n(C_\bullet)}$.

Proof. The chain map condition gives

$$f_n(\ker d_n^C) \subseteq \ker d_n^D, \qquad f_n(\operatorname{im} d_{n+1}^C) \subseteq \operatorname{im} d_{n+1}^D.$$

Thus the formula is well-defined on the quotient and is R-linear. Compatibility with composition and identities follows from the formula. □

Chain homotopies. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. A *chain homotopy* s from f to g is a family of R-linear maps $s_n : C_n \to D_{n+1}$ such that

$$f_n - g_n = d_{n+1}^D \circ s_n + s_{n-1} \circ d_n^C \qquad \text{for every } n.$$

In the following diagram, the middle vertical map is the sum of the two composites from C_n to D_n using a dashed arrow:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n - g_n} & \swarrow{\scriptstyle s_{n-1}} & & & \\
\cdots & \longrightarrow & D_{n+1} & \xrightarrow[d_{n+1}^D]{} & D_n & \xrightarrow[d_n^D]{} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

If such a family s exists, f and g are *chain homotopic*, written $s : f \simeq g$. For complexes concentrated in nonnegative degrees, $s_{-1} = 0$.

Proposition. Chain homotopic maps $f, g : C_\bullet \to D_\bullet$ induce the same maps on homology: $H_n(f) = H_n(g)$ for every n.

Proof. If $x \in \ker d_n^C$, the homotopy identity gives

$$f_n(x) - g_n(x) = d_{n+1}^D s_n(x) \in \operatorname{im} d_{n+1}^D.$$

Hence $[f_n(x)] = [g_n(x)]$. □

Chain homotopy equivalences. A chain map $f : C_\bullet \to D_\bullet$ is a *chain homotopy equivalence* if there is a chain map $g : D_\bullet \to C_\bullet$ such that

$$g \circ f \simeq \mathrm{id}_{C_\bullet}, \qquad f \circ g \simeq \mathrm{id}_{D_\bullet}.$$

The map g is a *homotopy inverse* of f. In this case the complexes are *chain homotopy equivalent*, written $f : C_\bullet \simeq D_\bullet$.

Proposition. A chain homotopy equivalence $f : C_\bullet \to D_\bullet$ induces an isomorphism $H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$ for every n.

Proof. Let g be a homotopy inverse of f. The preceding propositions give

$$H_n(g) \circ H_n(f) = H_n(g \circ f) = H_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{H_n(C_\bullet)},$$
$$H_n(f) \circ H_n(g) = H_n(f \circ g) = H_n(\mathrm{id}_{D_\bullet}) = \mathrm{id}_{H_n(D_\bullet)}.$$

The middle equality in each line uses $g \circ f \simeq \mathrm{id}_{C_\bullet}$ and $f \circ g \simeq \mathrm{id}_{D_\bullet}$: chain homotopic maps induce the same maps on homology. The first and last equalities follow from compatibility with composition and identity maps, respectively. Thus $H_n(g)$ is the inverse of $H_n(f)$. □

Remark. For later applications, merely inducing isomorphisms on homology is too weak: a chain map can lose this property after a non-exact additive covariant functor is applied degree by degree. Such a functor does preserve chain homotopies and chain homotopy equivalences, because it preserves sums and compositions. We will therefore apply the functor to homotopy equivalent resolutions and then use the preceding proposition to compare the resulting homology.

Example: comparing the two resolutions. Let $P_\bullet$ and $Q_\bullet$ be the underlying complexes of the standard and redundant-relation resolutions of $\mathbb{Z}/n\mathbb{Z}$ in Section 1.3, with $n \geq 2$. The diagrams below show the full augmented resolutions, with $\mathbb{Z}/n\mathbb{Z}$ in degree -1 and zeros at both ends. The maps f and g are extended by the identity in degree -1. Recall that

$$d_2(t) = (-2t, t), \qquad d_1(a, b) = n(a + 2b).$$

The chain maps. Define $f : P_\bullet \to Q_\bullet$ by $f_1(x) = (x, 0)$ and $f_0 = \mathrm{id}$, with all other degree components zero. Extend it to the augmented complexes by $f_{-1} = \mathrm{id}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_2=0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0=\mathrm{id}} & & \downarrow{\scriptstyle f_{-1}=\mathrm{id}} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[d_2]{} & \mathbb{Z}^2 & \xrightarrow[d_1]{} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

In the other direction, define $g : Q_\bullet \to P_\bullet$ by $g_1(a,b) = a + 2b$ and $g_0 = \mathrm{id}$, with all other degree components zero. Extend it to the augmented complexes by $g_{-1} = \mathrm{id}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^2 & \xrightarrow{d_1} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle g_2=0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0=\mathrm{id}} & & \downarrow{\scriptstyle g_{-1}=\mathrm{id}} & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\times n]{} & \mathbb{Z} & \xrightarrow[\varepsilon]{} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

The identities $d_1 f_1(x) = nx$, $n g_1(a,b) = d_1(a,b)$, and $g_1 d_2(t) = 0$ verify that the squares commute. Both maps respect the common augmentation $\varepsilon : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

The composites. gf. Since $g_1 f_1(x) = x$, we have $gf = \mathrm{id}_{P_\bullet}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\times n]{} & \mathbb{Z} & \xrightarrow[\varepsilon]{} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

and $\mathrm{id}_{P_\bullet} \simeq \mathrm{id}_{P_\bullet}$ trivially.

fg. Put $e(a,b) = (a + 2b, 0)$. Then $(fg)_2 = 0$, $(fg)_1 = e$, and $(fg)_0 = \mathrm{id}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^2 & \xrightarrow{d_1} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[d_2]{} & \mathbb{Z}^2 & \xrightarrow[d_1]{} & \mathbb{Z} & \xrightarrow[\varepsilon]{} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

A homotopy from fg to the identity. Define $s_1 : Q_1 \to Q_2$ by $s_1(a,b) = -b$, and set $s_j = 0$ for $j \neq 1$, including $s_{-1} = 0$. The dashed arrow is the only nonzero homotopy component; the vertical maps extend $fg - \mathrm{id}_{Q_\bullet}$ by zero in degree -1:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^2 & \xrightarrow{d_1} & \mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle -\mathrm{id}} & \overset{s_1}{\swarrow} & \downarrow{\scriptstyle e-\mathrm{id}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[d_2]{} & \mathbb{Z}^2 & \xrightarrow[d_1]{} & \mathbb{Z} & \xrightarrow[\varepsilon]{} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

Indeed,

$$\begin{aligned}
d_2 s_1(a,b) &= (2b, -b) = e(a,b) - (a,b) = (e - \mathrm{id})(a,b), \\
s_1 d_2(t) &= -t = -\mathrm{id}(t).
\end{aligned}$$

In degree 0, both sides of the homotopy identity are zero. The degree--1 identity is also zero since $s_{-1} = 0$. Thus $fg - \mathrm{id}_{Q_\bullet} = ds + sd$, so $fg \simeq \mathrm{id}_{Q_\bullet}$. Together with $gf = \mathrm{id}_{P_\bullet}$, this proves that f and g are homotopy inverses and $P_\bullet \simeq Q_\bullet$.

The same maps and homotopy remain valid after deleting the augmentation term $\mathbb{Z}/n\mathbb{Z}$, since $s_{-1} = 0$. The same construction extends to arbitrary projective resolutions. The proof makes the role of projectivity explicit: exactness of the target resolution supplies surjections onto kernels, and projectivity of the source terms supplies lifts through these surjections.

Theorem (Comparison of projective resolutions). Let $P_\bullet \xrightarrow{\varepsilon_P} M$ and $Q_\bullet \xrightarrow{\varepsilon_Q} N$ be projective resolutions of left R-modules. In the augmented complexes, M and N occupy degree -1, and all lower terms are zero. Their degree-0 differentials are the augmentations; the underlying complexes $P_\bullet, Q_\bullet$ still have $d_0^P = d_0^Q = 0$. For every R-linear map $u : M \to N$, the following hold.

1. *Existence of a lift.* There is a chain map $f : P_\bullet \to Q_\bullet$ satisfying

   $$\varepsilon_Q f_0 = u\varepsilon_P.$$

   Together, f and u define a chain map of augmented complexes

   $$\widetilde{u} = (f, u) : (P_\bullet \xrightarrow{\varepsilon_P} M) \longrightarrow (Q_\bullet \xrightarrow{\varepsilon_Q} N),$$

   called a *lift of* u. Its component is f_n in degree $n \geq 0$ and u in degree -1.

2. *Uniqueness up to chain homotopy.* Any two lifts $\widetilde{u} = (f, u)$ and $\widetilde{u}' = (f', u)$ are chain homotopic. The homotopy can be chosen in the form

   $$\widetilde{s} = (s, 0), \qquad \widetilde{s}_n = s_n \quad (n \geq 0), \qquad \widetilde{s}_{-1} = 0 : M \to Q_0,$$

   where $s = (s_n)_{n \geq 0}$ is a chain homotopy from f to f' on the underlying complexes $P_\bullet, Q_\bullet$. Such $\widetilde{s}$ is not necessarily unique, since the lift in projective modules need not be unique.

Proof. (Existence.) Start with the two augmented resolutions and the given map u:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\varepsilon_P} & M & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle u} & & \\
\cdots & \longrightarrow & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\varepsilon_Q} & N & \longrightarrow & 0.
\end{array}$$

We seek maps $f_n : P_n \to Q_n$ extending u to a chain map of these augmented complexes. First, $u\varepsilon_P : P_0 \to N$ is R-linear and $\varepsilon_Q : Q_0 \to N$ is an epimorphism. Projectivity of P_0 gives a lift $f_0 : P_0 \to Q_0$ making the following diagram commute:

$$\begin{array}{ccc}
P_0 & & \\
{\scriptstyle f_0}\downarrow & \searrow^{u\varepsilon_P} & \\
Q_0 & \xrightarrow[\varepsilon_Q]{} & N.
\end{array}$$

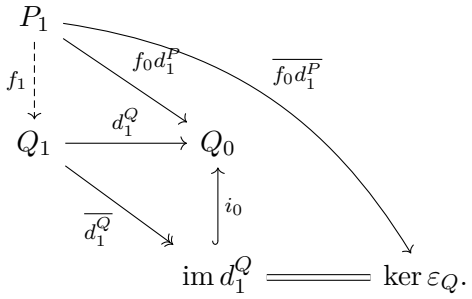
Thus $\varepsilon_Q f_0 = u\varepsilon_P$, which is the chain map condition in degree 0.

Next, consider $f_0 d_1^P : P_1 \to Q_0$. We check that

$$\varepsilon_Q f_0 d_1^P = u\varepsilon_P d_1^P = 0.$$

Thus its image lies in $\ker \varepsilon_Q$, giving a map $\overline{f_0 d_1^P} : P_1 \to \ker \varepsilon_Q$. Exactness identifies $\ker \varepsilon_Q = \operatorname{im} d_1^Q$. Write $i_0 : \operatorname{im} d_1^Q \hookrightarrow Q_0$ for the inclusion and $\overline{d_1^Q} : Q_1 \twoheadrightarrow \operatorname{im} d_1^Q$ for the induced epimorphism. Both bars change only the codomain of the original map.

Projectivity of P_1 supplies the dashed lift:



Under the displayed identification, $\overline{d_1^Q} f_1 = \overline{f_0 d_1^P}$. Composing with i_0 gives

$$d_1^Q f_1 = i_0 \overline{d_1^Q} f_1 = i_0 \overline{f_0 d_1^P} = f_0 d_1^P,$$

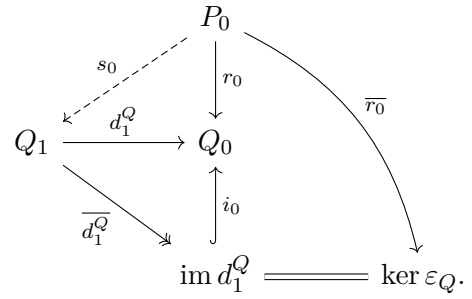
which is the chain map condition in degree 1. Repeating this construction gives all f_n, with $d_n^Q f_n = f_{n-1} d_n^P$ for $n \geq 1$. Thus $f = (f_n)$ is a chain map, and $\varepsilon_Q f_0 = u \varepsilon_P$ makes $\widetilde{u} = (f, u)$ the required augmented lift. The formal induction is omitted.

(Uniqueness up to chain homotopy.) Let $\widetilde{u} = (f, u)$ and $\widetilde{u}' = (f', u)$ be two lifts.

Degree 0. Set $s_{-1} = 0$ on the underlying complexes and put $r_0 = f_0 - f_0'$. Since both maps lift u,

$$\varepsilon_Q r_0 = u \varepsilon_P - u \varepsilon_P = 0.$$

Thus r_0 takes values in $\ker \varepsilon_Q = \operatorname{im} d_1^Q$. Projectivity of P_0 gives the dashed lift below; as before, bars change only codomains, and i_0 is the inclusion.



The lift satisfies $\overline{d_1^Q} s_0 = \overline{r_0}$. Composing with i_0 gives

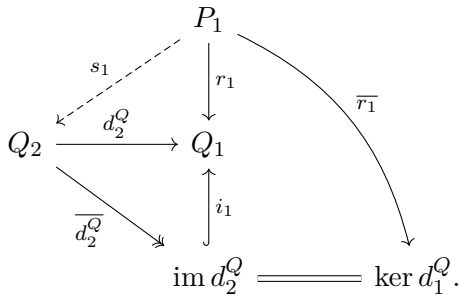
$$f_0 - f_0' = r_0 = d_1^Q s_0,$$

the homotopy identity in degree 0.

Degree 1. Define $r_1 = f_1 - f_1' - s_0 d_1^P : P_1 \to Q_1$. The chain map identities and the degree-0 identity give

$$\begin{aligned} d_1^Q r_1 &= (f_0 - f_0') d_1^P - d_1^Q s_0 d_1^P \\ &= (f_0 - f_0' - d_1^Q s_0) d_1^P = 0. \end{aligned}$$

Hence r_1 takes values in $\ker d_1^Q = \operatorname{im} d_2^Q$. Writing $i_1 : \operatorname{im} d_2^Q \hookrightarrow Q_1$ for the inclusion, projectivity of P_1 gives the next lift:



Composing $\overline{d_2^Q} s_1 = \overline{r_1}$ with i_1 gives $d_2^Q s_1 = r_1$, so

$$f_1 - f_1' = d_2^Q s_1 + s_0 d_1^P.$$

Degree 2. Put $r_2 = f_2 - f_2' - s_1 d_2^P : P_2 \to Q_2$. One extra term remains, but it still vanishes:

$$d_2^Q r_2 = (f_1 - f_1' - d_2^Q s_1) d_2^P = s_0 d_1^P d_2^P = 0.$$

Exactness and projectivity of P_2 again give a lift $s_2 : P_2 \to Q_3$ with $d_3^Q s_2 = r_2$. Continuing in the same way produces all s_n satisfying

$$f_n - f_n' = d_{n+1}^Q s_n + s_{n-1} d_n^P \qquad (n \geq 0).$$

Thus s is a chain homotopy from f to f' on the underlying complexes.

The formal induction is omitted.

Extend s to the augmented complexes by setting $\widetilde{s}_{-1} : M \to Q_0$ equal to zero. In degree 0, the additional term $\widetilde{s}_{-1}\varepsilon_P$ vanishes; in degree -1, the homotopy identity is

$$u - u = \varepsilon_Q \widetilde{s}_{-1} = 0.$$

Thus $\widetilde{s} = (s, 0)$ is a chain homotopy from $\widetilde{u}$ to $\widetilde{u}'$. □

Corollary (Uniqueness of projective resolutions). Any two projective resolutions of the same left R-module are chain homotopy equivalent, both as augmented complexes and after deleting the augmentation terms.

Proof. Label two copies of the same left R-module A by $A^{(1)}$ and $A^{(2)}$, and write

$$\widetilde{P}_\bullet = (P_\bullet \xrightarrow{\varepsilon_P} A^{(1)}), \qquad \widetilde{Q}_\bullet = (Q_\bullet \xrightarrow{\varepsilon_Q} A^{(2)}).$$

The map $\mathrm{id}_A : A^{(1)} \to A^{(2)}$ lifts by the comparison theorem, as does $\mathrm{id}_A : A^{(2)} \to A^{(1)}$ in the reverse direction:

$$\widetilde{f} = (f, \mathrm{id}_A) : \widetilde{P}_\bullet \longrightarrow \widetilde{Q}_\bullet,$$
$$\widetilde{g} = (g, \mathrm{id}_A) : \widetilde{Q}_\bullet \longrightarrow \widetilde{P}_\bullet.$$

Here id_A denotes the identity on the underlying left R-module A. The composites satisfy

$$\widetilde{g}\widetilde{f} = (gf, \mathrm{id}_A) \quad \text{lifts } \mathrm{id}_A : A^{(1)} \to A^{(1)},$$
$$\widetilde{f}\widetilde{g} = (fg, \mathrm{id}_A) \quad \text{lifts } \mathrm{id}_A : A^{(2)} \to A^{(2)}.$$

But the identity chain maps

$$\mathrm{id}_{\widetilde{P}_\bullet} = (\mathrm{id}_{P_\bullet}, \mathrm{id}_A), \qquad \mathrm{id}_{\widetilde{Q}_\bullet} = (\mathrm{id}_{Q_\bullet}, \mathrm{id}_A)$$

also lift these respective left R-linear maps. Uniqueness up to homotopy gives

$$\widetilde{g}\widetilde{f} \simeq \mathrm{id}_{\widetilde{P}_\bullet}, \qquad \widetilde{f}\widetilde{g} \simeq \mathrm{id}_{\widetilde{Q}_\bullet}.$$

Thus $\widetilde{P}_\bullet \simeq \widetilde{Q}_\bullet$. The theorem provides both homotopies with degree--1 component zero, so deleting $A^{(1)}$ and $A^{(2)}$ leaves $gf \simeq \mathrm{id}_{P_\bullet}$ and $fg \simeq \mathrm{id}_{Q_\bullet}$. Hence $P_\bullet \simeq Q_\bullet$ as well. □

The strict equality $gf = \mathrm{id}_{P_\bullet}$ in the preceding example is a special feature of the chosen maps; in general, both composites are only required to be homotopic to the respective identities.

Remark. In particular, f induces isomorphisms $H_n(P_\bullet) \cong H_n(Q_\bullet)$. These recover A in degree 0 and zero in positive degrees. As noted earlier, the useful conclusion for later applications is the chain homotopy equivalence: every additive covariant functor F between categories of left modules over possibly different rings preserves it, even when F is not exact. Thus $F(P_\bullet) \simeq F(Q_\bullet)$ and their homology is isomorphic in each degree. However, a homology isomorphism may not be preserved by F alone.

Remark. For computations, we choose a projective resolution as simple as possible. In practice, we use simple free resolutions.

2.3 Tensor functors and left derived functors

Categories. A *category* $\mathcal{C}$ consists of:

- a collection of objects;
- for each pair (X, Y) of objects, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms $f : X \to Y$;
- an identity morphism $\mathrm{id}_X : X \to X$ for each object X;
- composition $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, sending (g, f) to $g \circ f$.

These satisfy, whenever the compositions are defined,

$$h \circ (g \circ f) = (h \circ g) \circ f, \qquad \mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X \quad (f : X \to Y).$$

Covariant functors. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ assigns:

- to each object X, an object $F(X)$;
- to each morphism $f : X \to Y$, a morphism $F(f) : F(X) \to F(Y)$.

It should satisfy

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f).$$

The categories of interest.

- R-**Mod** has left R-modules as objects and R-linear maps as morphisms.
- **Ab** has abelian groups as objects and group homomorphisms as morphisms.

Identities and composition are the usual ones for maps. Every abelian group has a unique left $\mathbb{Z}$-module structure, its group homomorphisms are exactly the $\mathbb{Z}$-linear maps, and vice versa. Thus **Ab** is canonically isomorphic to $\mathbb{Z}$-**Mod**; we freely identify them. In particular, the preceding definitions of chain complexes, homology, and chain homotopies apply to **Ab**.

Additive functors. In both R-**Mod** and **Ab**, each morphism set is an abelian group under pointwise addition, and composition is bilinear (when treating morphism sets as left $\mathbb{Z}$-modules).

A covariant functor

$$F : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$$

is *additive* if, for all $f, g : M \to N$,

$$F(f+g) = F(f) + F(g).$$

Equivalently, its maps on morphism sets are group homomorphisms.

In particular, $F(0) = 0$ and $F(-f) = -F(f)$. We assume F is additive from now on.

Applying F degree by degree. For a chain complex $C_\bullet$ of left R-modules, write $F(C_\bullet)$ for

$$\cdots \longrightarrow F(C_{n+1}) \xrightarrow{F(d^C_{n+1})} F(C_n) \xrightarrow{F(d^C_n)} F(C_{n-1}) \longrightarrow \cdots .$$

This is a chain complex of abelian groups, since

$$F(d^C_n)F(d^C_{n+1}) = F(d^C_n d^C_{n+1}) = F(0) = 0.$$

Its homology $H_n(F(C_\bullet))$ is an abelian group, equivalently a left $\mathbb{Z}$-module; no additional left R-module structure is assumed. If $f : C_\bullet \to D_\bullet$ is a chain map, then $F(f) = (F(f_n))$ is a chain map $F(C_\bullet) \to F(D_\bullet)$, because

$$F(d^D_n)F(f_n) = F(d^D_n f_n) = F(f_{n-1}d^C_n) = F(f_{n-1})F(d^C_n).$$

Additivity and preservation of composition likewise carry chain homotopies to chain homotopies, and hence preserve chain homotopy equivalences. □

Main motivating example: tensor products. A *right R-module* is an abelian group N with an action $N \times R \to N$, written $(n, r) \mapsto nr$, satisfying

$$(n+n')r = nr + n'r, \quad n(r+r') = nr + nr',$$
$$(nr)r' = n(rr'), \qquad n1_R = n$$

for all $n, n' \in N$ and $r, r' \in R$. A map of right R-modules is *right R-linear* if it is additive and satisfies $f(nr) = f(n)r$. We write **Mod**-R for the catehory of right R-modules and these maps. Fix a right R-module N for the rest of this example.

Defining tensor products by universal property. Let M be a left R-module and A an abelian group. A map $\beta : N \times M \to A$ is *biadditive and R-balanced* if

$$\begin{aligned}\beta(n+n',m) &= \beta(n,m)+\beta(n',m),\\ \beta(n,m+m') &= \beta(n,m)+\beta(n,m'),\\ \beta(nr,m) &= \beta(n,rm)\end{aligned}$$

for all $n, n' \in N$, $m, m' \in M$, and $r \in R$. The *tensor product* of N and M over R is an abelian group $N \otimes_R M$ together with a biadditive, R-balanced map

$$\tau : N \times M \longrightarrow N \otimes_R M, \qquad \tau(n,m) = n \otimes m,$$

such that, for every abelian group A, each biadditive, R-balanced map $\beta : N \times M \to A$ factors uniquely through a group homomorphism $\overline{\beta} : N \otimes_R M \to A$:

$$\begin{array}{ccc} N \times M & \xrightarrow{\ \tau\ } & N \otimes_R M \\ & \searrow^{\beta} & \downarrow{\scriptstyle \overline{\beta}} \\ & & A. \end{array}$$

Equivalently, $\overline{\beta}(n \otimes m) = \beta(n,m)$.

Existence and uniqueness. Let L be the free abelian group on symbols $[n,m]$, indexed by $(n,m) \in N \times M$, and let K be the subgroup generated by

$$\begin{aligned}&[n+n',m]-[n,m]-[n',m],\\ &[n,m+m']-[n,m]-[n,m'],\\ &[nr,m]-[n,rm].\end{aligned}$$

Then L/K, with $\tau(n,m) = [n,m]+K$, has the required property: each β extends uniquely to a homomorphism $L \to A$, which vanishes on K and hence factors uniquely through L/K. The elements $n \otimes m$, called *pure tensors*, generate $N \otimes_R M$ as an abelian group.

If (T,τ) and (T',τ') both have the universal property, it gives unique homomorphisms $a : T \to T'$ and $b : T' \to T$ with $a\tau = \tau'$ and $b\tau' = \tau$. Uniqueness forces $ba = \mathrm{id}_T$ and $ab = \mathrm{id}_{T'}$. Thus the tensor product is unique up to a unique isomorphism compatible with its universal map. □

The tensor functor. For an R-linear map $f : M \to M'$, the map $(n,m) \mapsto n \otimes f(m)$ is biadditive and R-balanced, since

$$(nr) \otimes f(m) = n \otimes rf(m) = n \otimes f(rm).$$

The universal property therefore defines a group homomorphism

$$\mathrm{id}_N \otimes f : N \otimes_R M \longrightarrow N \otimes_R M', \qquad n \otimes m \longmapsto n \otimes f(m).$$

On pure tensors one checks

$$\mathrm{id}_N \otimes \mathrm{id}_M = \mathrm{id}_{N \otimes_R M}, \qquad (\mathrm{id}_N \otimes g)(\mathrm{id}_N \otimes f) = \mathrm{id}_N \otimes (g \circ f)$$

for composable f, g. Since pure tensors generate the tensor product, these are identities of homomorphisms.

Thus

$$N \otimes_R - : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \qquad M \longmapsto N \otimes_R M, \quad f \longmapsto \mathrm{id}_N \otimes f$$

is a covariant functor. It is additive: for $f, h : M \to M'$, biadditivity gives

$$\mathrm{id}_N \otimes (f + h) = (\mathrm{id}_N \otimes f) + (\mathrm{id}_N \otimes h). \qquad \square$$

A concrete calculation. Let $m, n \geq 2$, put $B = \mathbb{Z}/m\mathbb{Z}$ and $d = \gcd(m, n)$, and take $F = B \otimes_{\mathbb{Z}} -$. The universal property gives isomorphisms

$$B \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\cong} B, \qquad x \otimes k \longmapsto kx,$$
$$B \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} B/nB, \quad x \otimes \overline{k} \longmapsto kx + nB.$$

Their inverses send x to $x \otimes 1$ and $x + nB$ to $x \otimes \overline{1}$, respectively; the tensor relations make these maps well defined. Reduction modulo d identifies $B/nB \cong \mathbb{Z}/d\mathbb{Z}$.

Now, apply F to the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Under these identifications, the resulting sequence is

$$0 \longrightarrow B \xrightarrow{\times n} B \xrightarrow{q} B/nB \longrightarrow 0,$$

where q is the quotient map.

First observation: right exactness. The right-hand part remains exact: q is surjective and $\ker q = nB = \mathrm{im}(\times n)$. However, if $d > 1$, multiplication by n kills the nonzero element $\overline{m/d} \in B$, so the left-hand injection is lost.

Definition (Right and left exactness). A covariant functor $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is *right exact* if every exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is sent to an exact sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0.$$

It is *left exact* if it similarly preserves exactness of sequences $0 \to A \to B \to C$.

Proposition. For every right R-module N, the functor $N \otimes_R -$ is right exact.

Proof. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of left R-modules. Every pure tensor $x \otimes c$ lifts to $x \otimes b$ with $\beta(b) = c$, so $\mathrm{id}_N \otimes \beta$ is surjective. Put $J = \mathrm{im}(\mathrm{id}_N \otimes \alpha)$. There is an induced homomorphism

$$(N \otimes_R B)/J \longrightarrow N \otimes_R C.$$

Its inverse is induced by the biadditive, R-balanced map

$$N \times C \longrightarrow (N \otimes_R B)/J, \qquad (x, c) \longmapsto [x \otimes b], \quad \beta(b) = c.$$

This is independent of the choice of b: two choices differ by $\alpha(a)$, whose tensor with x belongs to J. Balancing follows from $xr \otimes b = x \otimes rb$ and $\beta(rb) = rc$. The universal property now gives the inverse, so $\ker(\mathrm{id}_N \otimes \beta) = J$, proving exactness. $\square$

Second observation: homology. Return to $F = (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} -$. Delete $\mathbb{Z}/n\mathbb{Z}$ and its image under F from the two sequences above. The remaining complexes are

$$\begin{aligned} P_\bullet: &\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0, \\ F(P_\bullet): &\quad 0 \longrightarrow B \xrightarrow{\times n} B \longrightarrow 0, \end{aligned}$$

with only nonzero terms in degrees 1 and 0. As before, $H_1(P_\bullet) = 0$ and $H_0(P_\bullet) \cong \mathbb{Z}/n\mathbb{Z}$.

However, for the transformed complex,

$$\begin{aligned} H_1(F(P_\bullet)) &= \ker(\times n : B \to B) = \langle \overline{m/d} \rangle \cong \mathbb{Z}/d\mathbb{Z}, \\ H_0(F(P_\bullet)) &= B/nB \cong F(\mathbb{Z}/n\mathbb{Z}), \\ H_i(F(P_\bullet)) &= 0 \qquad (i \geq 2). \end{aligned}$$

Here H_1 is exactly the kernel that measures the failure to preserve the original injection. Meanwhile, H_0 simply recovers the functor's value on the resolved left R-module. This suggests the following definition.

Left derived functors. Let $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ be an additive, covariant, right exact functor.

Proposition. For any projective resolution $P_\bullet \xrightarrow{\varepsilon} M$, the sequence

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\varepsilon)} F(M) \longrightarrow 0$$

is exact at $F(P_0)$ and $F(M)$, and the augmentation induces an isomorphism $H_0(F(P_\bullet)) \cong F(M)$.

Proof. The sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$ is exact, so its image under F is exact by right exactness. Thus $F(\varepsilon)$ is surjective and $\operatorname{im} F(d_1) = \ker F(\varepsilon)$. In the underlying complex $F(P_\bullet)$, the degree-0 differential is zero. The first isomorphism theorem therefore gives

$$H_0(F(P_\bullet)) = F(P_0)/\operatorname{im} F(d_1) = F(P_0)/\ker F(\varepsilon) \cong F(M),$$

where the isomorphism sends $[x]$ to $F(\varepsilon)(x)$. □

Exactness at $F(P_n)$ for $n \geq 1$ can fail. Homology in these degrees measures that failure.

Definition (Values on left modules). For each left R-module M, choose a projective resolution $P_\bullet \xrightarrow{\varepsilon} M$ and define

$$L_nF(M) := H_n(F(P_\bullet)) = \frac{\ker F(d_n)}{\operatorname{im} F(d_{n+1})} \qquad (n \geq 0).$$

Here $P_\bullet$ is the underlying complex, with $d_0 = 0$; the augmentation term $F(M)$ is omitted. The proposition gives $L_0F(M) \cong F(M)$.

Well-definedness on objects. Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of M. The comparison theorem gives maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ lifting id_M, with $gf \simeq \mathrm{id}_{P_\bullet}$ and $fg \simeq \mathrm{id}_{Q_\bullet}$. Since F is covariant and additive, it preserves chain homotopies; hence

$$F(g)F(f) \simeq \mathrm{id}_{F(P_\bullet)}, \qquad F(f)F(g) \simeq \mathrm{id}_{F(Q_\bullet)}.$$

It follows that

$$H_n(F(f)) : H_n(F(P_\bullet)) \xrightarrow{\cong} H_n(F(Q_\bullet)).$$

Any two comparison maps lifting id_M are chain homotopic, so this isomorphism is independent of the chosen comparison map. These isomorphisms compose transitively: a composite of comparison maps again lifts id_M. Comparing a resolution with itself gives the identity isomorphism. Thus $L_nF(M) = H_n(F(P_\bullet))$ is independent of the choice of projective resolution $P_\bullet$, up to canonical isomorphism. □

Action on morphisms and functoriality. Given an R-linear map $u : M \to M'$, choose a chain map $f : P_\bullet \to Q_\bullet$ lifting u, where $Q_\bullet$ resolves M'. Define

$$\begin{aligned} L_nF(u) = H_n(F(f)) : \quad & L_nF(M) \longrightarrow L_nF(M'), \\ & [x] \longmapsto [F(f_n)(x)]. \end{aligned}$$

This is independent of the lift: two lifts are chain homotopic, their images under F are chain homotopic, and hence they induce the same homomorphism on homology.

We must also compare different resolutions of both left R-modules. Let $P'_\bullet \to M$ and $Q'_\bullet \to M'$ be projective resolutions, and let $f' : P'_\bullet \to Q'_\bullet$ be a lift of u. Choose comparison maps

$$a : P_\bullet \to P'_\bullet, \qquad b : Q_\bullet \to Q'_\bullet$$

lifting id_M and $\mathrm{id}_{M'}$, respectively. Both bf and $f'a$ are maps $P_\bullet \to Q'_\bullet$ lifting u. Uniqueness in the comparison theorem gives $bf \simeq f'a$. Since F preserves chain homotopies, taking homology gives a commutative square:

$$\begin{array}{ccc} H_n(F(P_\bullet)) & \xrightarrow{H_n(F(f))} & H_n(F(Q_\bullet)) \\ {\scriptstyle H_n(F(a))}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle H_n(F(b))} \\ H_n(F(P'_\bullet)) & \xrightarrow[H_n(F(f'))]{} & H_n(F(Q'_\bullet)). \end{array}$$

The vertical maps are the canonical isomorphisms from the preceding paragraph. Thus the maps obtained using the two pairs of resolutions agree under these identifications. This proves independence of both the lift and the resolutions. □

The identity chain map lifts id_M. If $g : Q_\bullet \to S_\bullet$ lifts $v : M' \to M''$, then $g \circ f$ lifts $v \circ u$. We may therefore use these lifts to compute

$$\begin{aligned} L_nF(\mathrm{id}_M) &= \mathrm{id}_{L_nF(M)}, \\ L_nF(v \circ u) &= H_n(F(g) \circ F(f)) \\ &= L_nF(v) \circ L_nF(u). \end{aligned}$$

For parallel maps u, u', the sum of their lifts lifts $u + u'$; additivity of F and of the induced homology maps gives $L_nF(u + u') = L_nF(u) + L_nF(u')$. Thus, for every $n \geq 0$, we obtain an additive covariant functor

$$L_nF : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab},$$

called the *nth left derived functor* of F.

The commuting square above says that different choices of resolutions give functors related by a canonical natural isomorphism. In degree 0, compatibility with the augmentations identifies $L_0F(u)$ with $F(u)$, so $L_0F \cong F$. □

Third observation: tensoring with $\mathbb{Z}$. Return to the standard resolution of $\mathbb{Z}/n\mathbb{Z}$, but now take $N = \mathbb{Z}$ as a right $\mathbb{Z}$-module and put $F = \mathbb{Z} \otimes_{\mathbb{Z}} -$. For every left $\mathbb{Z}$-module M, the tensor relations give an isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}} M \xrightarrow{\cong} M, \qquad k \otimes x \longmapsto kx,$$

with inverse $x \mapsto 1 \otimes x$. These identifications commute with homomorphisms. Thus tensoring with $\mathbb{Z}$ preserves exactness, and the standard resolution becomes the same exact sequence under these identifications:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

After deleting the augmentation term, we therefore obtain

$$L_0F(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}, \qquad L_iF(\mathbb{Z}/n\mathbb{Z}) = 0 \quad (i \geq 1).$$

Here both left and right exactness are preserved. □

Definition (Exact functor). A covariant functor $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is *exact* if it sends every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0.$$

Equivalently, F is both left exact and right exact. In particular, $\mathbb{Z} \otimes_{\mathbb{Z}} -$ is exact.

Proposition. An exact functor preserves exactness of sequences of arbitrary length.

Proof. Consider any three consecutive terms $A \xrightarrow{a} B \xrightarrow{b} C$ with $\operatorname{im} a = \ker b$. Put $K = \ker b$ and $I = \operatorname{im} b$. Factor $a = \iota q$ and $b = jp$, where $q : A \twoheadrightarrow K$, $j : I \hookrightarrow C$, and

$$0 \longrightarrow K \xrightarrow{\iota} B \xrightarrow{p} I \longrightarrow 0$$

is short exact. Applying F preserves this short exact sequence. The short exact sequences associated with $\ker q \subseteq A$ and $I \subseteq C$ also show that $F(q)$ is surjective and $F(j)$ is injective. Consequently,

$$\operatorname{im} F(a) = \operatorname{im} F(\iota) = \ker F(p) = \ker F(b).$$

This proves exactness at each position, regardless of the sequence's length. □

Proposition (Vanishing conditions). Let $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ be additive, covariant, and right exact. Then $L_iF(M) = 0$ for every $i \geq 1$ if either

1. F is exact; or
2. M is projective.

In both cases, $L_0F(M) \cong F(M)$.

Proof. If F is exact, it preserves the exactness of any augmented projective resolution of M. Hence its underlying complex $F(P_\bullet)$ is exact in every positive degree, giving $L_iF(M) = 0$, $i \geq 1$.

If M is projective, choose the length-zero projective resolution

$$0 \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0,$$

with $P_0 = M$, augmentation id_M, and $P_i = 0$ for $i \geq 1$. After deleting the augmentation term and applying F, the complex has only $F(M)$ in degree 0. Its positive-degree homology is therefore zero, and its degree-0 homology is $F(M)$. □

2.4 Tor functors

Definition. Let N be a right R-module. For $i \geq 0$, define the *tor functor* as

$$\mathrm{Tor}_i^R(N, -) := L_i(N \otimes_R -),$$

and so for every left R-module A,

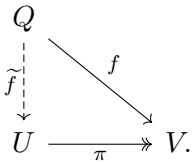
$$\mathrm{Tor}_i^R(N, A) = L_i(N \otimes_R -)(A) = H_i(N \otimes_R P_\bullet),$$

where $P_\bullet \xrightarrow{\varepsilon_P} A$ is a projective resolution of left R-modules. The augmentation term $N \otimes_R A$ is omitted before taking homology. By Section 2.3, these abelian groups are independent of the resolution up to canonical isomorphism, and

$$\mathrm{Tor}_0^R(N, A) \cong N \otimes_R A.$$

Tor is covariant in both variables. Functoriality in A is already part of the construction of left derived functors. A right R-linear map $v : N \to N'$ induces the chain map $v \otimes \mathrm{id}_{P_\bullet}$ and hence a map $\mathrm{Tor}_i^R(N, A) \to \mathrm{Tor}_i^R(N', A)$. One can check the well-definedness.

Definition (Projective right modules). A right R-module Q is *projective* if, for every epimorphism $\pi : U \to V$ between right moduels and every right R-linear map $f : Q \to V$, there is a right R-linear map $\widetilde{f} : Q \to U$ with $\pi\widetilde{f} = f$:



The lift need not be unique.

Projective resolutions of right modules. A *projective resolution of right R-modules* is an exact sequence of right R-modules and right R-linear maps

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \xrightarrow{\varepsilon_Q} N \longrightarrow 0,$$

with every Q_i projective. As before, $Q_\bullet$ denotes the complex with N deleted. Free right R-modules have the form $\bigoplus_{s \in S} e_s R$. The same basis-lifting proof makes them projective right R-modules, and the same successive free-cover construction gives a projective resolution of every right R-module.

Theorem (Balanced computation of Tor). Let N be a right R-module and A a left R-module. Let $P_\bullet \xrightarrow{\varepsilon_P} A$ be a projective resolution of left R-modules and $Q_\bullet \xrightarrow{\varepsilon_Q} N$ a projective resolution of right R-modules. For every $i \geq 0$, there is a canonical isomorphism

$$\operatorname{Tor}_i^R(N, A) = H_i(N \otimes_R P_\bullet) \cong H_i(Q_\bullet \otimes_R A),$$

compatible with maps in both N and A. Thus, either variable may be resolved.

Proof. See [Weibel, Theorem 2.7.2]. □

Fact (Tensoring with projective left or right modules). If N is a projective right R-module, then $N \otimes_R -$ is exact on left R-modules.

Proof. Projective right R-modules are flat; see [Weibel, §3.2, p. 68]. □

Remark. If $N \otimes_R -$ is exact on left R-modules, then N need no be a projective right R-module. It means that projectivity implies flatness, but the converse is not always true.

Theorem (Vanishing of Tor) $\operatorname{Tor}_i^R(N, A) = 0$ for every $i \geq 1$ if either

1. N is projective; or
2. A is projective.

In both cases, $\operatorname{Tor}_0^R(N, A) \cong N \otimes_R A$.

Proof. If N is projective, then $N \otimes_R -$ is exact, which gives the result.

If A is projective, choose the length-zero projective resolution will give the result. □

2.5 Group homology

The group ring. Fix a group G. Its *integral group ring* is

$$R = \mathbb{Z}G = \left\{ \sum_{g \in G} n_g g : n_g \in \mathbb{Z}, \quad n_g = 0 \text{ for all but finitely many } g \right\}.$$

In this section, Section 3.4, and Chapter 4, R always refers to $\mathbb{Z}G$; other sections are not.

Addition is coefficientwise, and multiplication is defined by

$$\left(\sum_g n_g g \right) \left(\sum_h m_h h \right) = \sum_{g,h} n_g m_h (gh).$$

This is an associative unital ring, with identity $1 \cdot 1_G$. Giving a left R-module A is equivalent to giving an abelian group with a G-action by additive automorphisms: restrict the left R-action to G, or extend the group action by

$$\left(\sum_g n_g g \right) a = \sum_g n_g (ga).$$

An additive map between such left R-modules is R-linear exactly when it commutes with the G-action. We study G through its actions on A: G is the object of study, and A is the coefficient left R-module.

Trivial left and right R-modules and the augmentation ideal. Give $\mathbb{Z}$ the trivial left and right actions $g \cdot k = k \cdot g = k$ for all $g \in G$ and $k \in \mathbb{Z}$. Define the *augmentation*

$$\varepsilon : R \longrightarrow \mathbb{Z}, \qquad \varepsilon\left(\sum_g n_g g\right) = \sum_g n_g = \left(\sum_g n_g g\right) \cdot 1 = 1 \cdot \left(\sum_g n_g g\right).$$

It is a surjective unital ring homomorphism: $\varepsilon(rs) = \varepsilon(r)\varepsilon(s)$ and $\varepsilon(1_G) = 1$.
Its kernel is the two-sided *augmentation ideal*

$$\begin{aligned} I := \ker \varepsilon &= \left\{ \sum_g n_g g \in R : \sum_g n_g = 0 \right\} \\ &= \langle g - 1 : g \in G \rangle_{\mathbb{Z}}. \end{aligned}$$

Indeed, every $g - 1$ lies in I, and an element with coefficient sum zero satisfies $\sum_g n_g g = \sum_g n_g (g - 1)$. The induced ring isomorphism is

$$R/I \xrightarrow{\cong} \mathbb{Z}, \qquad r + I \longmapsto \varepsilon(r),$$

with inverse $k \mapsto k1_G + I$, since $r - \varepsilon(r)1_G \in I$. The trivial left and right R-actions on $\mathbb{Z}$ are

$$r \cdot k = \varepsilon(r)k, \qquad k \cdot r = k\varepsilon(r).$$

Thus, the identification $R/I \cong \mathbb{Z}$ is also an isomorphism of left and right R-modules. □

Coinvariants. To force the G-action on A to be trivial, impose the relations $ga = a$. The resulting *coinvariants group* is

$$A_G := A/IA, \qquad IA = \langle ga - a : g \in G,\ a \in A \rangle_{\mathbb{Z}}.$$

Here IA consists of finite sums of elements ia, with $i \in I$ and $a \in A$; it is an R-submodule since I is a two-sided ideal. Write $\overline{a} = a + IA$. $\overline{ga} = \overline{a}$, so the induced action on A_G is trivial. It is the largest quotient module of A with trivial G-action. We regard A_G as an abelian group only later.

Proposition (Coinvariants group as a tensor product). There are isomorphisms

$$\begin{aligned} \phi_A : A_G &\longrightarrow \mathbb{Z} \otimes_R A, \qquad \overline{a} \longmapsto 1 \otimes a, \\ \psi_A : \mathbb{Z} \otimes_R A &\longrightarrow A_G, \qquad k \otimes a \longmapsto k\overline{a}, \end{aligned}$$

inverse to each other, where $\mathbb{Z}$ is the trivial *right* R-module.

Proof. The map ϕ_A is well defined because the additive map $a \mapsto 1 \otimes a$ kills each generator of IA:

$$1 \otimes (ga - a) = (1 \cdot g) \otimes a - 1 \otimes a = 0.$$

For ψ_A, the map $(k, a) \mapsto k\overline{a}$ is biadditive and R-balanced. Indeed, $\overline{ra} = \varepsilon(r)\overline{a}$, so

$$(k \cdot r)\overline{a} = k\varepsilon(r)\overline{a} = k\overline{ra}.$$

The tensor universal property therefore gives a well-defined ψ_A. Finally,

$$\psi_A\phi_A(\overline{a}) = \overline{a}, \qquad \phi_A\psi_A(k \otimes a) = 1 \otimes ka = k \otimes a.$$

Since pure tensors generate, the maps are inverse. □

The balancing relation transfers the action on A to $\mathbb{Z}$, making the action trivial.

The coinvariant functor. For an R-linear map $f : A \to B$, we have $f(IA) \subseteq IB$, so

$$f_G : A_G \longrightarrow B_G, \qquad f_G(\overline{a}) = \overline{f(a)}$$

is well defined. Induced quotient maps preserve identities, composition, and addition. Thus

$$(-)_G : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \qquad A \longmapsto A_G, \quad f \longmapsto f_G$$

is additive and covariant. □

A canonical natural isomorphism. Isomorphisms ϕ_A form a *natural isomorphism of functors*

$$\phi : (-)_G \xrightarrow{\cong} \mathbb{Z} \otimes_R -.$$

This means that each component ϕ_A is an isomorphism and that, for every R-linear map $f : A \to B$, the following square commutes:

$$\begin{array}{ccc} A_G & \xrightarrow[\cong]{\phi_A} & \mathbb{Z} \otimes_R A \\ {\scriptstyle f_G}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}} \otimes f} \\ B_G & \xrightarrow[\phi_B]{\cong} & \mathbb{Z} \otimes_R B. \end{array}$$

Both paths send $\overline{a}$ to $1 \otimes f(a)$. The isomorphism is *canonical* because its formula $\phi_A(\overline{a}) = 1 \otimes a$ requires no choice of basis or resolution. After identifying the values on objects by these maps, the actions of the two functors on morphisms agree as well.

This identification transfers right exactness from the tensor to the coinvariant. If $A \to B \to C \to 0$ is exact, tensoring gives an exact sequence $\mathbb{Z} \otimes_R A \to \mathbb{Z} \otimes_R B \to \mathbb{Z} \otimes_R C \to 0$. The naturality squares identify its terms and arrows with those of $A_G \to B_G \to C_G \to 0$, which is therefore exact. Thus $(-)_G$ is right exact. □

Definition (Group homology). For $n \geq 0$, the *nth homology of G with coefficients in A* is

$$H_n(G; A) := L_n((-)_G)(A).$$

Let $P_\bullet \to A$ be a projective resolution of left R-modules, and omit the augmentation term. Naturality applied to each differential $d_n : P_n \to P_{n-1}$ gives

$$\phi_{P_{n-1}} \circ (d_n)_G = (\mathrm{id}_{\mathbb{Z}} \otimes d_n) \circ \phi_{P_n}.$$

Hence the component maps form an *isomorphism of chain complexes*

$$\phi_{P_\bullet} : (P_\bullet)_G \xrightarrow{\cong} \mathbb{Z} \otimes_R P_\bullet.$$

Taking homology gives

$$\begin{aligned} H_n(G; A) &= H_n((P_\bullet)_G) \\ &\cong H_n(\mathbb{Z} \otimes_R P_\bullet) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A). \end{aligned}$$

Here $\mathbb{Z}$ in the tensor product is the trivial right R-module. In particular,

$$H_0(G; A) \cong A_G.$$

Computing with a resolution of $\mathbb{Z}$. By the balanced computation theorem, any projective resolution $Q_\bullet \to \mathbb{Z}$ of the trivial right R-module gives

$$H_n(G; A) \cong \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A) \cong H_n(Q_\bullet \otimes_R A).$$

Thus we can fix one resolution for G and use it for every coefficient module A. In Chapter 4 we will construct the *bar resolution*, an explicit free resolution of this trivial right R-module $\mathbb{Z}$.

Warning. Freeness of $\mathbb{Z}$ as a $\mathbb{Z}$-module may not imply the freeness and projectivity of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

If $G \neq \{1\}$, the abelian group $\mathbb{Z}$ with trivial G-actions is not free as either a left or a right $\mathbb{Z}G$-module; for $G = \{1\}$, we have $\mathbb{Z} \cong R$ is free of rank one as both a left and a right R-module.

Proof. Choose $g \in G$ with $g \neq 1$. Since distinct group elements are distinct basis elements of the free abelian group $\mathbb{Z}G$, we have $g - 1 \neq 0$ in R. For either trivial R-action, however, g acts as the identity, so

$$(g-1)k = 0, \qquad k(g-1) = 0 \qquad \text{for every } k \in \mathbb{Z}.$$

Suppose that $\mathbb{Z}$ were a free left R-module with basis $(e_s)_{s \in S}$. Since $\mathbb{Z} \neq 0$, the set S is nonempty. Choose $s \in S$. In a free left R-module, an element is zero exactly when all its basis coefficients are zero. The element $(g-1)e_s$ has the nonzero coefficient $g - 1$ at e_s, so $(g-1)e_s \neq 0$. This contradicts the fact that $g - 1$ annihilates every element of the trivial left R-module.

Similar for the right R-module case. Thus, neither the trivial left nor right R-module is free. □

Also, although $\mathbb{Z}$ is free as an abelian group, it need not be projective as a left R-module.

To sum up, a free resolution over $\mathbb{Z}$ cannot simply be used as a projective resolution over $\mathbb{Z}G$. This is also the reason why we need a bar resolution.

Proposition (Vanishing in positive degrees). For $n > 0$, the following hold.

1. If $G = \{1\}$, then $H_n(G; A) = 0$ for every abelian group A.
2. If A is projective over $\mathbb{Z}G$ (in particular, if it is free over $\mathbb{Z}G$), then $H_n(G; A) = 0$.

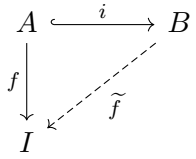
Proof. In the first case, the trivial right R-module $\mathbb{Z}$ is free of rank one, hence projective. Tensoring with a projective right R-module is exact. Thus $\mathbb{Z} \otimes_R -$ is exact, and the canonical natural isomorphism $(-)_G \cong \mathbb{Z} \otimes_R -$ shows that $(-)_G$ is exact as well, giving the result.

In the second case, use the length-zero projective resolution will give the result. □

3 From Injective Resolutions to Group Cohomology

3.1 Injective modules and resolutions

Definition by the extension property. A left R-module I is *injective* if, for every monomorphism $i : A \to B$ of left R-modules and every R-linear map $f : A \to I$, there exists an R-linear map $\widetilde{f} : B \to I$ such that $\widetilde{f}i = f$, that is, the following diagram commutes:



The extension need not be unique. This property is dual to projectivity.

Cochain complexes and cohomology. A *cochain complex* of left R-modules is a sequence of left R-modules and R-linear maps as

$$C^\bullet = (\ \cdots \longrightarrow C^{n-1} \xrightarrow{d_C^{n-1}} C^n \xrightarrow{d_C^n} C^{n+1} \longrightarrow \cdots), \qquad d_C^{n+1} d_C^n = 0.$$

Its *nth cohomology* is the left R-module defined as

$$H^n(C^\bullet) = \ker d_C^n / \operatorname{im} d_C^{n-1}.$$

Elements of $\ker d_C^n$ are *n-cocycles* or *coclosed n-cochains*; elements of $\operatorname{im} d_C^{n-1}$ are *n-coboundaries* or *coexact n-cochains*. Thus $H^n(C^\bullet) = 0$ precisely when the complex is exact at C^n. For a cochain complex of abelian groups, its cohomology is an abelian group.

Definition and notation. An *injective resolution* of a left R-module M, written $M \xrightarrow{\iota} I^\bullet$, is an exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} I^2 \longrightarrow \cdots$$

of left R-modules in which every I^n is injective. The map ι is the *coaugmentation*. The notation $I^\bullet$ alone denotes the underlying cochain complex

$$0 \longrightarrow I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} I^2 \longrightarrow \cdots,$$

with $I^n = 0$ for $n < 0$. It excludes M. The coaugmentation identifies M with $H^0(I^\bullet)$, and $H^n(I^\bullet) = 0$ for $n > 0$. In the full coaugmented complex, M occupies degree -1 and $d_I^{-1} = \iota$.

Existence. We use the following fact; see [Weibel, Corollary 2.3.11 and Exercise 2.3.5]. For left modules, apply the cited right-module statement to the opposite ring, which has the multiplication of R reversed.

Fact (Enough injectives). Every left R-module embeds into an injective left R-module.

Embed M into injective I^0, and $I^0/\iota(M)$ into injective I^1. Its composite with the quotient map defines $d_I^0 : I^0 \to I^1$, with $\ker d_I^0 = \iota(M)$. Repeating it gives an injective resolution. □

Remark. The enough-injectives theorem takes more work than the existence of free left R-modules, and explicit injective resolutions can be difficult to construct. For Ext and group cohomology, the equivalent computation theorem in Section 3.3 will let us use a projective, usually free, resolution. However, this does not replace injective resolutions for an arbitrary right derived functor. Some categories may not have enough injectives; more work is needed.

Cochain maps. A *cochain map* $f : C^\bullet \to D^\bullet$ is a family of R-linear maps $f^n : C^n \to D^n$ with

$$d_D^n f^n = f^{n+1} d_C^n \qquad \text{for every } n.$$

Equivalently, the squares below commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow[d_D^{n-1}]{} & D^n & \xrightarrow[d_D^n]{} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

Proposition. A cochain map induces an R-linear map

$$H^n(f) : H^n(C^\bullet) \longrightarrow H^n(D^\bullet), \qquad [x] \longmapsto [f^n(x)].$$

Moreover, $H^n(gf) = H^n(g)H^n(f)$ and $H^n(\mathrm{id}_{C^\bullet}) = \mathrm{id}_{H^n(C^\bullet)}$.

Proof. The cochain map identities send cocycles to cocycles and coboundaries to coboundaries. Hence the formula is well-defined; composition and identity maps are also preserved. □

Cochain homotopies. Let $f, g : C^\bullet \to D^\bullet$ be cochain maps. A *cochain homotopy* from f to g is a family of R-linear maps $s^n : C^n \to D^{n-1}$ such that

$$f^n - g^n = d_D^{n-1} s^n + s^{n+1} d_C^n \qquad \text{for every } n.$$

We write $s : f \simeq g$. For complexes concentrated in nonnegative degrees, $s^0 : C^0 \to D^{-1} = 0$ is zero. In the diagram, the middle vertical arrow is the sum of the two composites using a dashed arrow:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & C^{n+1} & \longrightarrow & \cdots \\
 & & & \swarrow{\scriptstyle s^n} & \downarrow{\scriptstyle f^n - g^n} & \swarrow{\scriptstyle s^{n+1}} & & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow[d_D^{n-1}]{} & D^n & \xrightarrow[d_D^n]{} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

Proposition. Cochain homotopic maps induce the same maps on cohomology.

Proof. For $x \in \ker d_C^n$, $f^n(x) - g^n(x) = d_D^{n-1} s^n(x) \in \operatorname{im} d_D^{n-1}$. Thus, $H^n(f)[x] = H^n(g)[x]$. □

Cochain homotopy equivalences. A cochain map $f : C^\bullet \to D^\bullet$ is a *cochain homotopy equivalence* if there is a cochain map $g : D^\bullet \to C^\bullet$ such that

$$gf \simeq \mathrm{id}_{C^\bullet}, \qquad fg \simeq \mathrm{id}_{D^\bullet}.$$

In this case, we write $f : C^\bullet \simeq D^\bullet$, and the map g is a *homotopy inverse* of f.

Proposition. A cochain homotopy equivalence induces isomorphisms on cohomology.

Proof. If g is a homotopy inverse of f, then

$$H^n(g)H^n(f) = H^n(gf) = H^n(\mathrm{id}_{C^\bullet}) = \mathrm{id}_{H^n(C^\bullet)},$$
$$H^n(f)H^n(g) = H^n(fg) = H^n(\mathrm{id}_{D^\bullet}) = \mathrm{id}_{H^n(D^\bullet)}.$$

The middle equalities use the two homotopies. Hence $H^n(g)$ is the inverse of $H^n(f)$. □

Theorem (Comparison of injective resolutions). Let $M \xrightarrow{\iota_I} I^\bullet$ and $N \xrightarrow{\iota_J} J^\bullet$ be injective resolutions of left R-modules. For every R-linear map $u : M \to N$, the following hold.

1. There is a cochain map $f : I^\bullet \to J^\bullet$ *extending* u, that is, $f^0\iota_I = \iota_J u$, and it makes

   $$\widetilde{u} = (u, f) : (M \xrightarrow{\iota_I} I^\bullet) \longrightarrow (N \xrightarrow{\iota_J} J^\bullet),$$

   with component u in degree -1 and f^n in degree $n \geq 0$ a cochain map.

   We call it an *extension* of u.

2. Any two such extensions (u, f) and (u, f') are cochain homotopic. The homotopy can be chosen with $\widetilde{s}^0 : I^0 \to N$ equal to zero; its remaining components $s^n : I^n \to J^{n-1}$, $n \geq 1$, give a homotopy $f \simeq f'$ after the coaugmentation terms are deleted.

Proof. Leave it as an exercise. It takes slightly more work than in the projective case. □

Corollary (Uniqueness of injective resolutions). Any two injective resolutions of a left R-module M are cochain homotopy equivalent, both as coaugmented and underlying complexes.

Proof. It simply follows from the comparison theorem, just like that in the projective case. □

Remark. As for projective resolutions, the useful conclusion is homotopy equivalence. An additive covariant functor preserves it, even when it does not preserve exactness.

3.2 Right derived functors

Let functor $F : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ be additive, covariant, and left exact. Thus it sends every short exact sequence $0 \to A \to B \to C \to 0$ to a sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

that is exact at $F(A)$ and $F(B)$; the last map need not be surjective.

Applying F to cochain complexes. Applying F degree by degree sends a cochain complex $C^\bullet$ to a cochain complex $F(C^\bullet)$ of abelian groups since:

$$F(d^{n+1})F(d^n) = F(d^{n+1}d^n) = F(0) = 0.$$

It sends cochain maps to cochain maps and cochain homotopies to cochain homotopies. Indeed, applying F to the homotopy identity gives

$$F(f^n) - F(g^n) = F(d_D^{n-1})F(s^n) + F(s^{n+1})F(d_C^n).$$

Consequently, F preserves cochain homotopy equivalences. The cohomology $H^n(F(C^\bullet))$ is an abelian group; we do not assume an additional left R-module structure. □

Proposition (Degree zero). For an injective resolution $M \xrightarrow{\iota} I^\bullet$, the sequence

$$0 \longrightarrow F(M) \xrightarrow{F(\iota)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

is exact at $F(M)$ and $F(I^0)$. In particular, $F(\iota)$ induces an isomorphism $F(M) \cong H^0(F(I^\bullet))$.

Proof. The sequence $0 \to M \xrightarrow{\iota} I^0 \xrightarrow{d^0} I^1$ is exact. Left exactness gives that $F(\iota)$ is injective and

$$\operatorname{im} F(\iota) = \ker F(d^0) = H^0(F(I^\bullet)),$$

where the last equality uses the underlying complex. The isomorphism sends x to $F(\iota)(x)$. □

Exactness at $F(I^n)$ for $n \geq 1$ may fail. Its failure is measured by cohomology in these degrees.

Definition (Values on left modules). For each left R-module M, choose an injective resolution $M \xrightarrow{\iota} I^\bullet$ and define

$$R^n F(M) := H^n(F(I^\bullet)) = \frac{\ker F(d^n)}{\operatorname{im} F(d^{n-1})} \qquad (n \geq 0).$$

Here $I^\bullet$ is the underlying complex, with $d^{-1} = 0$; the coaugmentation term $F(M)$ is omitted before taking cohomology. The preceding proposition gives $R^0F(M) \cong F(M)$.

Well-definedness on objects. Two injective resolutions $I^\bullet, J^\bullet$ of M admit comparison maps $f : I^\bullet \to J^\bullet$ and $g : J^\bullet \to I^\bullet$ extending id_M, with $gf \simeq \mathrm{id}_{I^\bullet}$ and $fg \simeq \mathrm{id}_{J^\bullet}$. Since F preserves these homotopies,

$$H^n(F(f)) : H^n(F(I^\bullet)) \xrightarrow{\cong} H^n(F(J^\bullet))$$

has inverse $H^n(F(g))$. Any two comparison maps extending id_M are homotopic, so the isomorphism is independent of that choice. Comparison maps compose to comparison maps, so these identifications also agree when a third resolution is introduced. Comparing a resolution with itself gives the identity isomorphism. Thus $R^nF(M)$ is independent of the injective resolution, up to canonical isomorphism. □

Action on morphisms and functoriality. Its definition and well-definedness are left as an exercise. One can prove it like in the projective case.

Thus, for every $n \geq 0$, we obtain an additive covariant functor

$$R^n F : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab},$$

called the *nth right derived functor* of F.

In degree zero, the identifications $R^0F \cong F$ is natural. □

Proposition (Two vanishing conditions). For every $n > 0$, we have $R^nF(M) = 0$ if either

1. F is exact; or
2. M is injective.

In both cases, $R^0F(M) \cong F(M)$.

Proof. An exact functor preserves exact sequences of arbitrary length, as proved in Section 2.3. Thus, in the first case, $F(I^\bullet)$ remains exact in positive degrees. In the second case, choose the length-zero injective resolution

$$0 \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0,$$

where the second copy is I^0 and $I^n = 0$ for $n > 0$. After omitting the first copy and applying F, only $F(M)$ in degree zero remains. □

3.3 Hom and Ext functors

The Hom functors. For left R-modules A and B, let

$$\mathrm{Hom}_R(A,B) = \{\phi : A \to B : \phi \text{ is } R\text{-linear}\}.$$

Pointwise addition makes this an abelian group. For a general noncommutative ring R, it need not carry the pointwise left R-module structure. Fixing A gives an additive covariant functor

$$\mathrm{Hom}_R(A,-) : R\textbf{-Mod} \longrightarrow \textbf{Ab}, \qquad v : B \to B' \longmapsto (v_* : \phi \mapsto v\phi).$$

Fixing B instead gives an additive *contravariant* functor: an arrow $u : A \to A'$ induces an arrow in the opposite direction,

$$u^* : \mathrm{Hom}_R(A',B) \longrightarrow \mathrm{Hom}_R(A,B), \qquad \phi \longmapsto \phi u.$$

It preserves identities and reverses the order of composition.

Fact (Exactness of Hom). For every short exact sequence $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ of left R-modules, the sequences

$$0 \longrightarrow \mathrm{Hom}_R(A,X) \xrightarrow{i_*} \mathrm{Hom}_R(A,Y) \xrightarrow{p_*} \mathrm{Hom}_R(A,Z),$$
$$0 \longrightarrow \mathrm{Hom}_R(Z,B) \xrightarrow{p^*} \mathrm{Hom}_R(Y,B) \xrightarrow{i^*} \mathrm{Hom}_R(X,B)$$

are exact. This is the meaning of left exactness for the two Hom functors, with arrows reversed in the contravariant case. If A is projective, the first sequence is also surjective at the last term. If B is injective, the second sequence is also surjective there. Thus $\mathrm{Hom}_R(A,-)$ is exact when A is projective, and $\mathrm{Hom}_R(-,B)$ is exact when B is injective.

Proof. See [Weibel, Proposition 1.6.8, Corollary 1.6.9, and Lemmas 2.2.3 and 2.3.4]. □

Definition (Ext). For $n \geq 0$, define

$$\mathrm{Ext}^n_R(A,B) := R^n\mathrm{Hom}_R(A,-)(B) = H^n\big(\mathrm{Hom}_R(A,I^\bullet)\big),$$

where $B \xrightarrow{\iota_I} I^\bullet$ is an injective resolution. The coaugmentation term $\mathrm{Hom}_R(A,B)$ is omitted before taking cohomology. Section 3.2 gives independence of this choice up to canonical isomorphism and

$$\mathrm{Ext}^0_R(A,B) \cong \mathrm{Hom}_R(A,B).$$

Proposition (Vanishing). $\mathrm{Ext}^n_R(A,B) = 0$ for all $n > 0$ if A is projective, or B is injective.

Proof. If A is projective, the functor $\mathrm{Hom}_R(A,-)$ is exact. If B is injective, use its length-zero injective resolution. In either case the vanishing conditions of Section 3.2 apply. □

Theorem (Equivalent computation of Ext). Let A and B be left R-modules, and let $P_\bullet \xrightarrow{\varepsilon_P} A$ be a projective resolution of left R-modules. Then, for every $n \geq 0$, there is a canonical isomorphism

$$\mathrm{Ext}^n_R(A,B) \cong H^n\big(\mathrm{Hom}_R(P_\bullet,B)\big),$$

compatible with maps in both variables.

Here the cochain complex on the right is

$$0 \longrightarrow \mathrm{Hom}_R(P_0, B) \xrightarrow{\delta^0} \mathrm{Hom}_R(P_1, B) \xrightarrow{\delta^1} \mathrm{Hom}_R(P_2, B) \longrightarrow \cdots,$$
$$\delta^n(\phi) = \phi d^P_{n+1}.$$

The term $\mathrm{Hom}_R(A, B)$ is omitted. Thus one may resolve the second variable injectively or the first variable projectively; both resolutions consist of left R-modules.

Proof. See [Weibel, Theorem 2.7.6]. □

In particular, computing Ext requires no explicit injective resolution: we may choose a convenient free resolution of its first argument.

3.4 Group cohomology

Fix a group G and put $R = \mathbb{Z}G$. As in Section 2.5, coefficients are left R-modules, or equivalently abelian groups with a G-action. Write $\varepsilon : R \to \mathbb{Z}$ for the augmentation and $I = \ker \varepsilon$ for its ideal. Here $\mathbb{Z}$ carries the trivial left R-module structure, so $R/I \cong \mathbb{Z}$ as left R-modules.

Invariants. The *invariants group* of a left R-module A are

$$A^G := \{a \in A : ga = a \text{ for every } g \in G\} = \{a \in A : Ia = 0\} =: A\,[I]\,.$$

The equality follows because I is generated by the elements $g - 1$. This is the largest submodule of A on which G acts trivially. We regard it as an abelian group only later.

The invariants group retains the elements already fixed by the action, whereas the coinvariant group imposes relations making the action trivial.

Proposition (Invariant as Hom). There are inverse isomorphisms of abelian groups

$$\theta_A : A^G \longrightarrow \mathrm{Hom}_R(\mathbb{Z}, A), \quad a \longmapsto (k \mapsto ka),$$
$$\eta_A : \mathrm{Hom}_R(\mathbb{Z}, A) \longrightarrow A^G, \qquad h \longmapsto h(1).$$

Proof. If $a \in A^G$, then $ra = \varepsilon(r)a$ for every $r \in R$. Hence $k \mapsto ka$ is R-linear, since

$$\theta_A(a)(r \cdot k) = \varepsilon(r)ka = r(ka) = r((\theta_A(a))(k)).$$

Conversely, an R-linear $h : \mathbb{Z} \to A$ satisfies $gh(1) = h(g \cdot 1) = h(1)$, so $h(1) \in A^G$. Finally, $h(k) = kh(1)$ proves that the maps are inverse. They are additive. □

The invariant functor. For an R-linear map $u : A \to B$, restriction gives

$$u^G : A^G \longrightarrow B^G, \qquad a \longmapsto u(a),$$

because $gu(a) = u(ga) = u(a)$ for $a \in A^G$. Restrictions preserve identities, composition, and addition. Thus

$$(-)^G : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \qquad A \longmapsto A^G, \quad u \longmapsto u^G$$

is additive and covariant. □

A canonical natural isomorphism. Isomorphisms θ_A form a *natural isomorphism of functors*

$$\theta : (-)^G \xrightarrow{\cong} \mathrm{Hom}_R(\mathbb{Z}, -).$$

Here $\mathbb{Z}$ is the trivial left R-module. For every R-linear $u : A \to B$, the following square commutes:

$$\begin{array}{ccc} A^G & \xrightarrow[\cong]{\theta_A} & \mathrm{Hom}_R(\mathbb{Z}, A) \\ {\scriptstyle u^G}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(\mathbb{Z},u)} \\ B^G & \xrightarrow[\theta_B]{\cong} & \mathrm{Hom}_R(\mathbb{Z}, B). \end{array}$$

Both paths send a to the map $k \mapsto ku(a)$. The isomorphism is *canonical* because its formula $\theta_A(a)(k) = ka$ requires no choice of basis or resolution. After identifying the values on objects by these maps, the actions of the two functors on morphisms agree as well.

This identification transfers left exactness. If $0 \to A \to B \to C$ is exact, applying $\mathrm{Hom}_R(\mathbb{Z}, -)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\mathbb{Z}, A) \longrightarrow \mathrm{Hom}_R(\mathbb{Z}, B) \longrightarrow \mathrm{Hom}_R(\mathbb{Z}, C).$$

The naturality squares identify its terms and arrows with those of $0 \to A^G \to B^G \to C^G$, which is therefore exact at A^G and B^G. Thus $(-)^G$ is left exact. □

Definition (Group cohomology). For $n \geq 0$, the *nth cohomology of G with coefficients in A* is

$$H^n(G; A) := R^n\big((-)^G\big)(A).$$

Let $A \to I^\bullet$ be an injective resolution of left R-modules, and omit the coaugmentation term. Naturality applied to each differential $d^n : I^n \to I^{n+1}$ gives

$$\theta_{I^{n+1}} \circ (d^n)^G = \mathrm{Hom}_R(\mathbb{Z}, d^n) \circ \theta_{I^n}.$$

Hence the component maps form an *isomorphism of cochain complexes*

$$\theta_{I^\bullet} : (I^\bullet)^G \xrightarrow{\cong} \mathrm{Hom}_R(\mathbb{Z}, I^\bullet).$$

Taking cohomology gives

$$\begin{aligned} H^n(G; A) &= H^n\big((I^\bullet)^G\big) \\ &\cong H^n\big(\mathrm{Hom}_R(\mathbb{Z}, I^\bullet)\big) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A). \end{aligned}$$

In particular,

$$H^0(G; A) \cong A^G.$$

□

Computing with a resolution of $\mathbb{Z}$. By the equivalent computation theorem for Ext, any projective resolution $P_\bullet \to \mathbb{Z}$ of the trivial *left* R-module gives

$$H^n(G; A) \cong H^n\big(\mathrm{Hom}_R(P_\bullet, A)\big).$$

The augmentation term is omitted, and the cochain differential is $\delta^n(h) = h \circ d_{n+1}$. We can therefore fix one resolution for G and use it for every coefficient left R-module A.

In Chapter 4 we will construct the *left bar resolution* for this purpose, alongside the right-module version used for homology.

Proposition (Vanishing in positive degrees). For $n > 0$, the following hold.

1. If $G = \{1\}$, then $H^n(G; A) = 0$ for every abelian group A.
2. If A is injective over $\mathbb{Z}G$, then $H^n(G; A) = 0$.

Proof. In the first case, $R = \mathbb{Z}G = \mathbb{Z}$, so the trivial left R-module $\mathbb{Z}$ is free of rank one, hence projective. Therefore $\mathrm{Hom}_R(\mathbb{Z}, -)$ is exact. The canonical natural isomorphism $(-)^G \cong \mathrm{Hom}_R(\mathbb{Z}, -)$ shows that $(-)^G$ is exact as well. Its higher right derived functors therefore vanish.

In the second case, use the injective resolution $0 \to A \xrightarrow{\mathrm{id}_A} A \to 0$. After omitting the coaugmentation term, the underlying complex and its invariant complex are concentrated in degree 0, so the positive-degree cohomology vanishes. □

Remark (Injectivity depends on the ring). $\mathbb{Q}$ is injective over $\mathbb{Z}$, but the trivial left $\mathbb{Z}G$-module $\mathbb{Q}$ need not be injective.

4 The Bar Resolution

Fix a group G and put $R = \mathbb{Z}G$. The *bar resolution* is an explicit free resolution of the trivial left or right R-module $\mathbb{Z}$, constructed from tuples in G. It at least provides an approach to the group homology and cohomology of each group. However, the computation may still be difficult. In some cases (for example, finite or infinite cyclic groups), we can find some much easier free resolutions of such $\mathbb{Z}$.

4.1 Left and right bar resolutions

We construct the left bar resolution and obtain the right one by changing sides using the involution $g \mapsto g^{-1}$ of $\mathbb{Z}G$.

The left bar modules. For $n \geq 0$, let

$$B_n^L = \bigoplus_{(g_1,\ldots,g_n)\in G^n} R[g_1|\cdots|g_n].$$

These are free left R-modules; the brackets are formal basis symbols indexed by ordered tuples, with repetitions and identity entries allowed. In degree zero, there is one empty bracket, so $B_0^L = R[\,] \cong R$.

We now try to construct the differentials that make a chain complex of B_n^L.

Degree zero: generators of the augmentation kernel. Give $\mathbb{Z}$ its trivial left R-action. The first surjection is

$$\varepsilon : B_0^L \longrightarrow \mathbb{Z}, \qquad \varepsilon(r[\,]) = \varepsilon_R(r),$$

where $\varepsilon_R : R \to \mathbb{Z}$ is the coefficient-sum map defined in the previous chapter. Its kernel is $I[\,]$, where $I = \ker \varepsilon_R$ is generated as a left ideal by the elements $g - 1$. Thus a free left R-module mapping onto this kernel can have one basis symbol $[g]$ for each $g \in G$, sent to $(g-1)[\,]$. This explains the choice of B_1^L and the R-linear map

$$d_1^L : B_1^L \longrightarrow B_0^L, \qquad d_1^L[g] = (g-1)[\,].$$

In particular, $\operatorname{im} d_1^L = \ker \varepsilon$.

Higher degrees. It is difficult to find such differentials from an algebraic point of view; we may use some geometry or topology.

Vertex coordinates. To define all higher differentials uniformly and prove exactness, we rewrite the same free left R-modules in vertex coordinates. Think of $[g_1|\cdots|g_n]$ as n successive steps, with vertices

$$1, \quad g_1, \quad g_1g_2, \quad \ldots, \quad g_1\cdots g_n.$$

There are $n+1$ vertices. Multiplying the bar symbol by $a \in G$ translates every vertex on the left.

Let E_n be the free abelian group on tuples $(x_0, \ldots, x_n) \in G^{n+1}$, with diagonal left G-action

$$a(x_0, \ldots, x_n) = (ax_0, \ldots, ax_n).$$

This makes E_n a left R-module. The coordinate changes on $\mathbb{Z}$-basis are

$$\begin{aligned} B_n^L &\longleftrightarrow E_n \\ a[g_1|\cdots|g_n] &\longleftrightarrow (a, ag_1, ag_1g_2, \ldots, ag_1\cdots g_n), \\ x_0[x_0^{-1}x_1|\cdots|x_{n-1}^{-1}x_n] &\longleftrightarrow (x_0, \ldots, x_n). \end{aligned}$$

Here $a \in G$. The first vertex recovers $a = x_0$, and successive vertices recover $g_i = x_{i-1}^{-1}x_i$. Thus these are inverse bijections of the abelian-group bases. They commute with the left G-actions and extend additively to left R-linear isomorphisms $B_n^L \cong E_n$. In particular, the tuples starting at 1 form a free left R-basis of E_n. The entries g_i describe steps $x_i = x_{i-1}g_i$; the left R-action translates all vertices simultaneously.

For example, $[g]$ corresponds to $(1, g)$, pictured as a formal line:

$$1 \xrightarrow{\quad [g] \quad} g$$

The differential formula $d_1^L[g] = (g-1)[\,]$ is somehow $\partial_1(1, g) = g - 1$.

The general differential: delete one vertex. The same rule applies in every degree: take the alternating sum of the tuples obtained by deleting one vertex. Define

$$\partial_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n), \qquad n \geq 1.$$

The hat means that the indicated entry is omitted. Deletion commutes with the left G-action, so these maps are left R-linear. In degree zero, the augmentation sends each (x_0) to 1.

Translate deletion of a vertex of $(1, g_1, g_1g_2, \ldots, g_1\cdots g_n)$ back into bar symbols will give the corresponding left R-linear differential

$$\begin{aligned} d_n^L[g_1|\cdots|g_n] = g_1[g_2|\cdots|g_n] &\\ + \sum_{i=1}^{n-1} (-1)^i [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] &\\ + (-1)^n [g_1|\cdots|g_{n-1}]. & \end{aligned}$$

We have now defined the augmented left bar complex

$$\cdots \longrightarrow B_2^L \xrightarrow{d_2^L} B_1^L \xrightarrow{d_1^L} B_0^L \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

since $d_{n-1}^L d_n^L = 0$ follows from $\partial_{n-1}\partial_n = 0$ (one can check) and $\varepsilon\partial_1 = 0$ is discussed before.

It remains to prove the exactness.

Proposition. The augmented left bar complex is a free resolution of the trivial left R-module $\mathbb{Z}$.

Proof. To prove exactness, define homomorphisms of abelian groups

$$\begin{aligned} &s_n : E_n \longrightarrow E_{n+1}, \quad s_n(x_0, \ldots, x_n) = (1, x_0, \ldots, x_n) \quad (n \geq 0), \\ &s_{-1} : \mathbb{Z} \longrightarrow E_0, \qquad\qquad s_{-1}(k) = k(1). \end{aligned}$$

Deleting the newly inserted first vertex gives the original tuple; all remaining terms cancel those in $s_{n-1}\partial_n$. That is,

$$\partial_{n+1}s_n + s_{n-1}\partial_n = \mathrm{id}_{E_n} \quad (n \geq 0), \qquad \varepsilon s_{-1} = \mathrm{id}_{\mathbb{Z}},$$

where $\partial_0 = \varepsilon$ in this augmented formula.

In particular, a cycle z satisfies $z = \partial_{n+1} s_n(z)$, and the augmentation is surjective. The augmented sequence is therefore exact as a sequence of abelian groups, hence also as a sequence of left R-modules. Each $E_n \cong B_n^L$ is free over R, as shown above. □

Remark. The maps s_n are $\mathbb{Z}$-linear, but generally not R-linear. For $a \in G$,

$$s_n(a(x_0, \ldots, x_n)) = (1, ax_0, \ldots, ax_n),$$
$$a\, s_n(x_0, \ldots, x_n) = (a, ax_0, \ldots, ax_n).$$

They prove exactness of the bar resolution, but do not give an R-linear contraction to which tensor or Hom over R could be applied. Thus this contraction does not force group homology or cohomology to vanish.

Changing left modules into right modules. The group ring $R = \mathbb{Z}G$ has an additive involution

$$\iota\left(\sum_g n_g g\right) = \sum_g n_g g^{-1}, \qquad \iota(rs) = \iota(s)\iota(r).$$

For a left R-module M, let M^{r} be the same abelian group with right action

$$m \cdot r = \iota(r)m; \qquad m \cdot g = g^{-1}m.$$

Indeed, $(m \cdot r) \cdot s = \iota(s)\iota(r)m = m \cdot (rs)$. Every left R-linear map becomes right R-linear under this rule. The reverse construction is $r \cdot m = m\iota(r)$, so changing sides twice recovers the original module and maps. Exact sequences are preserved, and a left free basis becomes a right free basis. Consequently, $(B_\bullet^L)^{\mathrm{r}} \to \mathbb{Z}$ is already a free right R-resolution. This uses the involution of $\mathbb{Z}G$; an arbitrary ring need not admit such an identification of sides.

The right bar resolution in vertex coordinates. There is no need for a separate construction or exactness proof. Write $E_n^L = E_n$ for the left tuple module above, and let E_n^R have the diagonal right action $(x_0, \ldots, x_n)a = (x_0a, \ldots, x_na)$. The maps

$$\Theta_n : (E_n^L)^{\mathrm{r}} \longrightarrow E_n^R, \qquad (x_0, \ldots, x_n) \longmapsto (x_0^{-1}, \ldots, x_n^{-1})$$

are right R-linear bijections: $(a^{-1}x_i)^{-1} = x_i^{-1}a$. They commute with deletion of vertices and with augmentation, hence give an isomorphism of augmented complexes. Thus the left and right bar resolutions are the same construction after changing sides. □

To write the right resolution in bar symbols, choose the tuples whose last vertex is 1:

$$[g_1|\cdots|g_n]_R \longleftrightarrow (g_1 \cdots g_n, g_2 \cdots g_n, \ldots, g_n, 1).$$

Every tuple is uniquely such a representative multiplied on the right by x_n, with $g_i = x_{i-1}x_i^{-1}$. Hence

$$B_n^R = \bigoplus_{(g_1,\ldots,g_n) \in G^n} [g_1|\cdots|g_n]_R R.$$

We write the left basis symbols as $[g_1|\cdots|g_n]_L$. The same deletion boundary becomes

$$\begin{aligned} d_n^R[g_1|\cdots|g_n]_R = {} & [g_2|\cdots|g_n]_R \\ & + \sum_{i=1}^{n-1} (-1)^i [g_1|\cdots|g_ig_{i+1}|\cdots|g_n]_R \\ & + (-1)^n [g_1|\cdots|g_{n-1}]_R g_n. \end{aligned}$$

In particular, $d_1^R[g]_R = [\,]_R(1-g)$; the sign comes from using representatives with last vertex 1. With augmentation $\varepsilon([\,]_R r) = \varepsilon_R(r)$, this is the free right resolution obtained above. The two bracket conventions are simply coordinates on the corresponding tuple complexes.

4.2 Explicit definitions of group homology and cohomology

Fix a left $R = \mathbb{Z}G$-module A. We first use the unnormalized free right resolution $B_\bullet^R \to \mathbb{Z}$ and free left resolution $B_\bullet^L \to \mathbb{Z}$. The balanced computation of Tor and the equivalent computation of Ext give

$$H_n(G;A) \cong H_n(B_\bullet^R \otimes_R A),$$
$$H^n(G;A) \cong H^n\big(\mathrm{Hom}_R(B_\bullet^L, A)\big).$$

In each case the augmentation term is omitted. We now write these complexes without referring to left or right R-linear maps on a resolution.

The bar chain complex. Put

$$C_n(G;A) := B_n^R \otimes_R A \cong \bigoplus_{G^n} A \qquad (n \geq 0).$$

An element is a finite sum of symbols $a[g_1|\cdots|g_n]$, where this notation means $[g_1|\cdots|g_n]_R \otimes a$ and is additive in $a \in A$. Since G^0 consists of one empty tuple, $C_0(G;A) = A$; we identify $a[]$ with a. For $n \geq 1$, the differential is

$$\begin{aligned}\partial_n\big(a[g_1|\cdots|g_n]\big) = {} & a[g_2|\cdots|g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i a[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ & + (-1)^n (g_n a)[g_1|\cdots|g_{n-1}],\end{aligned}$$

and $\partial_0 = 0$. The last term follows from the tensor relation

$$([g_1|\cdots|g_{n-1}]_R g_n) \otimes a = [g_1|\cdots|g_{n-1}]_R \otimes (g_n a).$$

For example,

$$\partial_1(a[g]) = a - ga,$$
$$\partial_2(a[g|h]) = a[h] - a[gh] + (ha)[g].$$

Thus

$$H_n(G;A) \cong \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

In degree zero, $\operatorname{im}\partial_1$ is generated by the elements $a - ga$, so this gives $H_0(G;A) = A_G$.

The bar cochain complex. An R-linear map $B_n^L \to A$ is determined by arbitrary values on its free basis. Consequently,

$$C^n(G;A) := \mathrm{Hom}_R(B_n^L, A) \cong \{f : G^n \to A\},$$

with pointwise addition. These are *all* functions, with no homomorphism condition or finite-support requirement. In degree zero we again have $C^0(G;A) = A$.

The differential $\delta^n(f) = f \circ d^L_{n+1}$ is

$$\begin{aligned}(\delta^n f)(g_1, \dots, g_{n+1}) = {} & g_1 f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \dots, g_n).\end{aligned}$$

Here $f(g_1, \dots, g_n)$ denotes the value of the corresponding R-linear map on $[g_1|\cdots|g_n]_L$. The first term uses R-linearity: $f(g_1[\cdots]_L) = g_1 f([\cdots]_L)$. Hence

$$H^n(G; A) \cong \ker \delta^n / \operatorname{im} \delta^{n-1},$$

where $\delta^{-1} = 0$. In particular, $H^0(G; A) = \ker \delta^0 = A^G$.

The coefficient action appears at the *last* face in the chain formula and at the *first* face in the cochain formula. This follows from our right and left bar conventions, respectively; the formulas should not be interchanged for a general action.

Homogeneous cochains. Using vertex coordinates, a degree-n cochain is an equivariant function $\Phi : G^{n+1} \to A$, meaning

$$\Phi(gx_0, \dots, gx_n) = g\Phi(x_0, \dots, x_n).$$

Its differential is simply alternating deletion:

$$(\delta^n \Phi)(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \Phi(x_0, \dots, \widehat{x_i}, \dots, x_{n+1}).$$

The correspondence with the functions $f : G^n \to A$ above is

$$\begin{aligned}\Phi(x_0, \dots, x_n) &= x_0 f(x_0^{-1}x_1, \dots, x_{n-1}^{-1}x_n), \\ f(g_1, \dots, g_n) &= \Phi(1, g_1, g_1g_2, \dots, g_1 \cdots g_n).\end{aligned}$$

These describe the same cochain complex: an R-linear map $E_n \to A$ is exactly an equivariant function on its tuple basis, extended additively. The coordinate isomorphisms $B^L_n \cong E_n$ give the two inverse formulas and identify the differentials. □

An R-linear coefficient map $u : A \to A'$ acts on chains by $a[g_1|\cdots|g_n] \mapsto u(a)[g_1|\cdots|g_n]$, and on cochains by $f \mapsto u \circ f$. These maps commute with the displayed differentials and induce the coefficient maps defined earlier.

What the derived-functor description adds. We could take the displayed bar complexes and their homology or cohomology as the definitions of $H_n(G; A)$ and $H^n(G; A)$. No choice of resolution would then appear in the definition. The earlier theory explains why these constructions measure the failure of coinvariants to be exact and of invariants to be exact. It also proves that we may replace the bar resolution by any more convenient projective resolution, with the same results and the same maps on coefficients. Thus the bar formulas provide an explicit construction; the derived-functor description explains its relation to the module-theoretic operations from which we started.

Summary

A free resolution records generators, relations, and successive relations among relations. Projective and injective resolutions provide the lifting and extension properties needed for comparison. Their uniqueness up to homotopy, together with preservation of homotopies by additive functors, makes derived functors independent of the chosen resolutions up to canonical isomorphism.

Fix a group of interest G, put $R = \mathbb{Z}G$, and let A be a coefficient left R-module. The following descriptions hold for every $n \geq 0$.

Group homology: from the definition to explicit complexes. Choose a projective resolution $P_\bullet \to A$ of left R-modules, and a projective resolution $Q^R_\bullet \to \mathbb{Z}$ of the trivial right R-module. Let $B^R_\bullet$ be the right bar resolutions of $\mathbb{Z}$. Then

$$\begin{aligned} H_n(G;A) := L_n\big((-)_G\big)(A) &= H_n\big((P_\bullet)_G\big) \\ &\cong H_n(\mathbb{Z} \otimes_R P_\bullet) = \operatorname{Tor}^R_n(\mathbb{Z}, A) \\ &\cong H_n(Q^R_\bullet \otimes_R A) \\ &\cong H_n(B^R_\bullet \otimes_R A) = H_n(C_\bullet(G;A)). \end{aligned}$$

The first line is the definition of the left derived functors of coinvariants. The natural identification $(-)_G \cong \mathbb{Z} \otimes_R -$ and the definition of Tor gives the second line. Balanced computation then allows us to resolve the trivial right module $\mathbb{Z}$ instead of the coefficient left module A. Finally, the right bar resolution are one particular free resolution that can replace $Q^R_\bullet$.

Group cohomology: from the definition to explicit complexes. Choose an injective resolution $A \to I^\bullet$ of left R-modules, and a projective resolution $Q^L_\bullet \to \mathbb{Z}$ of the trivial left R-module. Let $B^L_\bullet$ be the left bar resolution of $\mathbb{Z}$. Then

$$\begin{aligned} H^n(G;A) := R^n\big((-)^G\big)(A) &= H^n\big((I^\bullet)^G\big) \\ &\cong H^n\big(\operatorname{Hom}_R(\mathbb{Z}, I^\bullet)\big) = \operatorname{Ext}^n_R(\mathbb{Z}, A) \\ &\cong H^n\big(\operatorname{Hom}_R(Q^L_\bullet, A)\big) \\ &\cong H^n\big(\operatorname{Hom}_R(B^L_\bullet, A)\big) = H^n(C^\bullet(G;A)). \end{aligned}$$

Here the definition uses the right derived functors of invariants. The natural identification $(-)^G \cong \operatorname{Hom}_R(\mathbb{Z}, -)$ gives Ext. Its equivalent computation theorem replaces the injective resolution of A by a projective resolution of the trivial left module $\mathbb{Z}$. We may then choose the left bar resolution. Thus the computation of group cohomology using a free resolution of $\mathbb{Z}$ follows from the definition via an injective resolution of the coefficient module.

All complexes in the displays are *unaugmented*: $P_\bullet$ and $I^\bullet$ exclude A, while $Q^R_\bullet$, $Q^L_\bullet$, and the bar complexes exclude $\mathbb{Z}$. The displayed isomorphisms are canonical and natural in A; the different complexes need not themselves be isomorphic.

After passing to resolutions of $\mathbb{Z}$, we can fix the resolutions once for G and vary A. Changing a left action into a right action by $x \cdot g = g^{-1}x$ allows the right resolution to be obtained from a left one; the bar constructions above are related in precisely this way. The right resolution is used in $- \otimes_R A$, and the left one in $\operatorname{Hom}_R(-, A)$.

In bar coordinates, $C_n(G;A)$ consists of finite sums of symbols $a[g_1|\cdots|g_n]$, whereas $C^n(G;A)$ consists of all functions $G^n \to A$. In degree zero, all these descriptions recover

$$H_0(G;A) \cong A_G \cong \mathbb{Z} \otimes_R A, \qquad H^0(G;A) \cong A^G \cong \mathrm{Hom}_R(\mathbb{Z}, A).$$

Positive-degree homology and cohomology measure the failure of coinvariants and invariants, respectively, to be exact.

References

[Weibel] Charles A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, 1994. doi:10.1017/CBO9781139644136. The results cited above are in Chapters 1–3; group homology, cohomology, and the bar resolution are treated in Chapter 6. Online chapter texts: 1, 2, 3, 6.