

Prerequisites

HUANG Qinpei (with the assistance of AI)

October 9, 2026

Abstract

These preparatory notes supply the algebraic tools used in the lecture notes on group homology and cohomology. Concrete module constructions and their universal properties come first, followed by categories, naturality, and additive functors. Exact sequences, complexes, homotopies, and connecting maps complete the framework.

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The companion *Defining Group Homology and Cohomology* is called the *Main Notes* in cross-references. Chapters in this volume use ordinary numbering. The main notes prefix references to this volume with P: for example, Section P3.2 refers to Section 3.2 here.

1 Module Constructions

Fix a unital ring R . Unless a side is specified, modules are left R -modules. We use the elementary module language recalled in Section 2.1. Each construction is given concretely and then characterized by its maps; the general categorical language is developed in Chapter 2.

1.1 Smaller constructions: submodules, quotients, and the first isomorphism theorem

A universal property describes a construction through the maps into or out of it. It includes both an *existence* statement and a *uniqueness* statement. The distinguished structure maps are part of the construction.

Submodules: the ring is part of the question

Definition 1.1 (Submodule). A subset $N \subseteq M$ is an R -submodule if it is an additive subgroup and is stable under the action of R :

$$rn \in N \quad (r \in R, n \in N).$$

Equivalently, $0 \in N$, and $n - n' \in N$ and $rn \in N$ for all $n, n' \in N$ and $r \in R$.

The inherited addition and scalar action make N an R -module, and the inclusion $N \hookrightarrow M$ is R -linear. Conversely, the image of any injective R -linear map is a submodule.

Example 1.2 (Changing the ring changes the submodules). The subset $\mathbb{Z} \subseteq \mathbb{Q}$ is a $\mathbb{Z}$ -submodule, but it is not a $\mathbb{Q}$ -submodule: it contains 1 but not $\frac{1}{2} \cdot 1$. Likewise, for a field k , the line $k(1, 0) \subseteq k^2$ is a k -submodule, but it is not an $M_2(k)$ -submodule for the usual matrix action, since $E_{21}(1, 0) = (0, 1)$ is outside the line.

In fact, the only $M_2(k)$ -submodules of k^2 are 0 and k^2 .

Thus a statement about submodules must specify both the ring and its action. Knowing the underlying abelian group is not enough.

Intersections of submodules are submodules. The *submodule generated by a subset* $S \subseteq M$ is the smallest submodule containing it:

$$\langle S \rangle_R = \left\{ \sum_{j=1}^t r_j s_j : t \geq 0, r_j \in R, s_j \in S \right\}.$$

Only finite sums occur; the empty sum is 0. In particular, $\langle M \rangle_R = M$, so every module has a generating set. For submodules $N, L \subseteq M$, their sum is $N + L = \{n + \ell : n \in N, \ell \in L\}$.

Quotient modules

Definition 1.3 (Quotient module). If $N \subseteq M$ is an R -submodule, the quotient abelian group M/N is an R -module with operations

$$(m + N) + (m' + N) = (m + m') + N, \quad r(m + N) = rm + N.$$

The canonical map $q : M \rightarrow M/N$, $m \mapsto m + N$, is a surjective R -linear map with kernel N .

The scalar action is well-defined because $m - m' \in N$ implies $rm - rm' = r(m - m') \in N$. This is precisely why stability under R is required: a quotient of the underlying abelian group need not otherwise inherit the action.

For a submodule $N \subseteq M$, the quotient map $q : M \rightarrow M/N$ has the following universal property: for every R -linear map $u : M \rightarrow T$,

$$u(N) = 0 \implies \text{there is a unique } \bar{u} : M/N \rightarrow T \text{ with } \bar{u}q = u.$$

Explicitly, $\bar{u}(m + N) = u(m)$; the condition $u(N) = 0$ proves well-definedness. If $i : N \hookrightarrow M$ is the inclusion, this says, together with $qi = 0$, that q is the *cokernel* of i .

An element $m + N$ is a coset, not a chosen representative. The equation $m + N = 0$ means $m \in N$. Taking a quotient imposes the relations $n = 0$ for every $n \in N$, while preserving the module operations. For a left ideal $I \subseteq R$, the quotient R/I is a left module; to give it the usual quotient *ring* structure, I must also be a right ideal.

Kernels, images, and the first isomorphism theorem

For an R -linear map $f : M \rightarrow N$, define

$$\ker f = \{m : f(m) = 0\}, \quad \operatorname{im} f = \{f(m) : m \in M\}.$$

Both are submodules, by additivity and R -linearity.

For a module map $f : M \rightarrow L$, the inclusion $k : \ker f \hookrightarrow M$ satisfies $fk = 0$ and has the following universal property. For every module map $u : T \rightarrow M$,

$$fu = 0 \implies \text{there is a unique } v : T \rightarrow \ker f \text{ with } kv = u.$$

The map v is simply u with its codomain restricted to $\ker f$. This is the definition of a *kernel* in a category with zero morphisms.

Proposition 1.4 (First isomorphism theorem). *Every R -linear map $f : M \rightarrow N$ induces an isomorphism*

$$M/\ker f \xrightarrow{\cong} \operatorname{im} f, \quad m + \ker f \mapsto f(m).$$

Proof. Two representatives have the same image exactly when their difference lies in $\ker f$, so the formula is well-defined and injective. It is surjective onto $\operatorname{im} f$ by definition, and is linear because f is linear. $\square$

Every submodule inclusion is the kernel of its quotient map, and every surjective module map is, up to isomorphism, the cokernel of its kernel inclusion. The first isomorphism theorem identifies $M/\ker f$ with $\operatorname{im} f$ and explains this compatibility. These statements need not hold in an arbitrary category.

A general cokernel of $f : M \rightarrow L$ is $L/\operatorname{im} f$ with its quotient map q . The quotient property says that $qf = 0$, and every $u : L \rightarrow T$ with $uf = 0$ factors uniquely through q .

Submodules and quotients in categorical language. A morphism $i : N \rightarrow M$ is a *monomorphism* if $iu = iv$ implies $u = v$ for all parallel maps into N . A morphism $q : M \rightarrow N$ is an *epimorphism* if $uq = vq$ implies $u = v$ for all parallel maps out of N .

In module categories, monomorphisms are precisely injective maps, and epimorphisms are precisely surjective maps. Injectivity and surjectivity give the respective cancellation properties. Conversely, a noninjective map is not monic: its nonzero kernel inclusion and the zero map become equal after composition. If q is epic, the quotient map $N \rightarrow N/\text{im } q$ and the zero map agree after composing with q , so they agree. The quotient is therefore zero, and q is surjective.

A *subobject* of M is represented by a monomorphism $i : N \rightarrow M$; two such maps represent the same subobject when an isomorphism of their sources commutes with the maps to M . In module categories, subobjects are thus represented by actual submodules. The embedding is part of the data, just as a quotient includes its quotient map.

Key point. A product, a coproduct, a kernel, and a quotient are each specified by an object together with maps. Their universal properties characterize the entire package up to a unique compatible isomorphism.

1.2 Larger constructions: products and direct sums

Products: construction and universal property

Let $(M_i)_{i \in I}$ be a family of R -modules indexed by a set. The *direct product*

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i\}$$

has coordinatewise addition and scalar multiplication, with projections $p_i((m_j)) = m_i$.

Definition 1.5 (Product). A *product* of objects $(X_i)_{i \in I}$ in a category $\mathcal{C}$ is an object P with maps $p_i : P \rightarrow X_i$ such that, for every object T and maps $f_i : T \rightarrow X_i$, there is a unique $f : T \rightarrow P$ satisfying $p_i f = f_i$ for every i .

In $R\text{-Mod}$, the direct product has this property: the only possible map is $f(t) = (f_i(t))_{i \in I}$, and it is linear. Equivalently, composition with the projections gives a bijection

$$\text{Hom}_R\left(T, \prod_i M_i\right) \cong \prod_i \text{Hom}_R(T, M_i).$$

Here and below, Hom_R abbreviates morphisms in the category of left R -modules unless the side is specified otherwise.

Direct sums: construction and universal property

The *direct sum* is the submodule

$$\bigoplus_{i \in I} M_i = \left\{ (m_i) \in \prod_{i \in I} M_i : m_i = 0 \text{ for all but finitely many } i \right\}.$$

Its canonical inclusions $\iota_i : M_i \rightarrow \bigoplus_j M_j$ place an element in the i th coordinate and put zero in every other coordinate. Every element has uniquely determined coordinates m_i , all but finitely many zero, and equals the finite sum $\sum_i \iota_i(m_i)$.

Definition 1.6 (Coproduct). A *coproduct* of objects $(X_i)_{i \in I}$ is an object C with maps $\iota_i : X_i \rightarrow C$ such that, for every object T and maps $f_i : X_i \rightarrow T$, there is a unique $f : C \rightarrow T$ satisfying $f\iota_i = f_i$ for every i .

In $R\text{-Mod}$, the direct sum is the coproduct. The map is

$$f((m_i)) = \sum_i f_i(m_i),$$

which is defined because the input has finite support. Uniqueness follows from the expression of each element as $\sum_i \iota_i(m_i)$. Thus

$$\text{Hom}_R\left(\bigoplus_i M_i, T\right) \cong \prod_i \text{Hom}_R(M_i, T).$$

The right-hand side allows arbitrary families of maps; it is a product, even when the object on the left is a direct sum.

For finite I , every tuple has finite support, so the two constructions are the same module. For infinite I they need not coincide: for instance, $(1, 1, 1, \dots)$ belongs to $\prod_{i \geq 1} \mathbb{Z}$ but not to $\bigoplus_{i \geq 1} \mathbb{Z}$. There is no operation of summing an arbitrary infinite family in a module.

For two submodules $A, B \subseteq M$, saying that M is their *internal direct sum* means that $A \oplus B \rightarrow M$, $(a, b) \mapsto a + b$, is an isomorphism. Equivalently, $A + B = M$ and $A \cap B = 0$. In this case each m has a unique decomposition $m = a + b$.

For a finite family the same module is both product and coproduct, with its projections and inclusions. This is called a *biproduct*. The empty product and empty coproduct are both the zero module. It is a *zero object*: there is exactly one morphism from it to any object and exactly one morphism to it from any object.

1.3 Abelian-group constructions: Hom sets and tensor products

The Hom set is an abelian group

For R -modules M, N , addition is defined pointwise:

$$(f + g)(m) = f(m) + g(m), \quad (-f)(m) = -f(m).$$

The zero function is the identity for this addition. The sum is linear:

$$(f + g)(rm) = rf(m) + rg(m) = r(f + g)(m),$$

and additivity follows from the abelian group law in N . Consequently $\text{Hom}_R(M, N)$ is an abelian group. Composition is additive in both variables.

It is therefore always a $\mathbb{Z}$ -module, but for a noncommutative ring there is in general no pointwise R -module structure. The tempting definition $(rf)(m) = rf(m)$ gives

$$(rf)(sm) = rsf(m), \quad s(rf)(m) = srf(m),$$

which need not agree.

For example, take $M = N = R = M_2(k)$ as left regular modules and $f = \text{id}_R$. If $r = E_{12}$ and $s = E_{21}$, then $(rf)(s) = E_{11}$ whereas $s(rf)(1) = E_{22}$. Thus rf is not left R -linear.

There is always a pointwise module structure over the center

$$Z(R) = \{z \in R : zr = rz \text{ for every } r \in R\}.$$

Thus commutative rings give a pointwise R -action; over a field this recovers the vector space $\text{Hom}_k(V, W)$. For arbitrary R , no such canonical R -action is available in general, although particular examples can carry additional structure.

Remark (Hom and universal properties). Here Hom is constructed directly as a group of maps. It will also occur inside universal properties: a product is characterized by a statement about Hom into it, and a coproduct by a statement about Hom out of it. No additional universal property is needed to define the Hom group.

Tensor products: a construction by symbols and relations

Products and sums were first built from coordinates and then characterized by maps. We now follow the same pattern for tensor products. The resulting universal property will suggest defining a free module in the reverse order: specify the maps it should admit, then construct such an object.

Over a field k , a vector space has a compatible right action $v \cdot a = av$ as well as its left action. For a noncommutative ring, a left module need not carry a right action. Fix a right R -module N and a left R -module M .

Definition 1.7 (Tensor product by generators and relations). Form the abelian group of finite formal integer combinations

$$L = \bigoplus_{(n,m) \in N \times M} \mathbb{Z}[(n, m)].$$

Let K be the subgroup generated by

$$\begin{aligned} [(n + n', m)] - [(n, m)] - [(n', m)], \\ [(n, m + m')] - [(n, m)] - [(n, m')], \\ [(nr, m)] - [(n, rm)] \end{aligned}$$

for $n, n' \in N$, $m, m' \in M$, and $r \in R$. Define $N \otimes_R M = L/K$ and denote the class of $[(n, m)]$ by $n \otimes m$. This is an abelian group.

The map $\tau(n, m) = n \otimes m$ is additive in each variable and satisfies $(nr) \otimes m = n \otimes (rm)$, precisely because we imposed these relations. There is initially no scalar action on the output.

Definition 1.8 (Balanced biadditive map). A map $\beta : N \times M \rightarrow B$ to an abelian group is *biadditive and R -balanced* when

$$\begin{aligned} \beta(n + n', m) &= \beta(n, m) + \beta(n', m), \\ \beta(n, m + m') &= \beta(n, m) + \beta(n, m'), \\ \beta(nr, m) &= \beta(n, rm). \end{aligned}$$

The last equality moves a scalar between the two given actions; it does not require an R -action on B .

Proposition 1.9 (Universal property of the constructed tensor product). *Every balanced biadditive $\beta : N \times M \rightarrow B$ factors through a unique additive map $\bar{\beta} : N \otimes_R M \rightarrow B$:*

$$\begin{array}{ccc} N \times M & \xrightarrow{\tau} & N \otimes_R M \\ & \searrow \beta & \downarrow \bar{\beta} \\ & & B. \end{array}$$

Here the top and diagonal arrows are maps of underlying sets: they are additive in each variable separately, but generally not homomorphisms from the product group $N \times M$. The dashed arrow is a homomorphism of abelian groups. Thus composition with τ gives a bijection

$$\text{Hom}_{\mathbb{Z}}(N \otimes_R M, B) \cong \{\text{balanced biadditive maps } N \times M \rightarrow B\}.$$

Proof. Extend β to $b : L \rightarrow B$ by

$$b\left(\sum_{j=1}^t a_j [(n_j, m_j)]\right) = \sum_{j=1}^t a_j \beta(n_j, m_j).$$

The formal coefficients make this additive map well-defined. The three identities for β say that b vanishes on the generators of K , hence on K . The quotient property gives $\bar{\beta}$. Since the pure tensors generate L/K , the required values $\bar{\beta}(n \otimes m) = \beta(n, m)$ determine it uniquely. $\square$

Uniqueness of the tensor product

If (T, τ) and (T', τ') both have the universal property, it provides additive maps $u : T \rightarrow T'$ and $v : T' \rightarrow T$ with $u\tau = \tau'$ and $v\tau' = \tau$. Since $(vu)\tau = \tau = \text{id}_T \tau$, uniqueness gives $vu = \text{id}_T$. Similarly $uv = \text{id}_{T'}$. Thus there is a unique isomorphism $T \cong T'$ carrying every specified tensor $n \otimes m$ to the corresponding tensor. This is the same uniqueness argument used for other universal constructions.

An element of a tensor product is a finite sum of *pure tensors* $n \otimes m$. It need not itself be a single pure tensor, and such sums usually do not have unique expressions. For instance, balancing already identifies $(nr) \otimes m$ with $n \otimes (rm)$.

Remark (Which scalars act on the answer?). For the stated right-left inputs, the tensor product is naturally an abelian group. If R is commutative, every left module has the compatible right action $mr = rm$, and

$$r(n \otimes m) = (nr) \otimes m = n \otimes (rm)$$

gives a canonical R -module structure. The relations in the construction respect this formula; commutativity is what permits scalars to pass one another when checking balancing. Taking $R = k$ recovers the usual tensor product of vector spaces and its universal property for k -bilinear maps into a vector space.

More generally, a commuting left S -action on N induces $s(n \otimes m) = (sn) \otimes m$. The required compatibility is $(sn)r = s(nr)$: such an N is called an (S, R) -bimodule. We will use this for the regular bimodule R and its quotients by two-sided ideals.

The regular module acts as a tensor identity

Proposition 1.10. *For a left R -module A , there is an isomorphism*

$$\mu : R \otimes_R A \xrightarrow{\cong} A, \quad \mu(r \otimes a) = ra.$$

Here the tensor factor R uses its right regular action. For a right module N , there is similarly an isomorphism $N \otimes_R R \cong N$ sending $n \otimes r$ to nr .

Proof. The map $(r, a) \mapsto ra$ is biadditive and balanced, since $(rs)a = r(sa)$, so it induces μ . Define $\nu : A \rightarrow R \otimes_R A$ by $\nu(a) = 1 \otimes a$. It is additive, and $\mu\nu(a) = a$. Conversely,

$$\nu\mu(r \otimes a) = 1 \otimes ra = r \otimes a$$

by balancing. Pure tensors generate, so $\nu\mu = \text{id}$ on the whole tensor product. The right-module statement has the same proof, with inverse $n \mapsto n \otimes 1$. $\square$

The first tensor factor R also has its commuting left regular action. It therefore gives $R \otimes_R A$ a left R -action, under which μ and ν are R -linear. This extra structure comes from the bimodule R ; it is consistent with the general abelian-group definition.

A quotient ring imposes relations on the other factor

Proposition 1.11. *Let I be a two-sided ideal of R and let A be a left R -module. Write IA for the submodule of finite sums $\sum_j i_j a_j$ with $i_j \in I$. Then*

$$(R/I) \otimes_R A \xrightarrow{\cong} A/IA, \quad \bar{r} \otimes a \mapsto ra + IA.$$

Proof. The formula $\beta(\bar{r}, a) = ra + IA$ is independent of the representative of $\bar{r}$: replacing r by $r + i$ changes ra by $ia \in IA$. It is biadditive, and $\beta(\bar{r}s, a) = rsa + IA = \beta(\bar{r}, sa)$. It therefore induces an additive map μ from the tensor product.

The map $A \rightarrow (R/I) \otimes_R A$, $a \mapsto \bar{1} \otimes a$, kills IA . Indeed, for $i \in I$,

$$\bar{1} \otimes ia = \bar{i} \otimes a = 0,$$

and additivity handles finite sums. It consequently descends to $\nu : A/IA \rightarrow (R/I) \otimes_R A$. The two composites are identities:

$$\begin{aligned} \mu\nu(a + IA) &= a + IA, \\ \nu\mu(\bar{r} \otimes a) &= \bar{1} \otimes ra = \bar{r} \otimes a. \end{aligned}$$

Again, the second equality on pure tensors proves the assertion for all elements. These maps are also left R -linear using the left action inherited from the bimodule R/I . $\square$

Example 1.12 (Integer coefficients: quotients and torsion subgroups). For an abelian group A , the preceding propositions give

$$\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A, \quad (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} A \cong A/nA \quad (n \geq 1).$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$ forces $na = 0$ by taking a quotient of A . In contrast, evaluation at $\bar{1}$ gives

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A[n] = \{a \in A : na = 0\}.$$

Indeed, an additive map sends $\bar{1}$ to an element a with $na = 0$, and each such element defines the unique map $\bar{k} \mapsto ka$. The condition $na = 0$ makes this formula independent of the integer representative k . Thus Hom selects the elements already satisfying that equation. Section 1.1 of the *Main Notes* places this calculation in the general method of generators and relations. These are generally different groups: for $A = \mathbb{Z}$ and $n \geq 2$, $A/nA = \mathbb{Z}/n\mathbb{Z}$ whereas $A[n] = 0$. For $A = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 1$, both calculations give groups isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$, but this does not identify the two constructions for arbitrary A . On the tensor side,

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}.$$

The last equality follows from Bézout's identity. Explicitly, the pairing $(\bar{a}, \bar{b}) \mapsto \overline{ab}$ modulo d induces the isomorphism, sending $\bar{1} \otimes \bar{1}$ to $\bar{1}$. For example, $(\mathbb{Z}/4\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, whereas $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z}) = 0$.

This distinction anticipates group homology and cohomology. For a group G acting on A , the later tensor formula $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ imposes the relations $ga = a$, whereas $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ selects the elements for which those relations already hold. Here $\mathbb{Z}$ carries the trivial G -action, as a right module in the tensor formula and a left module in the Hom formula. These are coinvariants and

invariants. The groups A/nA and $A[n]$ also reappear in positive degrees for finite cyclic groups C_n with trivial coefficients, through the alternating zero and multiplication-by- n maps in their standard resolution. That calculation belongs to the study of cyclic groups in the main notes.

Key point. Tensor products impose balanced relations, and Hom out of a presented module imposes equations on possible generator images. Their functorial behavior is described in these preparatory notes, and their exactness is studied in Section 3.2 of the *Main Notes*.

2 Categorical Language

The constructions of Chapter 1 are characterized by maps into or out of them. Categories and functors organize these maps, while naturality compares the constructions.

2.1 Categories and examples

Definition 2.1 (Category). A *category* $\mathcal{C}$ consists of a collection $\text{Ob}(\mathcal{C})$ of objects, a set $\text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms for each ordered pair of objects, composition maps

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z), \quad (g, f) \longmapsto g \circ f,$$

and identity morphisms $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$. These satisfy

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \text{id}_Y \circ f = f = f \circ \text{id}_X,$$

whenever the indicated composites are defined.

Here the objects may form a proper class, as in **Grp**, but morphisms between any two fixed objects form a set. This is the usual *locally small* convention. A morphism in an abstract category need not be a function, nor must an object be a set with extra structure.

Definition 2.2 (Isomorphism inside a category). A morphism $f : X \rightarrow Y$ is an *isomorphism* if there exists $g : Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. We then write $X \cong Y$. In categories of groups or modules, these are precisely the bijective structure-preserving homomorphisms.

Why introduce categories? An algebraic object is studied partly through the maps it admits. For example, a group homomorphism can preserve structure even when it is neither injective nor surjective. A category keeps all such maps and their composition in view. It lets us formulate a construction once and ask whether it works consistently for every object and every morphism.

This language also distinguishes the information retained by different settings. A function between the underlying sets of two groups need not be a group homomorphism. Thus specifying objects without specifying morphisms does not determine the category.

Key point. Categories organize objects *together with* their allowed maps. Later, a functor will describe a construction compatible with those maps, and naturality will express compatibility between constructions.

Example 2.3 (Sets and functions). The category **Set** has sets as objects and functions as morphisms. Function composition and identity functions satisfy the category axioms. The following examples restrict the objects and the allowed maps by requiring algebraic structure.

Groups and the category Grp

Definition 2.4 (Group). A *group* is a set G with a multiplication $G \times G \rightarrow G$, $(g, h) \mapsto gh$, satisfying:

1. **Associativity:** $(gh)k = g(hk)$ for all $g, h, k \in G$.
2. **Identity:** there is $e_G \in G$ with $e_G g = g = g e_G$.
3. **Inverses:** each $g \in G$ has an element g^{-1} with $g g^{-1} = g^{-1} g = e_G$.

The group is *abelian* if $gh = hg$ for every $g, h \in G$.

For abelian groups we usually write the operation as $+$, the identity as 0 , and the inverse of a as $-a$. Examples include $(\mathbb{Z}, +)$ and $(\mathbb{Z}/n\mathbb{Z}, +)$ for $n \geq 1$. The symmetric group S_3 , with permutation composition, is nonabelian.

Definition 2.5 (Group homomorphism). A *group homomorphism* $f : G \rightarrow H$ is a function satisfying

$$f(gg') = f(g)f(g') \quad (g, g' \in G).$$

We denote the set of these maps by $\text{Hom}_{\mathbf{Grp}}(G, H)$.

The displayed condition already forces preservation of the identity and inverses. Indeed, $f(e_G) = f(e_G)^2$ implies $f(e_G) = e_H$, and $f(g)f(g^{-1}) = e_H$ implies $f(g^{-1}) = f(g)^{-1}$.

For example, $q : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, $a \mapsto \bar{a}$, is a homomorphism. Every homomorphism $f : \mathbb{Z} \rightarrow A$ into an abelian group is determined by $a = f(1)$, because $f(m) = ma$, where ma means repeated addition and its extension to negative integers.

The category axioms can now be checked concretely. Every group has an identity homomorphism id_G , and homomorphisms can be composed:

$$G \xrightarrow{f} H \xrightarrow{g} K, \quad (g \circ f)(x) = g(f(x)).$$

The composite is a homomorphism since $g(f(xx')) = g(f(x)f(x')) = g(f(x))g(f(x'))$.

The data in Grp. The *category of groups* consists of more than a collection of groups. It contains four kinds of data:

Objects	Groups $G, H, K, \dots$
Morphisms	Group homomorphisms $f : G \rightarrow H$, with specified source and target
Composition	$\text{Hom}_{\mathbf{Grp}}(H, K) \times \text{Hom}_{\mathbf{Grp}}(G, H) \rightarrow \text{Hom}_{\mathbf{Grp}}(G, K)$, sending (g, f) to $g \circ f$
Identities	The identity homomorphism $\text{id}_G : G \rightarrow G$ for each group G

Composition is associative, and composing with an appropriate identity does nothing. These facts hold because our maps are functions. Thus the data satisfy the category axioms.¹

Abelian groups and the category **Ab**

Definition 2.6 (The category of abelian groups). The category **Ab** has abelian groups as objects and ordinary group homomorphisms as morphisms. In additive notation, its morphisms satisfy

$$f(a + a') = f(a) + f(a').$$

Composition and identities are the same as in **Grp**.

In particular, for any abelian groups A, B ,

$$\boxed{\text{Hom}_{\mathbf{Ab}}(A, B) = \text{Hom}_{\mathbf{Grp}}(A, B).}$$

There is no extra condition on a morphism merely because the objects are abelian. This makes **Ab** a *full subcategory* of **Grp**: it selects some objects and retains every morphism between them.

The abelian condition also lets us add homomorphisms pointwise: $(f + g)(a) = f(a) + g(a)$. The result is again a homomorphism, so $\text{Hom}_{\mathbf{Ab}}(A, B)$ is itself an abelian group. Here this is an additional structure on the same morphism set.

¹For the general definitions, see the [Stacks Project](#), [Categories](#), Section 4.2. We allow the usual large algebraic categories.

Unital rings and the category **Ring**

Definition 2.7 ((Unital) Ring). A *ring* R is an abelian group $(R, +, 0)$ equipped with a multiplication satisfying, for all $r, s, t \in R$,

$$(rs)t = r(st), \quad r(s+t) = rs + rt, \quad (r+s)t = rt + st,$$

and with an identity 1_R satisfying $1_R r = r = r 1_R$. It is *commutative* if $rs = sr$ for every $r, s \in R$.

We do not assume commutativity unless explicitly stated. We allow the zero ring, in which $1_R = 0$; it has only the zero unital module. Examples of rings are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, polynomial rings $k[x]$ over a field k , and matrix rings $M_n(k)$. For $n \geq 2$, $M_n(k)$ is noncommutative: $E_{11}E_{12} = E_{12}$ but $E_{12}E_{11} = 0$.

Definition 2.8 ((Unital) Ring homomorphism). A *ring homomorphism* $\varphi : R \rightarrow S$ is a function satisfying

$$\varphi(r + r') = \varphi(r) + \varphi(r'), \quad \varphi(rr') = \varphi(r)\varphi(r'), \quad \varphi(1_R) = 1_S.$$

The category **Ring** has these rings as objects and these homomorphisms as morphisms, with ordinary composition and identity maps.

The requirement on 1 is part of our convention. For instance, $r \mapsto (r, 0)$ from a nonzero ring R to $R \times R$ preserves addition and multiplication but fails to preserve the identity $(1_R, 1_R)$, so it is not a morphism of **Ring**.

Every ring R has a unique morphism $\mathbb{Z} \rightarrow R$, given by $n \mapsto n1_R$. This observation will explain why every R -module has a canonical underlying $\mathbb{Z}$ -module.

Left modules, right modules, and their morphisms

Fix an object R of **Ring**. The choice of R is now part of the setting: the morphisms below are linear over this fixed ring.

Definition 2.9 (Left R -module). A *left R -module* is an abelian group $(M, +, 0)$ together with an action $R \times M \rightarrow M$, $(r, m) \mapsto rm$, satisfying

$$\begin{aligned} r(m + m') &= rm + rm', & (r + r')m &= rm + r'm, \\ (rr')m &= r(r'm), & 1_R m &= m. \end{aligned}$$

These conditions hold for all $r, r' \in R$ and $m, m' \in M$.

Thus a module is an abelian group with a compatible scalar action. Over a field it is exactly a vector space. Over a general ring, a nonzero scalar need not have a multiplicative inverse, and a module need not have a basis.

Definition 2.10 (Right R -module). A *right R -module* is an abelian group M with an action $M \times R \rightarrow M$, $(m, r) \mapsto mr$, satisfying

$$\begin{aligned} (m + m')r &= mr + m'r, & m(r + r') &= mr + mr', \\ m(rr') &= (mr)r', & m1_R &= m. \end{aligned}$$

The order in the associativity axiom matters. On a left module, $(rs)m = r(sm)$; on a right module, $m(rs) = (mr)s$.

Definition 2.11 (Module homomorphism). For left R -modules M, N , an R -linear map is an additive function $f : M \rightarrow N$ satisfying

$$f(rm) = rf(m).$$

For right R -modules, it is an additive function satisfying $f(mr) = f(m)r$.

The resulting categories are

$$\begin{aligned} R\text{-}\mathbf{Mod} &= \{\text{left } R\text{-modules and left } R\text{-linear maps}\}, \\ \mathbf{Mod}\text{-}R &= \{\text{right } R\text{-modules and right } R\text{-linear maps}\}. \end{aligned}$$

Here the braces describe the objects *and* morphisms informally. Composition and identities are again ordinary ones: for example, $(g \circ f)(rm) = g(rf(m)) = r(g \circ f)(m)$. We write the morphism sets as $\text{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ and $\text{Hom}_{\mathbf{Mod}\text{-}R}(M, N)$ to keep the side explicit.

Basic examples. The ring R acts on itself by multiplication on either side, giving the regular modules ${}_R R$ and R_R . The column space k^n is a left $M_n(k)$ -module by matrix multiplication; row vectors form a right $M_n(k)$ -module. Every left or right module has an underlying abelian group, and every module homomorphism is a homomorphism of those groups.

Key point. For a general ring, being additive is necessary but need not be sufficient for being R -linear. The scalar action is extra data.

The observation: abelian groups already carry integer scalars

Let A be an abelian group. Define, for $a \in A$,

$$n \cdot a = \begin{cases} \underbrace{a + \cdots + a}_{n \text{ terms}}, & n > 0, \\ 0, & n = 0, \\ -((-n) \cdot a), & n < 0. \end{cases}$$

The group laws give

$$n(a + b) = na + nb, \quad (n + m)a = na + ma, \quad (nm)a = n(ma), \quad 1a = a,$$

so A becomes a left $\mathbb{Z}$ -module.

Conversely, in any unital $\mathbb{Z}$ -module, the scalar action must be this one: for $n > 0$, $na = (1 + \cdots + 1)a = a + \cdots + a$, and the axioms determine the values for 0 and negative integers. Thus the action is uniquely determined by the underlying abelian group.

The correspondence also works for morphisms. If $f : A \rightarrow B$ is a group homomorphism, then repeated addition gives

$$f(na) = nf(a) \quad (n \in \mathbb{Z}).$$

Every group homomorphism between abelian groups is therefore $\mathbb{Z}$ -linear, and conversely. We have a correspondence of both *objects* and *maps*, compatible with composition.

Remark (Conventions on scalars and sides). All rings are associative and unital; ring maps preserve 1, and all modules are unital. The categories $R\text{-}\mathbf{Mod}$ and $\mathbf{Mod}\text{-}R$ contain all modules, without a finite generation assumption. For commutative R , a left module acquires the compatible right action $mr = rm$. For arbitrary R , this rule instead gives a right R^{op} -action, where $r \cdot_{\text{op}} s = sr$. The opposite ring reverses scalar multiplication; it does not reverse the direction of module homomorphisms.

2.2 Functors and naturality

Functors: constructions on objects and on morphisms

Definition 2.12 (Functor). A *covariant functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ assigns an object $F(X)$ to each object X , and a morphism $F(f) : F(X) \rightarrow F(Y)$ to each $f : X \rightarrow Y$, such that

$$F(\text{id}_X) = \text{id}_{F(X)}, \quad F(g \circ f) = F(g) \circ F(f).$$

Unless stated otherwise, “functor” means covariant in these notes.

The object assignment alone is not a functor: one must specify how maps are transported and verify both laws.

The preceding integer-action construction defines a functor

$$L : \mathbf{Ab} \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod}, \quad A \longmapsto A \text{ with the repeated-addition action}, \quad f \longmapsto f.$$

There is also a *forgetful functor*

$$U : \mathbb{Z}\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \quad M \longmapsto (M, +), \quad f \longmapsto f.$$

Both preserve identities and composition because they leave the underlying functions unchanged. More generally, there are forgetful functors $R\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab} \rightarrow \mathbf{Grp} \rightarrow \mathbf{Set}$; here $\mathbf{Set}$ has sets as objects and functions as morphisms. We write $U_{\mathbf{Ab}} : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$ for the functor that forgets the scalar action and retains the underlying abelian group; the same notation is used for right modules when the domain is clear. It acts on morphisms by retaining the underlying additive map.

Definition 2.13 (Opposite category and contravariant functor). The *opposite category* $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$, with $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ and reversed composition. A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. Thus $f : X \rightarrow Y$ gives $F(f) : F(Y) \rightarrow F(X)$, and

$$F(g \circ f) = F(f) \circ F(g), \quad F(\text{id}_X) = \text{id}_{F(X)}.$$

Example 2.14 (Precomposition reverses arrows). For a fixed abelian group B , the assignment $A \mapsto \text{Hom}_{\mathbf{Ab}}(A, B)$ sends $u : A \rightarrow A'$ to $u^* : f \mapsto f \circ u$. It is contravariant, whereas $A \mapsto \text{Hom}_{\mathbf{Ab}}(B, A)$ is covariant by postcomposition. The two variances will be essential when defining Ext .

Remark (Every functor preserves isomorphisms). If $gf = \text{id}_X$ and $fg = \text{id}_Y$, applying a functor shows that $F(f)$ and $F(g)$ are inverse maps, in the appropriate order. This needs no exactness assumption. Preserving kernels, quotients, or entire exact sequences is a substantially stronger requirement.

Naturality: compatibility with every morphism

We often have two constructions $F, G : \mathcal{C} \rightarrow \mathcal{D}$ and, for each X , a comparison map $F(X) \rightarrow G(X)$. To compare the constructions themselves, these maps must fit together as X varies.

Definition 2.15 (Natural transformation and natural isomorphism). A *natural transformation* $\eta : F \Rightarrow G$ is a family of morphisms $\eta_X : F(X) \rightarrow G(X)$ such that, for every $f : X \rightarrow Y$,

$$\boxed{G(f) \circ \eta_X = \eta_Y \circ F(f)}.$$

Equivalently, the following square commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y). \end{array}$$

It is a *natural isomorphism* if every component η_X is an isomorphism. We then write $F \cong G$.

In words, comparing first and then applying the map gives the same result as applying the map first and then comparing. “Natural” means this precise compatibility, not merely that a formula looks familiar or does not visibly use a choice.

Example 2.16 (Multiplication by an integer). Fix $n \in \mathbb{Z}$. The maps $\mu_{n,A} : A \rightarrow A$, $a \mapsto na$, form a natural transformation $\text{id}_{\mathbf{Ab}} \Rightarrow \text{id}_{\mathbf{Ab}}$, because $f(na) = nf(a)$ for every homomorphism f . Such a transformation need not be invertible: multiplication by 2 on $\mathbb{Z}$ is not surjective.

A useful natural isomorphism. Define $H : \mathbf{Ab} \rightarrow \mathbf{Ab}$ by

$$H(A) = \text{Hom}_{\mathbf{Ab}}(\mathbb{Z}, A), \quad H(f)(u) = f \circ u.$$

The Hom set has pointwise addition, so it is an abelian group, and postcomposition is additive. Identity and composition laws follow directly from those for functions.

Evaluation at 1 gives an isomorphism

$$\text{ev}_A : H(A) \xrightarrow{\cong} A, \quad u \mapsto u(1),$$

whose inverse sends a to the homomorphism $n \mapsto na$. For every $f : A \rightarrow B$, one has

$$\text{ev}_B(H(f)(u)) = (f \circ u)(1) = f(u(1)) = f(\text{ev}_A(u)).$$

Hence $\text{ev} : H \Rightarrow \text{id}_{\mathbf{Ab}}$ is a natural isomorphism. This says more than the separate existence of an isomorphism for each A : it says these isomorphisms respect every homomorphism between abelian groups.

For two contravariant functors the same definition is applied to $\mathcal{C}^{\text{op}}$: if $f : X \rightarrow Y$, naturality reads $G(f)\eta_Y = \eta_X F(f)$. The direction of the square follows from the variance; its content is always compatibility with every morphism.

2.3 Additive categories and functors

The module constructions of Chapter 1 provide a model for categories in which maps can be added and kernels, images, and quotients can be formed.

Definition 2.17 (Preadditive, additive, and abelian categories). A category is *preadditive* if each morphism set is an abelian group and composition is additive in each variable:

$$h(f + g) = hf + hg, \quad (f + g)k = fk + gk.$$

It is *additive* if it is preadditive and has all finite biproducts, including a zero object as the empty biproduct. Here a biproduct is simultaneously a product and a coproduct, with compatible projections and inclusions, as for a finite direct sum of modules. For two objects, these compatibility equations are

$$p_i \iota_i = \text{id}, \quad p_i \iota_j = 0 \ (i \neq j), \quad \iota_1 p_1 + \iota_2 p_2 = \text{id}.$$

An additive category is *abelian* if every morphism $f : X \rightarrow Y$ has a kernel and a cokernel, and the canonical map

$$\operatorname{coim} f \longrightarrow \operatorname{im} f$$

is an isomorphism, where $\operatorname{coim} f = \operatorname{coker}(\ker f \rightarrow X)$ and $\operatorname{im} f = \ker(Y \rightarrow \operatorname{coker} f)$.

Both $R\text{-}\mathbf{Mod}$ and $\mathbf{Ab}$ are abelian categories. Their Hom groups and finite direct sums supply additivity; kernels and cokernels are the usual submodules and quotients. The last axiom is precisely the isomorphism $X/\ker f \cong \operatorname{im} f$. These definitions abstract the module operations already established; they do not assert that every abstract object has elements.

Remark (The setting for complexes). Compatible zero morphisms suffice to write $d^2 = 0$ and define chain maps; preadditivity supplies the sums in the homotopy equation $f - g = dh + hd$. An abelian category is a sufficient setting for homology as cycles modulo boundaries. We work in abelian categories whenever homology is needed; the result is a homology object, and is a module or an abelian group in our main examples.²

Additive functors and the four Hom and tensor examples

Definition 2.18 (Additive functor). Let $\mathcal{C}, \mathcal{D}$ be additive categories. A covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *additive* if its map on every Hom group is a group homomorphism:

$$F(f + g) = F(f) + F(g), \quad F(0) = 0.$$

A contravariant functor is additive when the same condition holds, with $F(f) : F(Y) \rightarrow F(X)$ for $f : X \rightarrow Y$. Equivalently, it is an additive covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

The following four functors take values in $\mathbf{Ab}$, using the Hom groups and tensor products constructed in Section 1.3.

A fixed source: covariant Hom

For a fixed left R -module M , define

$$\operatorname{Hom}_R(M, -) : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \quad A \longmapsto \operatorname{Hom}_R(M, A).$$

A map $u : A \rightarrow B$ acts by postcomposition:

$$u_* : \operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}_R(M, B), \quad f \longmapsto u \circ f.$$

Associativity of composition proves $(vu)_* = v_*u_*$, and the identity acts as the identity. Since $(u + v)f = uf + vf$, this functor is additive.

A fixed target: contravariant Hom

For a fixed left R -module N , define

$$\operatorname{Hom}_R(-, N) : (R\text{-}\mathbf{Mod})^{\text{op}} \longrightarrow \mathbf{Ab}.$$

A map $u : A \rightarrow B$ acts by precomposition in the opposite direction:

$$u^* : \operatorname{Hom}_R(B, N) \longrightarrow \operatorname{Hom}_R(A, N), \quad f \longmapsto f \circ u.$$

²See the Stacks Project, Section 12.5 for abelian categories and Section 12.13 for complexes and homotopies.

Now $(vu)^* = u^*v^*$, and $f(u+v) = fu + fv$ proves additivity. Together, the two Hom functors form the bifunctor $\text{Hom}_R(-, -) : (R\text{-}\mathbf{Mod})^{\text{op}} \times R\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$. The operations commute because $v(fu) = (vf)u$.

Evaluation at 1_R gives $\text{Hom}_R(R, A) \cong U_{\mathbf{Ab}}(A)$, with inverse $a \mapsto (r \mapsto ra)$: every R -linear $f : R \rightarrow A$ satisfies $f(r) = rf(1_R)$. This isomorphism is natural: both routes applied to f give $u(f(1_R)) = (u \circ f)(1_R)$. Concretely, if $R = \mathbb{Z}$, $u : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ is the quotient, and $f_a(k) = ka$, both routes send f_a to $\bar{a}$.

A fixed right module: tensoring on the left

Fix a right R -module N . Using the tensor product constructed in Section 1.3, define

$$N \otimes_R - : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}, \quad A \longmapsto N \otimes_R A.$$

For $u : A \rightarrow B$, the map is

$$1_N \otimes u : N \otimes_R A \rightarrow N \otimes_R B, \quad n \otimes a \longmapsto n \otimes u(a).$$

The proposed map is induced by a biadditive balanced map: its balancing condition is $nr \otimes u(a) = n \otimes ru(a) = n \otimes u(ra)$. The universal property thus makes it well defined.

A fixed left module: tensoring on the right

Fix a left R -module M . There is likewise a covariant functor

$$- \otimes_R M : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Ab}.$$

A right-module map $v : N \rightarrow N'$ acts by

$$v \otimes 1_M : N \otimes_R M \rightarrow N' \otimes_R M, \quad n \otimes m \longmapsto v(n) \otimes m.$$

Here $v(nr) \otimes m = v(n)r \otimes m = v(n) \otimes rm$ verifies balancing. In both functors the identity, composition, and additivity laws follow on pure tensors and hence on all tensors. For example,

$$\begin{aligned} (1 \otimes u')(1 \otimes u) &= 1 \otimes (u'u), \\ 1 \otimes (u + u') &= (1 \otimes u) + (1 \otimes u') \end{aligned}$$

in the second line for parallel maps u, u' . The analogous equations hold when the first tensor variable changes.

More generally, $(v \otimes u)(n \otimes a) = v(n) \otimes u(a)$ makes $- \otimes_R -$ a bifunctor, covariant and additive in each variable separately. Its first argument is a right module and its second a left module. Over a noncommutative ring these domains are different categories, even though over $\mathbb{Z}$ both are canonically $\mathbf{Ab}$.

Remark (Working conventions for homological algebra). From now on, our working categories are abelian and our functors are additive. The concrete setting is module categories: $R\text{-}\mathbf{Mod}$ or $\mathbf{Mod}\text{-}R$, with values usually in $\mathbf{Ab} = \mathbb{Z}\text{-}\mathbf{Mod}$; the identification with $\mathbb{Z}$ -modules uses the integer action described above. Contravariant functors are viewed as functors from opposite categories. We allow an arbitrary abelian target $\mathcal{D}$ in general statements, but module categories suffice for the applications in the main notes. Statements under weaker hypotheses can be read in this working setting.

3 Exact Sequences and Complexes

We work with modules and additive functors, in the abelian-category setting of Section 2.3. Exactness leads to complexes and their homology; homotopy gives comparisons preserved by additive functors. Finally, a short exact sequence of complexes relates their homology through connecting maps.

3.1 Exact sequences and exact functors

Definition 3.1 (Exact sequences). A pair of module maps $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact at B* if $\text{im } f = \ker g$. A sequence is *exact* if this holds at every term with an incoming and an outgoing map. Thus

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

is exact precisely when $\text{im } d_{n+1} = \ker d_n$ for every indicated n . Such a sequence may be finite, infinite in one direction, or infinite in both directions; a sequence continuing through many terms is called a *long exact sequence*. A *short exact sequence* has the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0.$$

It is exact precisely when i is injective, q is surjective, and $\text{im } i = \ker q$.

The zero at an endpoint adds a real condition. For example, $0 \rightarrow A \rightarrow B \rightarrow C$ asserts injectivity at the first nonzero term and exactness at B , but does not require $B \rightarrow C$ to be surjective. Dually, $A \rightarrow B \rightarrow C \rightarrow 0$ requires surjectivity onto C , but does not require $A \rightarrow B$ to be injective. Exactness at B always implies $gf = 0$; the converse fails when $\text{im } f$ is strictly smaller than $\ker g$. For instance, two consecutive zero maps on a nonzero module have zero composite and are not exact at their middle term.

For $n \geq 2$, the sequence $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$ is short exact: multiplication by n is injective, its image is $n\mathbb{Z}$, and the quotient map is surjective with precisely that kernel. Exactness does not assert that there is a homomorphism choosing representatives of all cosets. We examine this extra requirement next.

Direct summands and retracts

Sections and retractions turn lifting and extension properties into direct-sum decompositions.

Definition 3.2 (Direct summand and retract). A module P is a *direct summand* of a module B if $B \cong P \oplus K$ for some module K . It is a *retract* of B if there are maps $i : P \rightarrow B$ and $p : B \rightarrow P$ such that $pi = \text{id}_P$. For a specified submodule $P \subseteq B$, being an internal direct summand means that the specified inclusion admits such a retraction.

The equation $pi = \text{id}_P$ immediately implies that i is injective and p is surjective. It also identifies the projection $ip : B \rightarrow B$ onto $i(P)$: one has $(ip)^2 = ip$.

Lemma 3.3 (Direct summands are exactly retracts). *The two conditions on the object P in the preceding definition are equivalent. More precisely, maps i, p with $pi = \text{id}_P$ give*

$$B = i(P) \oplus \ker p \quad \text{internally, and hence} \quad B \cong P \oplus \ker p.$$

Proof. If $\alpha : P \oplus K \xrightarrow{\cong} B$ is given, take $i(a) = \alpha(a, 0)$ and $p = \pi_P \alpha^{-1}$. Then $pi = \text{id}_P$.

Conversely, suppose that $pi = \text{id}_P$. Here are two ways to establish the claimed decomposition.

Explicit inverse maps. Define

$$\begin{aligned} \Phi : P \oplus \ker p &\longrightarrow B, & \Phi(a, k) &= i(a) + k, \\ \Psi : B &\longrightarrow P \oplus \ker p, & \Psi(b) &= (p(b), b - i(p(b))). \end{aligned}$$

The second coordinate of $\Psi(b)$ belongs to $\ker p$, since $p(b - i(p(b))) = p(b) - p(b) = 0$. Both maps are linear, and

$$\Phi\Psi(b) = i(p(b)) + b - i(p(b)) = b.$$

For $(a, k) \in P \oplus \ker p$, the equation $p(k) = 0$ gives

$$\Psi\Phi(a, k) = (a, i(a) + k - i(a)) = (a, k).$$

Thus the displayed maps are mutually inverse isomorphisms.

Internal direct sum. Every $b \in B$ has the expression $b = i(p(b)) + (b - i(p(b)))$, with the two terms in $i(P)$ and $\ker p$. Their sum is therefore B . If $i(a) \in \ker p$, then $a = pi(a) = 0$, so the intersection is zero. Hence every element has a unique expression as a sum of one element from each submodule, which is exactly the internal direct-sum condition. $\square$

Splitting a specified short exact sequence

For a fixed short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0,$$

a *section* of q is a map $s : C \rightarrow B$ with $qs = \text{id}_C$. A *retraction* of i is a map $r : B \rightarrow A$ with $ri = \text{id}_A$. The direction of each identity matters: a section is a right inverse of q , while a retraction is a left inverse of i .

Lemma 3.4 (Splitting lemma). *For this specified short exact sequence, the following are equivalent:*

1. The map q has a module section.
2. The map i has a module retraction.
3. There is an isomorphism $\Phi : A \oplus C \xrightarrow{\cong} B$ such that $\Phi(a, 0) = i(a)$ and $q\Phi(a, c) = c$.

When these conditions hold, the sequence is called *split exact*. Condition (3) says that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{\pi_C} & C \longrightarrow 0 \\ & & \parallel & & \downarrow \cong \Phi & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{q} & C \longrightarrow 0. \end{array}$$

Proof. **A section gives the compatible decomposition.** Suppose $qs = \text{id}_C$. Put $\Phi(a, c) = i(a) + s(c)$. For $b \in B$, one has $q(b - s(q(b))) = 0$, so exactness gives a unique $a \in A$ with $i(a) = b - s(q(b))$. Define

$$r(b) = i^{-1}(b - s(q(b))), \quad \Psi(b) = (r(b), q(b)).$$

Here i^{-1} means the inverse isomorphism $i(A) \rightarrow A$, used only on elements of $i(A) = \ker q$. Thus r and Ψ are linear. Directly,

$$ri = \text{id}_A, \quad rs = 0, \quad \Phi\Psi(b) = b, \quad \Psi\Phi(a, c) = (a, c).$$

The map Φ is therefore an isomorphism with the required compatibilities, and r is a retraction.

A retraction gives a section. Conversely, suppose $ri = \text{id}_A$. We claim that the restriction $q|_{\ker r} : \ker r \rightarrow C$ is an isomorphism. If $b \in \ker r$ and $q(b) = 0$, write $b = i(a)$ by exactness. Then $0 = r(b) = a$, so $b = 0$. For surjectivity, choose a lift $b \in B$ of $c \in C$. The element $b - i(r(b))$ lies in $\ker r$ and still maps to c . Thus the restriction is bijective and linear. Its inverse, followed by $\ker r \hookrightarrow B$, defines a section $s : C \rightarrow B$.

A compatible decomposition gives both maps. Finally, if Φ is as in (3), set $s(c) = \Phi(0, c)$ and $r = \pi_A \Phi^{-1}$. The stated compatibility equations imply $qs = \text{id}_C$ and $ri = \text{id}_A$. $\square$

Splitting concerns the *actual arrows* i and q . An unspecified isomorphism $B \cong A \oplus C$ does not by itself assert the compatibility in the diagram. In general the abstract isomorphism type of the middle module alone does not determine whether the given sequence splits.

For a concrete illustration, let $B = \bigoplus_{j \geq 1} (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ and embed $A = \mathbb{Z}/2\mathbb{Z}$ as the subgroup generated by 2 in the first $\mathbb{Z}/4\mathbb{Z}$ coordinate. Set $C = B/i(A)$. The quotient replaces that one $\mathbb{Z}/4\mathbb{Z}$ summand by a $\mathbb{Z}/2\mathbb{Z}$ summand, so, by reindexing the countably many summands, $C \cong B$ and $A \oplus C \cong B$. Nevertheless, no retraction $r : B \rightarrow A$ exists: if x is the generator of that first $\mathbb{Z}/4\mathbb{Z}$ coordinate, then $r(i(\bar{1})) = r(2x) = 2r(x) = 0$, contrary to $ri = \text{id}_A$.

Key point. To prove that a short exact sequence splits, exhibit a section, a retraction, or a direct-sum isomorphism compatible with its specified injection and surjection. The splitting lemma converts any one of these data into the other two.

Exactness of functors

Definition 3.5 (Left, right, and exact functors). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an additive covariant functor between abelian categories. It is *left exact* if every exact sequence $0 \rightarrow A \xrightarrow{u} B \xrightarrow{v} C$ gives an exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C),$$

and *right exact* if every exact sequence $A \xrightarrow{u} B \xrightarrow{v} C \rightarrow 0$ gives an exact sequence

$$F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \longrightarrow 0.$$

It is *exact* if it is both left and right exact. In the first definition v need not be epic; in the second u need not be monic.

For an additive contravariant functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$, use the same definitions in $\mathcal{C}^{\text{op}}$. Thus left exactness takes $A \rightarrow B \rightarrow C \rightarrow 0$ to $0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A)$, and right exactness takes $0 \rightarrow A \rightarrow B \rightarrow C$ to $F(C) \rightarrow F(B) \rightarrow F(A) \rightarrow 0$. All displayed arrows are induced by F .

Remark (Testing short exact sequences). It suffices to test either property on short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, keeping the appropriate zero at the output. Here is why the endpoint omitted from the input definition causes no difficulty.

For the left exact test, first apply it to a monomorphism and its cokernel: F preserves monomorphisms. Given an exact sequence $0 \rightarrow A \rightarrow B \xrightarrow{v} C$, factor $v = jp$ through its image I , with $p : B \twoheadrightarrow I$ and $j : I \hookrightarrow C$. The test applied to $0 \rightarrow A \rightarrow B \rightarrow I \rightarrow 0$, together with the monicity of $F(j)$, gives

$$\text{im } F(A \rightarrow B) = \ker F(p) = \ker(F(j)F(p)) = \ker F(v).$$

It also gives the required injection $F(A) \rightarrow F(B)$.

For the right exact test, short exact sequences first show that F preserves epimorphisms. Given an exact $A \xrightarrow{u} B \rightarrow C \rightarrow 0$, factor $u = jp$ through $K = \text{im } u$, where $p : A \rightarrow K$ is epic and $j : K \hookrightarrow B$ is monic. Applying the test to $0 \rightarrow K \rightarrow B \rightarrow C \rightarrow 0$ gives

$$\text{im } F(u) = \text{im}(F(j)F(p)) = \text{im } F(j) = \ker F(B \rightarrow C),$$

and $F(B) \rightarrow F(C)$ is epic. The converse is immediate from the definitions; the contravariant cases follow in $\mathcal{C}^{\text{op}}$.

Consequently left exact covariant functors preserve monomorphisms, and right exact covariant functors preserve epimorphisms. Contravariantly these become, respectively, epimorphisms sent to monomorphisms and monomorphisms sent to epimorphisms. Preserving only these endpoint conditions is weaker than exactness; every functor preserves isomorphisms, as already noted in Section 2.2.

Remark (Why the endpoint conditions matter). Exactness only at the middle of $A \rightarrow B \rightarrow C$ need not be preserved by a functor that is merely left or right exact. For $n \geq 2$, $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ sends the middle-exact fragment $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$ to $0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$, which is not exact at its middle term. The input lacks injectivity of its first map. Dually, $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} -$ sends the middle-exact fragment $0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$ to $0 \rightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z}$; the input lacks surjectivity of its last map. These computations use Section 1.3. The asserted one-sided exactness of Hom and tensor is proved in Section 3.2 of the *Main Notes*.

Proposition 3.6 (Exact functors preserve exact sequences of every length). *An exact functor preserves exactness at every position of a finite or infinite exact sequence, reversing arrows in the contravariant case.*

Proof. For an exact fragment $A \xrightarrow{f} B \xrightarrow{g} C$, put $K = \text{im } f = \ker g$ and $J = \text{im } g$. Factor $f = ip$ and $g = jq$ as

$$A \xrightarrow{p} K \xrightarrow{i} B, \quad B \xrightarrow{q} J \xrightarrow{j} C,$$

where p, q are epic and i, j monic. Applying F to $0 \rightarrow K \xrightarrow{i} B \xrightarrow{q} J \rightarrow 0$ gives $\text{im } F(i) = \ker F(q)$. By the preservation of epimorphisms and monomorphisms just proved,

$$\text{im } F(f) = \text{im } F(i) = \ker F(q) = \ker F(g).$$

This checks each position independently, so the sequence need not be finite. The same argument in $\mathcal{C}^{\text{op}}$ proves the contravariant case. In an abstract abelian category the equalities are equalities of subobjects; for modules they are equalities of submodules. $\square$

3.2 Complexes and homology

We work with module complexes; the same constructions make sense in an abelian category, as in Section 2.3.

Chain complexes and maps

A chain complex retains the equation $d^2 = 0$ from an exact sequence, while allowing images to be smaller than kernels. This includes the sequences obtained by applying a functor that does not preserve exactness.

Definition 3.7 (Nonnegative chain complex). A nonnegative *chain complex* $C_{\bullet}$ of left R -modules is a family $(C_n)_{n \geq 0}$ with maps $d_n^C : C_n \rightarrow C_{n-1}$ for $n \geq 1$, satisfying $d_n^C d_{n+1}^C = 0$. We extend it by zero in negative degrees and put $d_0^C = 0$.

The same definition in all integer degrees gives $\text{Ch}(\mathcal{A})$; a subscript ≥ 0 specifies the nonnegative complexes, which are our default examples. Superscripts such as d^C identify the complex; the differential always lowers degree.

Definition 3.8 (Chain map). A *chain map* $f : C_\bullet \rightarrow D_\bullet$ is a family of module maps $f_n : C_n \rightarrow D_n$ satisfying

$$d_n^D f_n = f_{n-1} d_n^C \quad (n \geq 1).$$

Equivalently, each square in the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_2 & \xrightarrow{d_2^C} & C_1 & \xrightarrow{d_1^C} & C_0 \longrightarrow 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \cdots & \longrightarrow & D_2 & \xrightarrow{d_2^D} & D_1 & \xrightarrow{d_1^D} & D_0 \longrightarrow 0. \end{array}$$

Identity maps and composition are degree-wise: $(gf)_n = g_n f_n$, with the chain-map equation following from $d_n^E g_n f_n = g_{n-1} d_n^D f_n = g_{n-1} f_{n-1} d_n^C$. Thus complexes and chain maps form the category $\text{Ch}_{\geq 0}(R\text{-Mod})$. Degree-wise sums are also chain maps, by additivity of composition.

Cycles, boundaries, and induced maps

For every degree n , define submodules

$$Z_n(C) = \ker d_n^C, \quad B_n(C) = \text{im } d_{n+1}^C.$$

Their elements are *cycles* and *boundaries*. Since $d_n d_{n+1} = 0$, one has $B_n(C) \subseteq Z_n(C)$, so define

$$H_n(C) = Z_n(C)/B_n(C).$$

Two cycles determine the same class exactly when their difference is a boundary. A complex is exact at C_n exactly when $H_n(C) = 0$, and it is *acyclic* if every $H_n(C)$ is zero.

Proposition 3.9 (Homology is a functor). *For every $n \geq 0$, a chain map $f : C_\bullet \rightarrow D_\bullet$ induces an R -linear map*

$$H_n(f) : H_n(C) \longrightarrow H_n(D), \quad [c] \longmapsto [f_n(c)].$$

These maps define a functor $H_n : \text{Ch}_{\geq 0}(R\text{-Mod}) \rightarrow R\text{-Mod}$.

Proof. For a cycle c , the chain-map equation gives $d_n^D f_n(c) = f_{n-1} d_n^C(c) = 0$; for a boundary $d_{n+1}^C(x)$, it gives $f_n d_{n+1}^C(x) = d_{n+1}^D f_{n+1}(x)$, again a boundary. Thus f_n restricts to cycles and carries boundaries into boundaries, so it induces the stated linear map on the quotient, independent of the representative. The formulas on classes immediately give $H_n(\text{id}_C) = \text{id}_{H_n(C)}$ and $H_n(gf) = H_n(g)H_n(f)$. □

The same construction works in an abelian category, using the factorization $B_n(C) \rightarrow Z_n(C)$ and its cokernel in place of elementwise quotients.

Example 3.10 (The same terms can have different homology). Fix a nonzero module M . Put $C_1 = C_0 = M$ and all other terms zero. With $d_1 = 0$, one has $H_1(C) = M = H_0(C)$. With $d_1 = \text{id}_M$, both homology groups vanish. Both choices satisfy $d^2 = 0$; the objects alone do not determine the complex or its homology.

Example 3.11 (A complex with a nonzero boundary subgroup). Over $\mathbb{Z}$, the complex with terms $\mathbb{Z}$ in degrees 2, 1, 0 and differentials $d_2 = \times 2$, $d_1 = 0$ satisfies $d_1 d_2 = 0$. But $Z_1 = \mathbb{Z}$ and $B_1 = 2\mathbb{Z}$, so $H_1 \cong \mathbb{Z}/2\mathbb{Z}$. The element 1 is a cycle that is not a boundary.

Cochain complexes and their degree convention

Definition 3.12 (Cochain complex and cohomology). A *cochain complex* in an abelian category is a family $(C^n)_{n \in \mathbb{Z}}$ with morphisms $d^n : C^n \rightarrow C^{n+1}$ satisfying $d^{n+1}d^n = 0$. A cochain map $f : C^\bullet \rightarrow D^\bullet$ has components $f^n : C^n \rightarrow D^n$ satisfying

$$d_D^n f^n = f^{n+1} d_C^n.$$

Define

$$Z^n(C) = \ker d^n, \quad B^n(C) = \operatorname{im} d^{n-1}, \quad H^n(C) = Z^n(C)/B^n(C).$$

The equation $d^2 = 0$ gives $B^n \subseteq Z^n$. In modules, the elements of Z^n and B^n are *cocycles* and *coboundaries*. A nonnegative cochain complex has $C^n = 0$ for $n < 0$, hence $B^0 = 0$.

The direction distinguishes the two notations:

$$\begin{aligned} \text{chain: } & C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}, \\ \text{cochain: } & C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1}. \end{aligned}$$

The same commuting-square argument shows that cochain maps preserve cocycles and coboundaries, inducing $H^n(f)$. With degree-wise identities and composition, cochain complexes form a category and H^n is a functor. Reindexing by $\hat{C}_n = C^{-n}$ and $\hat{d}_n = d^{-n}$ converts cochains to chains, with $H_n(\hat{C}) = H^{-n}(C)$; this uses all integer degrees even when $C^\bullet$ is nonnegative.

Why Hom reverses the direction

Fix a left R -module N . If $C_\bullet$ is a chain complex of left R -modules, define

$$\begin{aligned} \operatorname{Hom}_R(C_\bullet, N)^n &= \operatorname{Hom}_R(C_n, N), \\ \delta^n : \operatorname{Hom}_R(C_n, N) &\longrightarrow \operatorname{Hom}_R(C_{n+1}, N), \quad \delta^n(\phi) = \phi d_{n+1}. \end{aligned}$$

Precomposition reverses the arrow $C_{n+1} \rightarrow C_n$, and $\delta^{n+1}\delta^n(\phi) = \phi d_{n+1}d_{n+2} = 0$. Thus Hom converts this chain complex into a cochain complex of abelian groups. A chain map $f : C_\bullet \rightarrow D_\bullet$ induces the cochain map in the reversed direction by $\psi \mapsto \psi f_n$.

Conversely, if $I^\bullet$ is a cochain complex, the groups

$$\begin{aligned} K_n &= \operatorname{Hom}_R(I^n, N), \\ \partial_n : K_n &\longrightarrow K_{n-1}, \quad \partial_n(\phi) = \phi d_I^{n-1} \end{aligned}$$

form a chain complex: $\partial_{n-1}\partial_n = 0$ follows from $d_I^{n-1}d_I^{n-2} = 0$. In both constructions the target is fixed, and the contravariance is in the complex being mapped *from*. For a noncommutative ring these Hom groups are naturally abelian groups; no unspecified left R -module structure is being used.

Evaluation on homology classes

Let $C_\bullet$ be a complex of abelian groups and let A be an abelian group. Evaluation induces a biadditive pairing

$$H_n(C) \times H^n(\operatorname{Hom}_{\mathbb{Z}}(C_\bullet, A)) \longrightarrow A, \quad ([z], [\phi]) \longmapsto \phi(z).$$

Indeed, the cocycle condition says $\phi d_{n+1} = 0$, so replacing z by $z + d_{n+1}w$ does not change the value. Replacing ϕ by $\phi + \psi d_n$ also changes nothing, because $d_n z = 0$. It follows that there is a natural homomorphism

$$H^n(\operatorname{Hom}_{\mathbb{Z}}(C_{\bullet}, A)) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(C), A).$$

This need not be an isomorphism. For example, let $C_1 = C_0 = \mathbb{Z}$ with d_1 multiplication by $n \geq 2$. Then $H_1(C) = 0$, whereas $H^1(\operatorname{Hom}_{\mathbb{Z}}(C, \mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$. Every evaluation on $H_1(C)$ is zero, although the cohomology group is nonzero. The universal coefficient theorems, discussed later in Section 6.4 of the *Main Notes*, account for this additional cohomological information. Evaluation alone does not assert a perfect duality between homology and cohomology.

3.3 Homotopy and additive functors

Chain homotopy permits maps to differ by a controlled combination of differentials. This will later compare resolutions that have different terms while resolving the same module.

Definition 3.13 (Chain homotopy). For maps $f, g : C_{\bullet} \rightarrow D_{\bullet}$ of nonnegative chain complexes, a *chain homotopy* from f to g is a family $h_n : C_n \rightarrow D_{n+1}$ such that

$$f_n - g_n = d_{n+1}^D h_n + h_{n-1} d_n^C, \quad h_{-1} = 0.$$

We write $f \simeq g$. The homotopy raises degree by one, so both terms on the right have the same source and target as f_n . In degree zero the equation is $f_0 - g_0 = d_1^D h_0$. For integer-indexed complexes the same equation is imposed in every degree, without the convention $h_{-1} = 0$.

Proposition 3.14 (Homotopic maps induce the same homology maps). *If $f \simeq g$, then $H_n(f) = H_n(g)$ for every n .*

Proof. For a cycle $c \in C_n$, the homotopy equation becomes

$$f_n(c) - g_n(c) = d_{n+1}^D h_n(c) + h_{n-1}(d_n^C(c)) = d_{n+1}^D h_n(c).$$

The difference is a boundary, so $[f_n(c)] = [g_n(c)]$ in $H_n(D)$. Every homology class has a cycle representative; hence the induced maps agree on every class. In an arbitrary abelian category, restrict the homotopy identity to $\ker d_n^C$: the difference factors through $\operatorname{im} d_{n+1}^D$, so its map on the quotient defining homology is zero. $\square$

Definition 3.15 (Cochain homotopy). For cochain maps $f, g : C^{\bullet} \rightarrow D^{\bullet}$, a *cochain homotopy* is a family $h^n : C^n \rightarrow D^{n-1}$ satisfying

$$f^n - g^n = d_D^{n-1} h^n + h^{n+1} d_C^n.$$

It lowers degree by one. For nonnegative cochain complexes we put $h^0 = 0$; the degree-zero equation is then $f^0 - g^0 = h^1 d_C^0$.

If $z \in C^n$ is a cocycle, then

$$f^n(z) - g^n(z) = d_D^{n-1} h^n(z),$$

which is a coboundary. Thus cochain-homotopic maps induce equal cohomology maps in every degree. This proves the cochain assertion directly, including degree zero: there the displayed difference vanishes on every cocycle because the incoming differential is zero. The definitions of cochain homotopy equivalence and contractibility are obtained by using this homotopy relation in the definitions below.

Homotopy is an equivalence relation on chain maps: use 0 for reflexivity, $-h$ for symmetry, and add homotopies for transitivity. It also respects composition. If $a : C' \rightarrow C$ and $b : D \rightarrow D'$ are chain maps, the maps $b_{n+1}h_n a_n$ give a homotopy from bfa to bga , by the chain map equations. Thus we can form the *homotopy category* $K_{\geq 0}(R\text{-}\mathbf{Mod})$: its objects are complexes and its morphisms are chain maps modulo chain homotopy.

Homology induces a map

$$\mathrm{Hom}_{K_{\geq 0}(R\text{-}\mathbf{Mod})}(C, D) \longrightarrow \prod_{n \geq 0} \mathrm{Hom}_R(H_n(C), H_n(D)). \quad (\mathrm{H})$$

To ask whether equal homology maps imply homotopy is to ask whether this map is injective for the particular pair of complexes. The answer is not always yes.

Example 3.16 (Equal homology maps need not imply homotopy, even for free terms). Fix $n \geq 2$. Let $P_1 = P_0 = \mathbb{Z}$ with $d_1^P = \times n$, and let $Q_1 = \mathbb{Z}$ with $Q_j = 0$ for $j \neq 1$. All their terms are free, and both complexes are nonnegative and bounded. Define $f : P \rightarrow Q$ by $f_1 = \mathrm{id}_{\mathbb{Z}}$, $f_0 = 0$ and zero in every other degree. It is a chain map, since both sides of $d_1^Q f_1 = f_0 d_1^P$ are zero.

The only nonzero homology of P is $H_0(P) = \mathbb{Z}/n\mathbb{Z}$; the only nonzero homology of Q is $H_1(Q) = \mathbb{Z}$. Hence every induced $H_j(f)$ is zero, just as for the zero chain map. But a homotopy $f \simeq 0$ would require, in degree one,

$$\mathrm{id}_{\mathbb{Z}} = d_2^Q h_1 + h_0 d_1^P = h_0 \circ (\times n).$$

Every homomorphism $h_0 : \mathbb{Z} \rightarrow \mathbb{Z}$ is multiplication by an integer k , so this would say $1 = kn$, which is impossible.

More precisely, a chain map here is any integer multiple of f_1 , and two such multiples are homotopic exactly when their coefficients differ by a multiple of n . Therefore $\mathrm{Hom}_{K_{\geq 0}(\mathbf{Ab})}(P, Q) \cong \mathbb{Z}/n\mathbb{Z}$, whereas the right side of (H) is zero. Having free terms alone does not make homology faithful.

Homotopy equivalence and contractibility

Definition 3.17 (Homotopy equivalence and contractibility). A chain map $f : C \rightarrow D$ is a *chain homotopy equivalence* if there is a chain map $g : D \rightarrow C$ with $gf \simeq \mathrm{id}_C$ and $fg \simeq \mathrm{id}_D$. A complex is *contractible* if $\mathrm{id}_C \simeq 0$. A map inducing isomorphisms on all homology objects is called a *quasi-isomorphism*.

The preceding proposition shows that every homotopy equivalence is a quasi-isomorphism and every contractible complex is acyclic. Neither converse holds for arbitrary complexes.

Example 3.18 (The disk complex is explicitly contractible). For any module M , let $D_1 = D_0 = M$, $d_1 = \mathrm{id}_M$, and all other terms zero. Set $h_0 : D_0 \rightarrow D_1$ equal to id_M and $h_j = 0$ otherwise. In degree zero, $d_1 h_0 = \mathrm{id}_{D_0}$; in degree one, $h_0 d_1 = \mathrm{id}_{D_1}$. The equations in the other degrees are zero. Thus $\mathrm{id}_D = dh + hd$, an explicit contraction.

The maps between D and the zero complex are zero, and this contraction makes their composite on D homotopic to its identity. Thus D is homotopy equivalent to zero despite having nonzero terms. More generally, the two-term complex $M \xrightarrow{\mathrm{id}} M$ in degrees $r, r-1$ contracts by $h_{r-1} = \mathrm{id}_M$. Such disk complexes describe extra terms that can be added to a resolution without changing its homotopy type.

Example 3.19 (Acyclic need not mean contractible). Place the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in degrees 2, 1, 0, with q the quotient map, and call the resulting complex $C_\bullet$. It is acyclic, so id_C and 0 induce the same homology maps. A contraction would require $d_1 h_0 = \text{id}_{\mathbb{Z}/2\mathbb{Z}}$, where $h_0 : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$. But every such homomorphism is zero, so no contraction exists. The map $C \rightarrow 0$ is consequently a quasi-isomorphism that is not a homotopy equivalence.

Remark. For complexes of vector spaces over a field, two chain maps induce the same maps on homology if and only if they are chain homotopic; this uses splittings of the cycle and boundary subspaces and need not hold for complexes whose terms are merely free modules.

Additive functors preserve homotopy

Proposition 3.20 (Additive functors on complexes and homotopies). *Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an additive covariant functor. Applying F to every term, differential, and component of a chain map defines an additive functor*

$$\text{Ch}(F) : \text{Ch}(\mathcal{C}) \longrightarrow \text{Ch}(\mathcal{D}),$$

which restricts to nonnegative complexes and preserves homotopies, homotopy equivalences, and contractibility. The same holds for cochain complexes.

An additive contravariant functor instead takes chain complexes to cochain complexes and cochain complexes to chain complexes, reversing maps between them; it also preserves homotopies, homotopy equivalences, and contractibility. Exactness is not required.

Proof. For a covariant additive F , the relation $d_{n-1}d_n = 0$ gives

$$F(d_{n-1})F(d_n) = F(d_{n-1}d_n) = F(0) = 0.$$

Zero objects are preserved since $\text{id}_{F(0)} = F(\text{id}_0) = 0$, so the construction also respects nonnegative complexes.

For a chain map $f : C_\bullet \rightarrow E_\bullet$, applying F to $d_n^E f_n = f_{n-1}d_n^C$ gives

$$F(d_n^E)F(f_n) = F(f_{n-1})F(d_n^C).$$

Identities and composition are preserved degree-wise, and $F(f_n + g_n) = F(f_n) + F(g_n)$ proves additivity on chain maps. If $h_n : C_n \rightarrow E_{n+1}$ is a homotopy from f to g , then

$$F(f_n) - F(g_n) = F(d_{n+1}^E)F(h_n) + F(h_{n-1})F(d_n^C).$$

Thus $F(h_n)$ gives the image homotopy; the cochain case is identical with homotopies lowering degree.

For contravariant F , define

$$F(C)^n = F(C_n), \quad \delta_{F(C)}^n = F(d_{n+1}^C).$$

Reversing composition verifies the cochain and cochain-map identities. For $h : f \simeq g$, put $k^n = F(h_{n-1}) : F(E_n) \rightarrow F(C_{n-1})$. Applying F to the homotopy equation, again reversing composition, gives

$$F(f_n) - F(g_n) = \delta_{F(C)}^{n-1}k^n + k^{n+1}\delta_{F(E)}^n,$$

the cochain homotopy equation, with $k^0 = 0$ in the nonnegative case.

Conversely, for a cochain complex put $F(C)_n = F(C^n)$ and $\partial_n^{F(C)} = F(d_C^{n-1})$. A cochain homotopy $h^n : C^n \rightarrow E^{n-1}$ becomes the chain homotopy $k_n = F(h^{n+1}) : F(E^n) \rightarrow F(C^{n+1})$, with $k_{-1} = 0$ in the nonnegative case, by the same calculation. In all cases, homotopy inverse maps and contractions remain such after applying F . $\square$

For the four module functors of Section 2.3, the induced differentials follow from their actions on maps:

complex	differential on a typical element
$\text{Hom}_R(M, C_\bullet)$	$f \mapsto d_C \circ f$
$\text{Hom}_R(C_\bullet, N)$	$f \mapsto f \circ d_C$
$N^\Gamma \otimes_R C_\bullet$	$n \otimes c \mapsto n \otimes d_C c$
$Q_\bullet \otimes_R M$	$q \otimes m \mapsto d_C q \otimes m$

Here $C_\bullet$ consists of left modules, $Q_\bullet$ of right modules, M, N are fixed left modules, and N^Γ is a fixed right module. Only the second row is a cochain complex; its degree- n differential is precomposition with d_{n+1}^C .

Since $\mathcal{D}$ is abelian, homology can be applied after a covariant additive functor, giving $H_n \circ \text{Ch}(F)$. A homotopy equivalence gives an isomorphism after both operations. The analogous statement uses cohomology for a contravariant functor. Agreement of the original homology groups alone is insufficient.

Example 3.21 (Acyclicity alone does not survive an additive functor). Use the acyclic complex $C_\bullet$ from Example 3.19, and apply $F = (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} -$. The natural tensor identifications give

$$F(C_\bullet) : \quad 0 \longrightarrow \underset{\text{degree 2}}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{0} \underset{\text{degree 1}}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{id}} \underset{\text{degree 0}}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0.$$

Tensoring multiplication by 2 gives zero, whereas tensoring the quotient q gives the identity in these coordinates. Consequently,

$$H_2(F(C_\bullet)) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(F(C_\bullet)) = H_0(F(C_\bullet)) = 0.$$

Thus the quasi-isomorphism $C_\bullet \rightarrow 0$ becomes a map $F(C_\bullet) \rightarrow 0$ that is not a quasi-isomorphism. There is no contradiction with preservation of contractibility: the earlier example proved that $C_\bullet$ is not contractible.

This is why the comparison theorem for resolutions will construct maps inverse up to homotopy. Applying an additive functor and then homology produces the same invariant from either resolution; an isomorphism between their original homology groups would not justify this conclusion.

Key point. Additive functors preserve homotopies and homotopy equivalences, even when they destroy exactness. This distinction is essential to the definition of derived functors.

3.4 Long exact sequences of complexes

Let $\mathcal{A}$ be an abelian category, in the sense of Section 2.3. Write $\text{Ch}(\mathcal{A})$ for the category of chain complexes indexed by $\mathbb{Z}$ and chain maps. Nonnegative complexes form the subcategory in which the negative-degree objects are zero.

Proposition 3.22. *The category $\text{Ch}(\mathcal{A})$ is abelian. Kernels, cokernels, images, and finite direct sums are computed degree by degree. The same statements hold for nonnegative chain complexes and for cochain complexes.*

Proof. Chain maps add degree-wise and composition is bilinear. The zero complex and degree-wise biproducts supply the remaining additive structure.

For a chain map $f : C \rightarrow D$, put $K_n = \ker f_n$ with inclusion $j_n : K_n \rightarrow C_n$. The chain map equation gives

$$f_{n-1}d_C j_n = d_D f_n j_n = 0.$$

The kernel property defines $d_K : K_n \rightarrow K_{n-1}$ by $j_{n-1}d_K = d_C j_n$; monicity of the inclusions gives $d_K^2 = 0$. For a chain map $u : X \rightarrow C$ with $fu = 0$, the unique degreewise factorizations through j_n commute with differentials. Hence $K \rightarrow C$ is the kernel in the category of complexes.

Dually, put $Q_n = \text{coker } f_n$ and write $\pi_n : D_n \rightarrow Q_n$. The equality $\pi_{n-1}d_D f_n = \pi_{n-1}f_{n-1}d_C = 0$ defines d_Q by $d_Q \pi_n = \pi_{n-1}d_D$. Epicity of π_n gives $d_Q^2 = 0$, and the degreewise quotient maps give the cokernel universal property for complexes.

Images and coimages are therefore computed degreewise. Their canonical comparison is a chain map whose components are isomorphisms in $\mathcal{A}$; the componentwise inverses also commute with differentials. Thus the comparison is an isomorphism of complexes, proving that the category is abelian. These constructions preserve zeros in negative degrees; reindexing gives the cochain statement. $\square$

Definition 3.23 (Short exact sequence of complexes). A sequence

$$0 \longrightarrow C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \longrightarrow 0$$

is *short exact* if i and p are chain maps and each sequence $0 \rightarrow C'_n \xrightarrow{i_n} C_n \xrightarrow{p_n} C''_n \rightarrow 0$ is exact in $\mathcal{A}$. Thus adjacent degrees form a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \longrightarrow & 0 \\ & & \downarrow d' & & \downarrow d & & \downarrow d'' & & \\ 0 & \longrightarrow & C'_{n-1} & \xrightarrow{i_{n-1}} & C_{n-1} & \xrightarrow{p_{n-1}} & C''_{n-1} & \longrightarrow & 0. \end{array}$$

The analogous definition for cochain complexes uses differentials from degree n to degree $n + 1$.

A *single complex* is exact, or acyclic, when its homology vanishes. A *sequence of complexes* is exact degreewise; its individual complexes need not be acyclic. For example, a short exact sequence of modules placed in degree zero remains a short exact sequence of complexes, with those modules as its degree-zero homology.

Theorem 3.24 (Long exact sequence in homology). A short exact sequence of chain complexes in an abelian category,

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0,$$

has natural connecting morphisms $\partial_n : H_n(C'') \rightarrow H_{n-1}(C')$. Together with the induced homology maps, they give a long exact sequence

$$\cdots \longrightarrow H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \longrightarrow \cdots.$$

For nonnegative complexes its right end is

$$H_0(C') \xrightarrow{i_*} H_0(C) \xrightarrow{p_*} H_0(C'') \longrightarrow 0.$$

Proof. We give the module proof; Appendix A justifies the same argument in an arbitrary abelian category. Degree subscripts on i, p, d are suppressed when determined by their sources.

Constructing the connecting map. Take $z \in C''_n$ with $d''z = 0$. Surjectivity of p_n supplies $b \in C_n$ such that $p(b) = z$. Then

$$p(db) = d''p(b) = d''z = 0.$$

Exactness in degree $n - 1$ gives a unique $a \in C'_{n-1}$ such that $i(a) = db$. This element is a cycle, since

$$i(d'a) = di(a) = d^2b = 0$$

and i_{n-2} is injective. Define

$$\partial_n[z] = [a].$$

Thus we lift a cycle, differentiate, and take the preimage under the injection. The lift itself need not be a cycle.

Independence of the lift. If $\tilde{b}$ is a second lift of the same z , then $\tilde{b} - b = i(u)$ for some $u \in C'_n$. Therefore

$$d\tilde{b} = db + i(d'u) = i(a + d'u).$$

The new answer differs from a by a boundary.

Independence of the representative. If $\tilde{z} = z + d''w$ with $w \in C''_{n+1}$, choose $v \in C_{n+1}$ with $p(v) = w$. The element $b + dv$ lifts $\tilde{z}$, and $d(b + dv) = db$. Thus this particular new lift gives the same a ; independence of the lift handles every other choice. Hence ∂_n is well defined.

Additivity and linearity. Lifts b_1, b_2 of cycles z_1, z_2 give the lift $b_1 + b_2$ of $z_1 + z_2$. If $db_j = i(a_j)$, its differential is $i(a_1 + a_2)$. This proves additivity. Also, rb lifts rz and $d(rb) = i(ra)$, so ∂_n is R -linear for complexes of left R -modules.

Exactness at $H_n(C)$. Since $pi = 0$, we have $p_*i_* = 0$. Conversely, suppose $b \in C_n$ is a cycle and $p_*[b] = 0$. Then $p(b) = d''w$ for some $w \in C''_{n+1}$. Lift w to $v \in C_{n+1}$. The element $b - dv$ lies in $\ker p = \text{im } i$, so $b - dv = i(a)$ for some $a \in C'_n$. It is a cycle; hence $i(d'a) = d(b - dv) = 0$ and $d'a = 0$. Therefore $[b] = i_*[a]$, proving $\ker p_* = \text{im } i_*$.

Exactness at $H_n(C'')$. If b is a cycle in C_n , it is already a cycle lift of $p(b)$, so the connecting construction gives $\partial_n p_*[b] = 0$. Conversely, suppose $\partial_n[z] = 0$. Choose b and a as in the construction. Since $[a] = 0$, write $a = d'u$ with $u \in C'_n$. Now

$$d(b - i(u)) = i(a) - i(d'u) = 0, \quad p(b - i(u)) = z.$$

Thus $[z] = p_*[b - i(u)]$, proving $\ker \partial_n = \text{im } p_*$.

Exactness at $H_{n-1}(C')$. The equality $i(a) = db$ in the definition of ∂_n shows $i_*\partial_n = 0$. Conversely, let $a \in C'_{n-1}$ be a cycle whose class maps to zero in $H_{n-1}(C)$. Then $i(a) = db$ for some $b \in C_n$. The element $z = p(b)$ is a cycle, since $d''z = p(db) = pi(a) = 0$. Using b as its lift gives $\partial_n[z] = [a]$. Hence $\ker i_* = \text{im } \partial_n$. These three checks, for all n , prove exactness at every position. In the nonnegative case $H_{-1}(C') = 0$, which gives the stated tail.

Naturality. Suppose there is a morphism of short exact sequences of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C' & \xrightarrow{i} & C & \xrightarrow{p} & C'' \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & D' & \xrightarrow{j} & D & \xrightarrow{q} & D'' \longrightarrow 0. \end{array}$$

If b lifts a cycle z and $db = i(a)$, then $\beta(b)$ lifts $\gamma(z)$ and $d\beta(b) = \beta(db) = \beta i(a) = j\alpha(a)$. Consequently the connecting square commutes:

$$\begin{array}{ccc} H_n(C'') & \xrightarrow{\partial_n} & H_{n-1}(C') \\ H_n(\gamma) \downarrow & & \downarrow H_{n-1}(\alpha) \\ H_n(D'') & \xrightarrow{\partial_n} & H_{n-1}(D'). \end{array}$$

The remaining squares commute by functoriality of homology, proving naturality of the whole long exact sequence. $\square$

Theorem 3.25 (Long exact sequence in cohomology). *A short exact sequence of cochain complexes in an abelian category $0 \rightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \rightarrow 0$ induces a natural long exact sequence*

$$\cdots \rightarrow H^n(C') \xrightarrow{i^*} H^n(C) \xrightarrow{p^*} H^n(C'') \xrightarrow{\delta^n} H^{n+1}(C') \rightarrow \cdots.$$

For nonnegative cochain complexes it begins $0 \rightarrow H^0(C') \rightarrow H^0(C) \rightarrow H^0(C'') \rightarrow H^1(C')$.

Proof. Reindex by $\hat{C}_n = C^{-n}$ and apply Theorem 3.24; a decrease of one in chain degree is an increase of one in cochain degree. Explicitly, for a cocycle $z'' \in C''^n$, choose $z \in C^n$ with $p(z) = z''$. There is a unique $w \in C'^{n+1}$ with $i(w) = dz$, and $d'w = 0$. The formula is

$$\delta^n[z''] = [w].$$

Well-definedness, exactness at all three types of terms, and naturality follow from the chain theorem under this reindexing. $\square$

The superscripts on i^* and p^* here denote the maps induced in cohomology by the given cochain maps. They do not reverse those arrows. A reversal occurs when the cochain maps themselves were obtained from a contravariant Hom functor.

Example 3.26 (Executing the connecting-map construction). Fix $n \geq 2$ and let $C_1 = C_0 = \mathbb{Z}$, with differential d_1 multiplication by n and all other terms zero. Multiplication by n in each degree is an injective chain map, and gives a short exact sequence of complexes

$$0 \rightarrow C_\bullet \xrightarrow{\times n} C_\bullet \rightarrow C_\bullet/nC_\bullet \rightarrow 0.$$

The quotient complex has $\mathbb{Z}/n\mathbb{Z}$ in degrees 1 and 0 and zero differential. Therefore

$$H_1(C) = 0, \quad H_0(C) = \mathbb{Z}/n\mathbb{Z}, \quad H_1(C/nC) = H_0(C/nC) = \mathbb{Z}/n\mathbb{Z}.$$

For $\bar{r} \in H_1(C/nC)$ choose the integer lift $r \in C_1$. Its differential in the middle complex is $nr \in C_0$. The injection from the left complex is also multiplication by n , so the unique element mapping to nr under that injection is r in the left C_0 . Consequently

$$\partial_1 : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}, \quad \partial_1(\bar{r}) = \bar{r}.$$

It is the identity isomorphism. The relevant portion of the long exact sequence is

$$0 \rightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}/n\mathbb{Z} \rightarrow 0.$$

The first copy is $H_1(C/nC)$; the next three are respectively $H_0(C)$ of the left complex, $H_0(C)$ of the middle complex, and $H_0(C/nC)$. Applying the differential and taking the preimage under the injection are separate steps. Omitting the second would incorrectly turn this nonzero connecting map into zero.

The main notes now apply these tools to projective and injective resolutions, and then to derived functors and group (co)homology.

A The long exact sequence in an abelian category

This appendix completes the proof of Theorem 3.24 in an arbitrary abelian category. Consider a short exact sequence of complexes

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0.$$

We construct $\partial_n : H_n(C'') \rightarrow H_{n-1}(C')$ using morphisms instead of representatives of elements, and then justify the diagram chases in Section 3.4.

The extra construction is a *pullback*. For maps $q : X \rightarrow Y$ and $t : T_0 \rightarrow Y$, their pullback has maps to X and T_0 whose composites to Y agree, and every other such pair factors uniquely through it. In an abelian category it is the kernel of $(q, -t) : X \oplus T_0 \rightarrow Y$. Below we also verify that pulling back an epimorphism gives an epimorphism.

Proof of the abelian-category statement. Pull back $C_n \rightarrow C''_n$ along the inclusion $Z_n(C'') \hookrightarrow C''_n$, obtaining an epimorphism $\rho : T \twoheadrightarrow Z_n(C'')$ and a map $b : T \rightarrow C_n$. The differential db factors through $i : C'_{n-1} \hookrightarrow C_{n-1}$ and then through $Z_{n-1}(C')$. Write the resulting map as $a : T \rightarrow Z_{n-1}(C')$. On $\ker \rho \cong C'_n$, the map a is d' , with codomain restricted to cycles. Its composite with $Z_{n-1}(C') \rightarrow H_{n-1}(C')$ is therefore zero. It follows that this composite factors uniquely through ρ , giving $\phi : Z_n(C'') \rightarrow H_{n-1}(C')$.

The map $C_{n+1} \rightarrow T$ whose two components are $d : C_{n+1} \rightarrow C_n$ and $d''p : C_{n+1} \rightarrow Z_n(C'')$ shows that $\phi d''p = 0$, because $d^2 = 0$. Since $p : C_{n+1} \rightarrow C''_{n+1}$ is epic, $\phi d'' = 0$. Thus ϕ annihilates $B_n(C'')$ and factors uniquely through $H_n(C'')$. This is the required morphism ∂_n .

For precision, the lifting arguments in the exactness and naturality checks remain valid categorically as follows. Pullbacks of an epimorphism in an abelian category are epimorphisms. To verify this, let $q : X \twoheadrightarrow Y$ and $t : T_0 \rightarrow Y$. Their pullback is $K = \ker(q, -t) \subseteq X \oplus T_0$. If $u : T_0 \rightarrow U$ annihilates the projection $K \rightarrow T_0$, then $(0, u) : X \oplus T_0 \rightarrow U$ factors through the epimorphism $(q, -t)$. The resulting map $v : Y \rightarrow U$ satisfies $vq = 0$, so $v = 0$ and $u = 0$. Taking u to be the cokernel map proves that $K \rightarrow T_0$ is epic.

Accordingly, whenever a map into a quotient needs a lift, pull back the quotient map and work after this epimorphism of source objects. Finitely many lifts in a chase require only finitely many such pullbacks. Equality can be checked after an epimorphism. Moreover, a map factors through a subobject if it does so after an epimorphism: compose with the subobject's cokernel, and use that its kernel is the original subobject. To prove each kernel-image equality in the module proof of Theorem 3.24, start with the inclusion of that kernel, lift its map to cycles by such a pullback, and carry out the corresponding calculation. It gives a factorization through the stated image after an epimorphism, hence before it as well. The reverse containment is the vanishing of the adjacent composite already checked. This justifies all three exactness arguments in an arbitrary abelian category. The same rule proves the naturality square, while the quotient construction has supplied the morphism and its well-definedness directly. $\square$

Reindexing by $\hat{C}_n = C^{-n}$ gives the corresponding result for cochain complexes, as in Theorem 3.25.