Minimal Surface and Plateau's Problem in $\mathbb{R}^3$

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Preface

This brief lecture note aims to provide a basic understanding and intuition about what minimal surfaces are and how 3D printers can be used to create them.

The definitions and theorems mainly follow the book written by Do Carmo, [1]

1 Regular Surface and its curvature

1.1 Definitions

Definition 1.1 (Regular Surface)**.** A subset $\mathrm{S} \subset \mathrm{R}^n$ is a regular surface if, for each $\mathrm{p} \in \mathrm{S}$, there exists a neighborhood V in R^3 and a map $\mathbf{x} : \mathrm{U} \to \mathrm{V} \cap \mathrm{S}$ of an open set $\mathrm{U} \subset \mathrm{R}^2$ onto $\mathrm{V} \cap \mathrm{S} \subset \mathrm{R}^3$ such that:

1. $\mathbf{x}$ is differentiable. This means that if we write

$$\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)), \quad (u, v) \in U,$$

For completeness, we will give the formal definition of the tangent space of the regular surface the functions $\mathrm{x}_j(\mathrm{u}, \mathrm{v})$ have continuous partial derivatives of all orders in U for all $j \in \{1, 2, 3\}$.

2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1 , this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : \mathrm{V} \cap \mathrm{S} \to \mathrm{U}$ which is continuous.

3. (The regularity condition.) For each $\mathrm{q} \in \mathrm{U}$, the differential $d\mathbf{x}_{\mathrm{q}} : \mathrm{R}^2 \to \mathrm{R}^3$ is one-to-one.

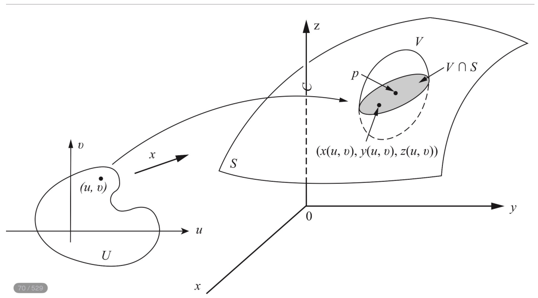

Figure 1: Regular surface when $n = 3$ [1]

Actually, condition 3 guarantees the existence of the tangent space everywhere on the surface.

Example 1.2 (Surface of Revolution)**.** Let $S \subset R^3$ be the set obtained by rotating a regular connected plane curve C about an axis in the plane which does not meet the curve; we shall take the xz plane as the plane of the curve and the z axis as the rotation axis. Let:

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

be a parametrization for C and denote by u the rotation angle about the z axis. Thus, we obtain a map

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

from the open set $U = \left\{(u, v) \in R^2; 0 < u < 2\pi, a < v < b\right\}$ into S (Fig. 2-18).

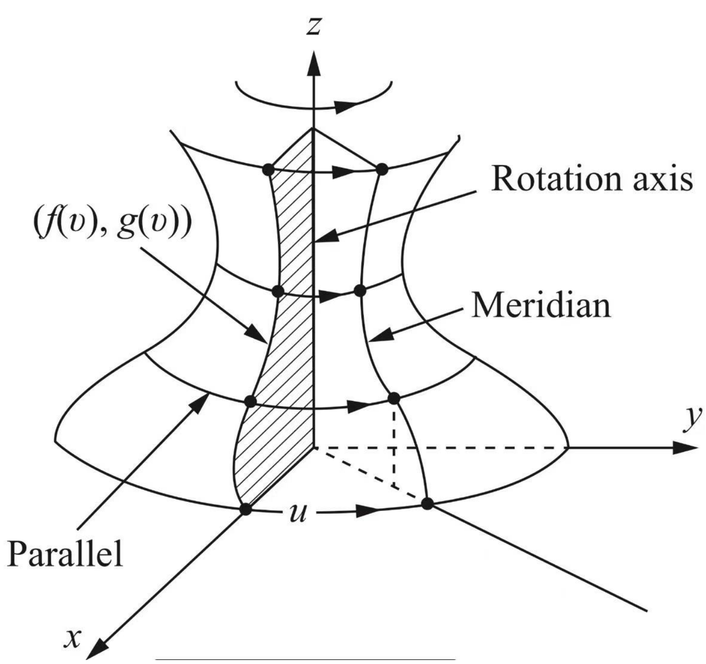

Figure 2: Surface of revolution[1]

This is intuitive and gives us a way to generate a regular surface from a regular curve.

For completeness, we will state the formal definitions here, but to have a basic intuition about what a tangent space T_pS at point $p \in S$ is, we can think of it as the best linear approximation of the plane to the surface S at p.

Definition 1.3 (Ruled Surface)**.** a ruled surface is a surface in $\mathbb{R}^3$ is described by a parametric representation of the form:

$$\mathbf{x}(u, v) = \mathbf{c}(u) + v\mathbf{r}(u)$$

for u varying over an interval and v ranging over the reals.

Definition 1.4 (Tangent Space, tangent vector)**.** A **tangent vector** to S, at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

The **tangent space** to S at p, denoted by T_pS, is the set of all tangent vectors at p.

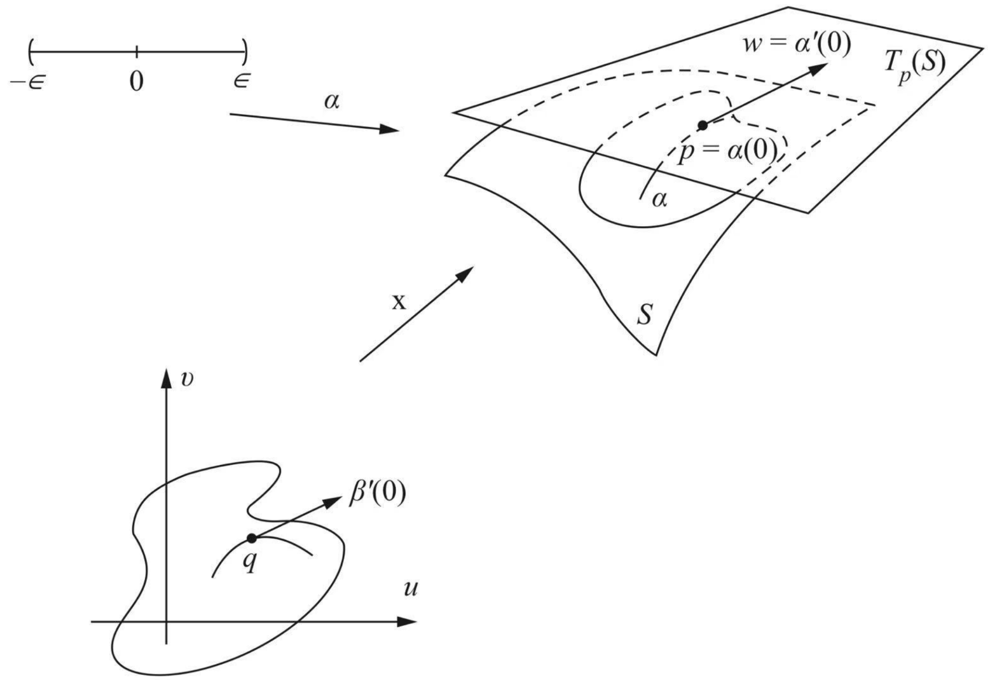

Figure 3: Tangent Vector[1]

Proposition 1.5. *Let* $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ *be a parametrization of a regular surface* S *and let* $q \in U$*. The vector subspace of dimension 2,*

$$dx_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S *at* $\mathbf{x}(q)$*.*

By this proposition, we can easily give an explicit expression of the T_pM if we know the local chart of the surface. i.e.

$$T_pM = dx_q(\mathbb{R}^2) = \text{span}\{X_u, X_v\} \text{ where } (u, v) \in \mathbb{R}^2$$

Definition 1.6 (First Fundamental Form)**.** [4] Let M be a regular surface and u, v tangent vectors in the tangential space T_pM at p. The inner product $I_p\langle u, v\rangle := \langle u, v\rangle$ is called the **first fundamental form**.

Here is just the Euclidean inner product in $\mathbb{R}^n$; in this way, we have a concrete way of measuring the **length** of the tangent vectors.

Definition 1.7 (Unit Normal Vector)**.** Let M be a regular surface, T_pM be the tangent space at p and (U, X) be the local chart, then the **unit normal vector** is defined as follows:

$$N(p) = \frac{X_u \times X_v}{\mid X_u \times X_v \mid}$$

It is easy to verify that $N(p)$ is well-defined, as the tangent space is well-defined, and we restrict the length of the vector. It can also be written as a mapping:

$$N : M \longrightarrow \mathbb{S}^2$$

Intuitively, the unit normal vector describes the 'direction' of the tangent space, so the idea of defining the **curvature** is to measure the rate of change of the $N(p)$ along p.

The following definition is to define how to measure the rate of change of the $N(p)$.

Definition 1.8 (Covariant Derivative)**.** Let Y be a smooth vector field on $p \in V \subset S$ and $v \in T_pS$, the covariant derivative is defined as follows:

$$D_yY := (\frac{d(Y \circ \alpha)}{dt} \mid_t = 0)^T = (dY_p(y))^T$$

where $\alpha : (-\epsilon, \epsilon) \to S,\ \alpha(0) = p, \alpha'(0) = v$

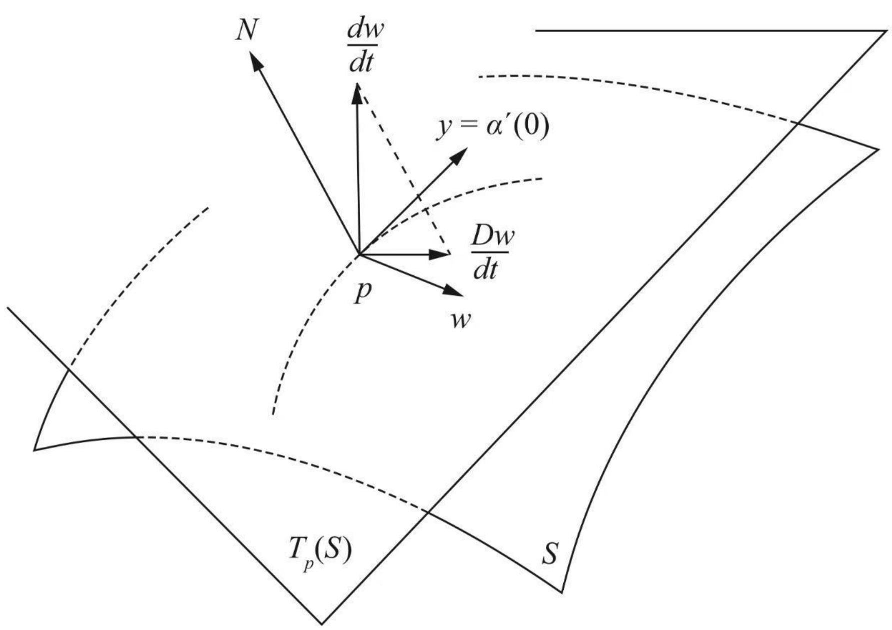

Figure 4: Covariant derivative[1]

Definition 1.9 (Shape Operator)**.** We define the shape operator S in the following way:

$$\begin{aligned} S(p) : T_pM &\longrightarrow T_pM \\ v &\longrightarrow D_v(N(p)) \end{aligned}$$

We can rewrite the definition:

$$S(p)(v) = (dN_p(v))^T = dN_p(v) \text{ since } N(p) \text{ is of constant length}$$

The shape operator is simply the differential of N at the point p.

We omit the formal definition of the general differential between two regular surfaces, and describe dN_p here:

The linear map $dN_p : T_pS \to T_pS$ operates as follows: For each tangent vector $v \in T_pS$, it has a parametrized curve $\alpha(t)$ in S with $\alpha(0) = p, \alpha'(0) = v$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere $\mathbb{S}^2$, this amounts to restricting the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0)) = dN_p(v)$ is a vector in T_pS.

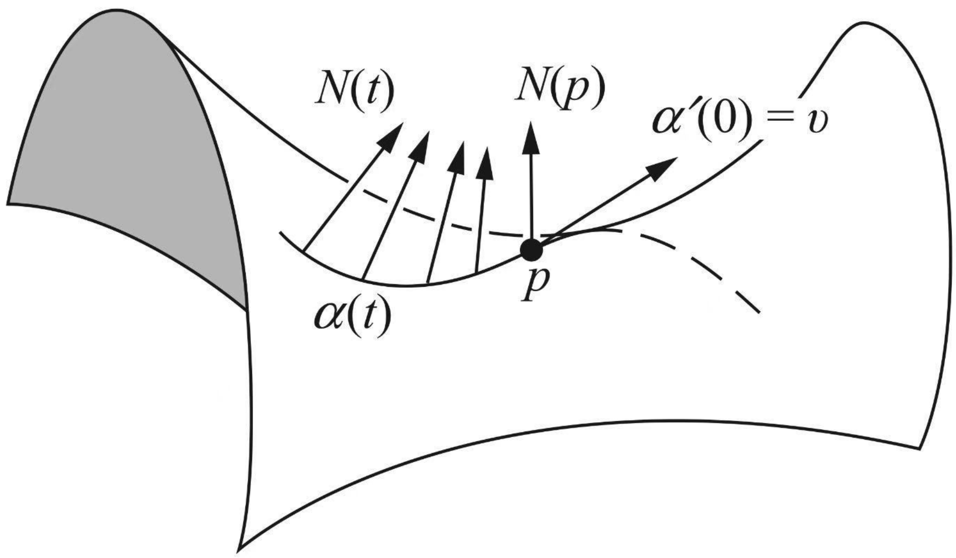

Figure 5: Differential of the map N[1]

Intuitively, it illustrates how the unit normal vector changes its direction as it aligns with the vector v.

Moreover, it is not hard to see that $S = dN_p$ is a **self-adjoint linear map**, so we can apply the techniques we learnt in linear algebra to it.

Definition 1.10 (Principal curvature[4])**.** The eigenvalues $\kappa_1(p), \kappa_2(p)$ of the shape operator $S : T_p\mathcal{S} \to T_p\mathcal{S}$ are called the **principal curvatures**.

Definition 1.11. The functions $H, K : \mathcal{M} \to \mathbb{R}$

$$H(p) = \frac{1}{m} \operatorname{tr}(S(p))$$
$$K(p) = \det(S(p))$$

denote the **mean** and **Gaussian** curvature, respectively

Choosing an orthonormal eigenbasis of the shape operator S yields a diagonal matrix with $\text{diag}(S) = (\kappa_1, \kappa_2)$ and therefore, the following relationship between the mean, Gaussian, and principal curvature can be derived.

Proposition 1.12. *Let $H(p), K(p)$ denote the mean and Gaussian curvature, respectively. Then one has*

$$\begin{aligned} H(p) &= \frac{1}{2}(\kappa_1 + \kappa_2) \\ K(p) &= \kappa_1 \kappa_2. \end{aligned}$$

Actually, the sign of the mean curvature is related to the orientation, but since we are considering the case of a minimal surface, we can omit it.

1.2 Computations

In the first Calculus course, we have learned how to compute the length of a curve and the Area of surfaces using the fundamental values E, F, G. Here, we provide their formal definitions.

Definition 1.13 (Coefficients of the first fundamental form)**.** Let $X : U \to S$ be the local patch of the regular surface S,we define:

$$\begin{cases} E(u,v) = \langle X_u, X_u \rangle \\ F(u,v) = \langle X_u, X_v \rangle \\ G(u,v) = \langle X_v, X_v \rangle \end{cases}$$

Definition 1.14 (Area)**.** Let S be the regular surface and $X : U \to S$ be the local patch, then we define the Area of the region R:

$$Area(R) := \int_{X^{-1}(R)} |X_u \times X_v| \, dudv$$

By simple computations, we have:

$$Area(R) := \int_{X^{-1}(R)} \sqrt{EG - F^2} \, dudv$$

Definition 1.15 (Coefficients of the second fundamental form)**.** Let $X : U \to S$ be the local patch of the regular surface S,we define:

$$\begin{cases} e(u,v) = \langle N, X_{uu} \rangle \\ f(u,v) = \langle N, X_{uv} \rangle \\ g(u,v) = \langle N, X_{vv} \rangle \end{cases}$$

By these coefficients, we can give a formula for these two curvatures

$$H(p) = \frac{eG + Eg - 2fF}{2(EG - F^2)}$$

$$K(p)=\frac{eg-f^2}{EG-F^2}$$

1.3 Gauss-Bonnet Theorem

Theorem 1.16 (GLOBAL GAUSS-BONNET THEOREM.[1]). *Let* $\mathrm{R}\subset\mathrm{S}$ *be a regular region of an oriented surface and let* $\mathrm{C_1},\dots,\mathrm{C_n}$ *be the closed, simple, piecewise regular curves which form the boundary* $\partial\mathrm{R}$ *of* R *. Suppose that each* $\mathrm{C_i}$ *is positively oriented and let* $\theta_1,\dots,\theta_{\mathrm{p}}$ *be the set of all external angles of the curves* $\mathrm{C_1},\dots,\mathrm{C_n}$*. Then*

$$\sum_{\mathrm{i=1}}^{\mathrm{n}}\int_{\mathrm{c_i}}\mathrm{k_g}(\ \mathrm{s})\mathrm{ds}+\iint_{\mathrm{R}}\mathrm{K}\mathrm{d}\sigma+\sum_{l=1}^{\mathrm{p}}\theta_l=2\pi\chi(\mathrm{R}),$$

We will present a weak version of the Gauss-Bonnet Theorem, which is one ofcritical important theorems we wfromlearn from the classical basic curve and surface course.

Theorem 1.17 (Corollary of the Gauss-Bonnet Theorem[1]). *Let* S *be an orientable compact surface; then:*

$$\iint_S K d\sigma=2\pi\chi(S)$$

The left-hand side is also called the ***total curvature*** *on the surface* S*.*

2 Minimal Surface from differential viewpoint

Definition 2.1 (Minimal Surface). A regular surface S is called a **minimal surface** if the mean curvature H vanishes for every point on S.

Definition 2.2 (Normal variation[4]). Let $x:\mathcal{U}\to\mathbb{R}^3$ be a regular local surface, $\mathcal{S}\subset\mathcal{U}$ be a bounded region in $\mathcal{U}$, and let $N(u,v)$ denote the unit normal vector to $x(u,v)$ for any $(u,v)\in\mathcal{U}$. The **normal variation** of x under any differentiable mapping $h:S\to\mathbb{R}$ and $\epsilon>0$ is defined as

$$\begin{aligned}X:(-\epsilon,\epsilon)\times\mathcal{S}&\longrightarrow\mathbb{R}^3\\(t,(u,v))&\longmapsto x(u,v)+t\times h(u,v)\times N(u,v).\end{aligned}$$

In the following we use the abbreviation $X^t(u,v):=X(t,(u,v))$. The normal variation X^{t} describes for each t a slightly deformed (local) surface in normal direction and hence, to indicate the t different first fundamental forms, we use the notation.

$$E^t=\left\langle X_u^t,X_u^t\right\rangle,F^t=\left\langle X_u^t,X_v^t\right\rangle\text{ and }G^t=\left\langle X_v^t,X_v^t\right\rangle.$$

In preparation for the main result in this section, the relation of a vanishing mean curvature to a minimal surface area, the so-called first variation of a surface, will be derived.

Lemma 2.3. *Let* $\mathcal{A}(t)=\int_S\|X_{\mathrm{u}}^t\times X_{\mathrm{v}}^t\|\,dudv$ *denote the area of each normal variation* $X^t(S)$*. Then the first variation of* A*. i.e.,* $\left(\frac{dA}{dt}\right)\big|_{t=0}$*, is*

$$\left(\frac{d\mathcal{A}}{dt}\right)\bigg|_{t=0}=-2\int_S hH\sqrt{EG-F^2}d(u,v),$$

where H *is the mean curvature of* S*.*

Proof. . First, we describe the area operator:

$$\mathcal{A}(t)\left(X^t(S)\right) = \int_S \left\|X_{\mathrm{u}}^t \times X_v^t\right\| dudv = \int_S \sqrt{E^tG^t - \left(F^t\right)^2} dudv$$

Starting with E^t, G^t, and F^t, these will be subsequently calculated.

$$\begin{aligned} E^t &= \left\langle X_u^t, X_u^t \right\rangle \\ &= \langle x_u + th_uN + thN_u, x_u + th_uN + thN_u \rangle \\ &= \langle x_u + x_u \rangle + \langle x_u, th_uN \rangle + \langle x_u, thN_u \rangle + \langle th_uN, x_u \rangle + \langle thN_u, x_u \rangle + \mathcal{O}\left(t^2\right) \\ &= E + 2\langle x_u, th_uN \rangle + 2\langle x_u, thN_u \rangle + \mathcal{O}\left(t^2\right) \\ &= E + 2th_u \underbrace{\langle x_u, N \rangle}_{=0} - 2th\left(N, x_{uu}\right) + \mathcal{O}\left(t^2\right) \\ &= E - 2the + \mathcal{O}\left(t^2\right). \end{aligned}$$

analogously,

$$\begin{aligned} F^t &= F - 2thf + \mathcal{O}\left(t^2\right) \text{ and} \\ G^t &= G - 2thg + \mathcal{O}\left(t^2\right) \end{aligned}$$

Where e, f, and g are the coefficients of the second fundamental form.

$$\begin{aligned} E^tG^t - \left(F^t\right)^2 &= \left(E - 2the + \mathcal{O}\left(t^2\right)\right)\left(G - 2thg + \mathcal{O}\left(t^2\right)\right) - \left(F - 2thf + \mathcal{O}\left(t^2\right)\right)^2 \\ &= EG - F^2 - 2thgE - 2theG + 4thfF + \mathcal{O}\left(t^2\right) \\ &= EG - F^2 - 2th(eG - 2fF + gE) + \mathcal{O}\left(t^2\right) \\ &= EG - F^2 - 4thH\left(EG - F^2\right) + \mathcal{O}\left(t^2\right) \\ &= \left(EG - F^2\right)(1 - 4thH) + \mathcal{O}\left(t^2\right) \end{aligned}$$

hence,

$$\begin{aligned} \sqrt{E^tG^t - \left(F^t\right)^2} &= \sqrt{\left(EG - F^2\right)(1 - 4thH) + \mathcal{O}\left(t^2\right)} \\ &= \sqrt{\left(EG - F^2\right)}\sqrt{(1 - 4thH) + \mathcal{O}\left(t^2\right)} \\ &= \sqrt{\left(EG - F^2\right)}\sqrt{(1 - 2thH)^2 + \mathcal{O}\left(t^2\right)} \\ &= \sqrt{\left(EG - F^2\right)}(1 - 2thH) + \mathcal{O}\left(t^2\right) \end{aligned}$$

As a conclusion, the first variation is given as

$$\begin{aligned} \left.\left(\frac{d\mathcal{A}}{dt}\right)\right|_{t=0} &= \left.\left(\frac{d}{dt}\int_S \sqrt{\left(EG - F^2\right)}(1 - 2thH) + \mathcal{O}\left(t^2\right) d(u,v)\right)\right|_{t=0} \\ &= -2\int_S hH\sqrt{\left(EG - F^2\right)} \end{aligned}$$

□

This is just complicated computations.

Theorem 2.4. *Let $x : \mathcal{U} \to \mathbb{R}^3$ be a regular surface and $\mathcal{S} \subset \mathcal{U}$. Then x is a critical point of the Area functional on $\mathcal{S}$ if and only if the mean curvature H vanishes.*

Proof. First, let the mean curvature be identically zero, i.e., $H = 0$, then by the result in the previous lemma, the first variation is zero for each h and therefore, $X^0 = x$ is a critical point of the Area functional $\mathcal{A}$.

For the other direction, assume the contrary, i.e., $\left(\frac{d\mathcal{A}}{dt}\right)\big|_{t=0} = 0$ for any differentiable $S \to \mathbb{R}$ and there is a point $p := (\bar{u}, \bar{v}) \in S$ such that the mean curvature $H(p) \neq 0$. nce h is arbitrary and smooth, choose h such that $h(p) = H(p)$ in a small neighborhood (p) around p and zero outside. Again, by (3.6) and the positive definiteness of the first fundamental form,

$$\left(\frac{d\mathcal{A}}{dt}\right)\bigg|_{t=0} = -2 \int_s \underbrace{H^2 \sqrt{EG - F^2}}_{\geq 0} d(u, v) < 0.$$

This contradicts the assumption that the first variation equals zero. Hence, the mean value $H(p) = 0$ for any arbitrary p.

□

Since x is only a critical point, it is uncertain whether the obtained surface is actually minimal. There are cases where x is not necessarily a minimum.

2.1 Equivalent definitions of the minimal surface

One can refer to Chapter 2.2 of the article [3] written by Meeks, William, and Pérez, Joaquín, for more equivalent definitions. To be explicit, eight equivalent definitions.

In this lecture note, we only use two.

Definition 2.5 (Isothermal Parametrization). Let U be an open subset of S, a parametrization $X : U \to \mathbb{R}^3$ is called an isothermal parametrization if:

$$E = G \text{ and } F = 0$$

In this section, we prove a classical and fundamental theorem in differential geometry:

Theorem 2.6 (Existence of Isothermal Coordinates). *Let S be a regular surface and $p \in S$. Then there exists a neighborhood U of p and a smooth parametrization*

$$\mathbf{x} : V \subset \mathbb{R}^2 \to U \subset S$$

such that the first fundamental form has the conformal form

$$I = E(du^2 + dv^2), \qquad E > 0.$$

In other words, (u, v) are isothermal coordinates.

The proof of this theorem is analytic and lacks geometric intuition, so it is presented here or in Do Carmo's textbook cite do2016differential. However, this theorem is useful since isothermal parametrization is very useful in the context of minimal surfaces.

Proposition 2.7. *Let $\mathcal{X} = \mathcal{X}(u,v)$ be a (local) regular parametrized surface and assume that $\mathcal{X}$ is isothermal. Then*

$$\triangle\mathcal{X} = 2\lambda^2 H$$

where $\lambda^2 = \langle \mathcal{X}_u, \mathcal{X}_u \rangle = \langle \mathcal{X}_v, \mathcal{X}_v \rangle$

By these properties of isothermal parametrization, we can easily deduce the following corollary:

Corollary 2.8. *Let $\mathcal{X} = \mathcal{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ be a (local) regular parametrized surface and assume that $\mathcal{X}$ is isothermal. Then*

$$\mathcal{X} \text{ is minimal } \iff \text{ coordinates function } x, y, z \text{ are harmonic}$$

where $\lambda^2 = \langle \mathcal{X}_u, \mathcal{X}_u \rangle = \langle \mathcal{X}_v, \mathcal{X}_v \rangle$

Definition 2.9 (Def 2.8 in [3])**.** A surface $M \subset \mathbb{R}^3$ is minimal if and only if its stereographically projected Gauss map $g : M \longrightarrow \mathbb{C} \cup \{\infty\} := \mathbb{P}^1$ is meromorphic with respect to the underlying Riemann surface structure/complex atlas.

3 Minimal Surface from complex viewpoint

After understanding what a minimal surface is, it is natural to ask if there is a general form to parametrize all minimal surfaces. The following definition and proposition will give a general form.

The original statement of this parametrization needs some background knowledge on real differential forms and holomorphic forms on a Riemann surface; for convenience, we will skip the formal definitions of these complex concepts.

A holomorphic 1-form can be just expressed as $\omega = f(z)dz$.

Definition 3.1 (Weierstrass–Enneper Parameterization[7])**.** where g is meromorphic and f is analytic, such that wherever g has a pole of order m and f has a zero of order $2m$, and letc_1, c_2, c_3 be constants.

Then the surface with coordinates (x_1, x_2, x_3) is minimal, where the x_k are defined using the real part of a complex integral, as follows:

$$\begin{aligned} x_k(\zeta) &= \mathrm{Re}\left\{ \int_0^\zeta \varphi_k(z)\, dz \right\} + c_k, \qquad k = 1, 2, 3 \\ \varphi_1 &= f(1 - g^2)/2 \\ \varphi_2 &= if(1 + g^2)/2 \\ \varphi_3 &= fg \end{aligned}$$

here Re denotes the real part, $z \in \mathbb{C}$.

This representation guarantees that the resulting surface has zero mean curvature.

Idea of the Proof In complex analysis, we know that:

Harmonic functions are real parts of holomorphic functions.

And the parametrization X of the surface are all harmonic. So we know:

$$X = \mathcal{R} \int (\text{holomorphic 1-forms})$$

That is to solve:

$$X = \mathcal{R} \int (\Phi_1, \Phi_2, \Phi_3)$$

With:

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$$

The key result of the parametrzation is that, every solution to the above has the following form:

$$\begin{aligned}
\Phi_1 &= \frac{1}{2}\left(\frac{1}{G} - G\right) dh \\
\Phi_2 &= \frac{i}{2}\left(\frac{1}{G} + G\right) dh \\
\Phi_3 &= dh
\end{aligned}$$

And it turns out that the function G here is the Gauss Map of the surface.

What is more important is that every nonplanar minimal surface defined over a connected domain can be given a parametrization of this type.

Proposition 3.2 (Lemma8.2 in [5]). *Every simply-connected minimal surface in $\mathbb{R}^3$ can be represented in the* ***Weierstrass–Enneper Parameterization*** *as defined above.*

We will skip the proof of the above proposition and directly use it.

4 Classical examples of the minimal surface

After all this discussion, we can finally provide concrete examples using the parametrization above.

4.1 Catenoid

Definition 4.1 (Catenoid). Choose:

$$g(z) = z, \qquad f(z) = \frac{1}{z^2}, \qquad M = \mathbb{C}^*, \qquad z = u + iv$$

Substituting into the Weierstrass representation yields the classical catenoid:

$$X(r, \theta) = (\cosh u \cos\theta,\ \cosh u \sin\theta,\ u), \qquad r = e^u.$$

Theorem 4.2. *The catenoid is the only nontrivial minimal surface of revolution in $\mathbb{R}^3$*

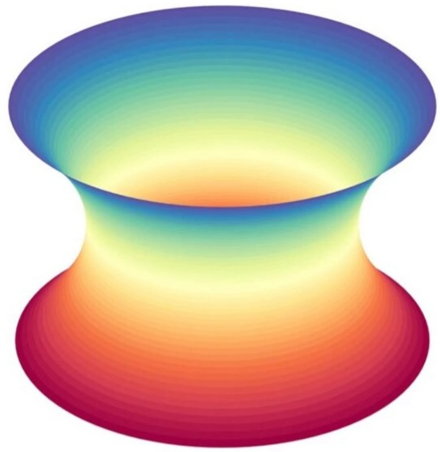

Figure 6: Catenoid

4.2 Helicoid

Definition 4.3 (Helicoid)**.** A common Weierstrass data is:

$$g(z) = z, \qquad f(z) = \frac{1}{z}, \qquad M = \mathbb{C}^*, \qquad z = u + iv$$

Then the resulting surface is:

$$X(u, v) = (\sinh v \cos u,\ \sinh v \sin u,\ u),$$

The standard helicoid.

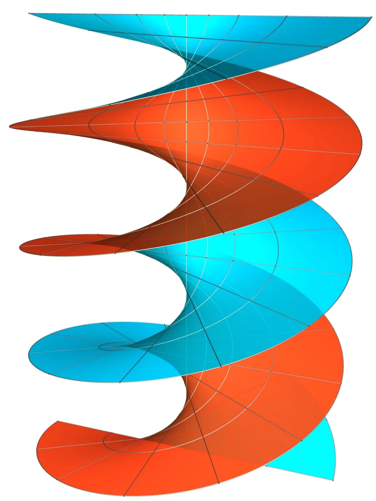

Figure 7: Helicoid

Theorem 4.4 (Catalan,1842)**.** *Helicoid is the only nontrivial ruled minimal surface in* $\mathbb{R}^3$

An interesting fact is that there exists an isometric transformation between these two surfaces, the animation can be seen by clicking here

4.3 Enneper's Surface

Definition 4.5 (Enneper's Surface of order n)**.** Based on the representation above, let $f(z) = 1$ and $g(z) = z^n$. Then we have:

$$\begin{cases} \phi_1 = \dfrac{1 - z^{2n}}{2} \\ \phi_2 = i\dfrac{1 + z^{2n}}{2} \\ \phi_3 = z^n \end{cases}$$

So the coordinates function (x_1, x_2, x_3) will be:

$$\begin{cases} x_1(\xi) = Re(\dfrac{\xi}{2} + \dfrac{\xi^{2n+1}}{2(2n+1)}) + c_1 \\ x_2(\xi) = Re(i(\dfrac{\xi}{2} + \dfrac{\xi^{2n+1}}{2(2n+1)})) + c_2 \\ x_3(\xi) = Re(\dfrac{\xi^{n+1}}{n+1}) + c_3 \end{cases}$$

Now we try to reparametrize it using the real numbers by $\xi = u + iv$ and do a binomial expansion, we have:

$$\begin{cases} x_1(u,v) = u - \dfrac{u^{2n+1} - \binom{2n+1}{2}u^{2n-1}v^2 + \binom{2n+1}{4}u^{2n-3}v^4 - \cdots + (-1)^n uv^{2n}}{2n+1} + c_1 \\ x_2(u,v) = -v + (-1)^{n+1}\dfrac{v^{2n+1} - \binom{2n+1}{2}v^{2n-1}u^2 + \binom{2n+1}{4}u^{2n-3}v^4 - \cdots + uv^{2n}}{2n+1} + c_2 \\ x_3(u,v) = \dfrac{u^{n+1} - \binom{n-1}{2}u^{n-1}v^2 + \cdots}{n+1} + c_3 \end{cases}$$

The surface with the above parametrization is called **Enneper's Surface of order** n, denoted as Enn_n[4].

Example 4.6 (Parametrization of Enn_1)**.** For convenience, we multiply some constants and let $\vec{c} = (c_1, c_2, c_3) = \vec{0}$ to make it more readable.

$$\begin{cases} x_1(u,v) = u - \dfrac{u^3}{3} - uv^2 \\ x_2(u,v) = -v + \frac{v^3}{3} - u^2 v, \\ x_3(u,v) = u^2 - v^2 \end{cases}$$

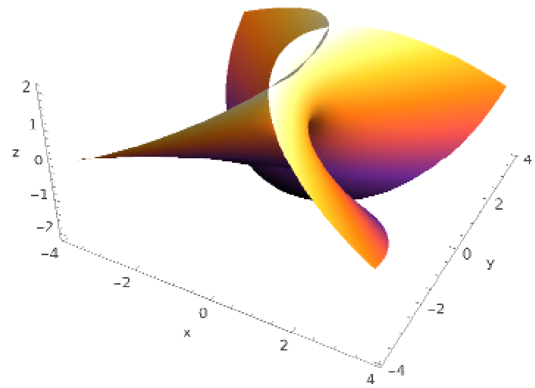

Figure 8: *Enn*$_1$ using mathematica

Theorem 4.7 (Theorem 9.4 in [5])**.** *There are only two complete regular minimal surfaces whose total curvature is* -4π*. These are the catenoid and Enneper's surface.*

Example 4.8 (Parametrization of Enn_2)**.** For convenience, we multiply some constants and let $\vec{c} = (c_1, c_2, c_3) = \vec{0}$ to make it more readable.‘

$$\begin{cases} x_1(u,v) = u - \dfrac{u^5 - 10u^3v^2 + 5uv^4}{5} \\ x_2(u,v) = -v + \dfrac{v^5 - 10v^3u^2 + 5uv^4}{5} \\ x_3(u,v) = 2\dfrac{u^3 - 3uv^2}{3} \end{cases}$$

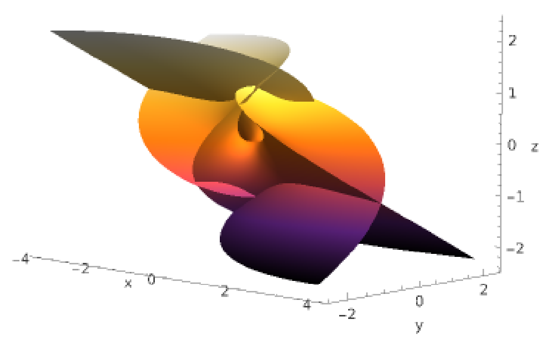

Figure 9: *Enn*$_2$ using mathematica

These examples are the most classical minimal surfaces in $\mathbb{R}^3$; in fact, besides Enneper's surface, which will self-intersect, the rest two nontrivial surfaces, along with the plane, are thought to be the only complete, embedded minimal surfaces in $\mathbb{R}^3$. Then Costa came up with his surface in his phd thesis in 1982.

4.4 Costa's Surface

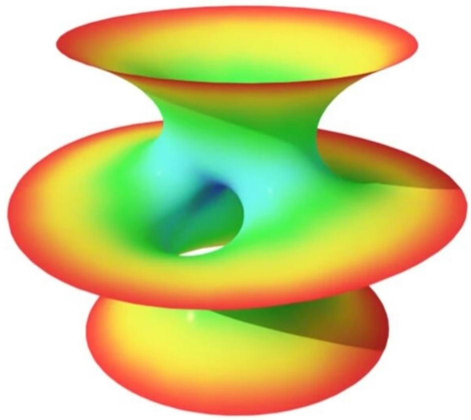

Figure 10: Costa's Surface

To introduce the history, we must look back at the progress of research. Karcher's lecture note[2] has a great introduction.

In the topology class, we have already learned about the 1-point compactification of $\mathbb{R}^n$; it has a deeper theory behind it.

Definition 4.9 (Topological Ends)**.** Let X be a topological space, and suppose that

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$$

is an ascending sequence of compact subsets of X whose interiors cover X. Then X has one end for every sequence

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots,$$

where each U_n is a connected component of $X \backslash K_n$. The number of ends does not depend on the specific sequence (K_i) of compact sets; there is a natural bijection between the sets of ends associated with any two such sequences.

Using this definition, a neighborhood of an end (U_i) is an open set V such that $V \supset U_n$ for some n. Such neighborhoods represent the neighborhoods of the corresponding point at infinity in the end compactification (this "compactification" is not always compact; the topological space X has to be connected and locally connected).

Theorem 4.10 (Osserman[5])**.** *If a complete minimal surface in $\mathbb{R}^3$ has finite total curvature, then it is conformally equivalent to*

$$\Sigma = \bar{\Sigma} \backslash \{p_1, \ldots, p_n\}$$

where:

- *$\bar{\Sigma}$ is a compact Riemann surface*

- *Each removed point p_i corresponds to one end*

So, Ends become the punctures of a compact Riemann surface

Theorem 4.11 (Schoen,1983 No-Go Theorem[6])**.** *The only complete minimal immersions $M^n \subset \mathbf{R}^{n+1}$, which are regular at infinity and have two ends, are the catenoids and pairs of planes.*

Here is a good animation of the progress.

Theorem 4.12 (Hoffman-Meeks Halfspace Theorem)**.** *A properly embedded complete minimal surface cannot lie in a halfspace. A meromorphic embedding must therefore have at least two catenoid ends, one up, one down.*

This means we can hope to find a minimal torus with at least three punctures. For ease, they try to find case three. A more symmetric example will be easier to find.

Therefore, we hope that the surface has **two catenoid ends (up and down)** and **a planar end between them**. How could a (non-minimal) torus with just one planar end look like?

It is easier the other way round: **a plane with one handle is the same as a torus with one planar end**. Such a surface is simple enough to be imagined as consisting of two vertical planes of reflectional symmetry. This excludes all the tori that do not have a reflection-symmetric fundamental domain.

At this stage, we have three distinguished points on the torus: the intersection of the two symmetry planes meets the surface in three points. We puncture our torus in two of these points, the outside ones. Then we can easily deform these punctures into half-catenoids (up and down). This surface resembles the final minimal surface already strongly.

We omit the fact that if one has an embedded minimal surface, then the ends are ordered by height, and the vertical limit normals point alternatingly up and down.

For our model, this means: we can assume two simple poles of the Gauss map at the catenoid punctures and at least a double zero at the middle planar end. At the last of the three distinguished points on the symmetry axis, the normal is also vertical; our model surface suggests another pole. The simplest Gauss map compatible with the present picture is then of degree three. We need to assume that at the planar end G has a triple zero, because a double zero there and just one other zero not on the symmetry axis is not compatible with the two vertical symmetry planes.

If we draw what we have into the rectangular fundamental domain of our torus, we see: **three simple poles** and **one triple zero** on the half lattice points. The inverse of the derivative of the Weierstraß $\wp$-function has the same zeros and poles.

Therefore, we arrived at a candidate for the Gauss map: $G = \rho/\wp'$. Assuming that this is correct, we now need a differential which has simple poles at the catenoid ends and a simple zero at the planar end (namely to give double poles to Gdh or dh/G at embedded ends). And dh needs another zero to compensate the third pole of G (at the last special point on the axis). If we write $dh = H \cdot dz$ then we have met such a function H when we discussed the doubly periodic embedded surfaces. (We also recognize it as $H = (\wp - \wp($ branch point $))/\wp'$.) In other words, up to constant factors we have determined the Weierstraß data of the wanted surface. Scaling dh by a real number only changes the size of the minimal surface; the phase of dh on the symmetry lines (curvature lines) is determined by our earlier discussion. If we multiply the Gauss map by $e^{\mathbf{i}\varphi}$ then we only rotate the surface. The one remaining question is what is the value of the real constant ρ in $G = \rho/\wp'$? This factor is called the "Lopez-Ros parameter" because they made significant use of it in the proof of their theorem quoted above. With the help of the symmetries, we find that, in general, the Weierstraß data obtained so far have two horizontal periods. Only on the more symmetric square torus can they be closed by choosing the parameter ρ correctly. This can be done with the intermediate value theorem.

This surface was found in 1982 by C. Costa in his thesis.

Later Hoffman-Meeks found that one can also have other rectangular tori with three punctures minimally embedded; one has to deform the planar middle end into a catenoid end, i.e., one has to split the triple zero of the Gauss map into three simple zeros and the position of the other vertical points is the additional parameter needed to close two periods.

5 Plateau's Problem

The Plateau's Problem is very easy to understand. The famous mathematician Jesse Douglas solved the problem and won the 1936 Fields Medal.the problem and won the 1936 Fields Medal. His official grounds for the medal are: **Did important work on the Plateau problem, which is concerned with finding minimal surfaces connecting and determined by some fixed boundary.**

Definition 5.1 (Plateau's Problem)**.** Given a boundary, does there exist an area minimizing minimal surface spanning it?

More mathematically, this problem can be stated as:

Definition 5.2 (Plateau's Problem theoretically)**.** Let $\Gamma \in \mathbb{R}^3$ be a curve, let $\mathcal{F}$ be the set of surfaces that spanning the curve Γ, let:

$$A_\Gamma := \inf_{M \in \mathcal{F}} \text{Area}(M)$$

If there exists a surface $M \in \mathcal{F}$ such that $\text{Area}(M) = A_\Gamma$, then this surface solves Plateau's problem.

But this is still too general or lack of restriction to solve. So we need to formalise it more.

Definition 5.3 (Douglas's Plateau Problem)**.** Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve and let $D \subset \mathbb{R}^2$ be the open unit disk. Let $\mathcal{F}_\Gamma$ be the set of maps $u : \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$. $\quad \left(\iint_D \|\nabla u\|^2 dxdy < \infty\right)$
- $u : \partial D \to \Gamma$ is monotone and onto.

Let $\mathcal{A}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \text{Area}(u(D)) \in (0, \infty)$. Then there exists a map $w \in \mathcal{F}_\Gamma$ such that

$$\text{Area}(w(D)) = \mathcal{A}_\Gamma.$$

Idea of the Proof The difficulty of the proof lies in two aspects:Areaginenumerate

Bounding the area of the surface $A(S)$ doesn't give much information about the surface itself.

The group of conformal maps of the disk is not compact, and the Dirichlet energy is invariant under conformal transformation.

For the first part, the large F_Γ can be restricted by using the isothermal parametrization we have introduced before:

Let $u : D \longrightarrow \mathbb{R}^3$ be an element in F_Γ, then we have the explict expression of the area:

$$A(u) = \iint_D \sqrt{|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2}\, dxdy$$

The Dirichlet energy $E(u)$ of the parametrization is:

$$E(u) = \frac{1}{2}\iint_D |u_x|^2 + |u_y|^2\, dxdy$$

Let F_Γ^{iso} be the set of all $u \in F_\Gamma$ which is also an isothermal parametrization.

Remark 5.4. Take $u \in F_\Gamma$, $E(u) \geq A(u)$, when $u \in F_\Gamma^{iso}$, the equality holds.

Similarly, we can define the lower bound among the isothermal parametrization.

$$A_\Gamma^{iso} := \inf_{u \in F_\Gamma^{iso}} A(u)$$

It is easy to check that $A_\Gamma^{iso} = A_\Gamma$ since:

- $A_\Gamma^{iso} \geq A_\Gamma$ since $F_\Gamma^{iso} \subset F_\Gamma$
- $A_\Gamma^{iso} \geq A_\Gamma$ since given $u \in F_\Gamma$, there exists $u^* \in F_\Gamma^{iso}$ s.t. $u^*(D) = u(D)$. Just another way of expressing the existence of the global isothermal parametrization.

More generally, we can define:

$$\epsilon_\Gamma := \inf_{u \in F_\Gamma} E(u) \quad \epsilon_\Gamma^{iso} := \inf_{u \in F_\Gamma^{iso}} E(u)$$

By the remark above and $F_\Gamma^{iso} \subset F_\Gamma$, we have:

$$A_\Gamma = A_\Gamma^{iso} = \epsilon_\Gamma^{iso}, \quad \epsilon_\Gamma^{iso} \geq \epsilon_\Gamma \geq A_\Gamma$$

So:

$$\epsilon_\Gamma = A_\Gamma$$

Therefore, the first problem is no longer a concern, as we no longer need to address it.

For the second part, we need to understand why the non-compactness is a problem.

Let $Conf(D)$ be the set of all conformal transformations of the unit disk D itself. Take $\phi_k : D \longrightarrow D$ as the conformal map of D by the following formula:

$$\phi_k(z) = e^{i\theta}\frac{z - a}{1 - \bar{a}z} = \frac{\frac{k-1}{k} - z}{1 - \frac{k-1}{k}z}$$

Then, let $u \in F_\Gamma$, we can construct $u_k = u(\phi_k(z))$ as a sequence with the same Dirichlet energy but not convergent.

The trouble caused by $Conf(D)$ is because of the symmetry of the unit disk D.

Recall the theorem we have learnt in Complex Analysis, three points fix a conformal map of D itself; it is natural to add on the new condition:

$$u(1,0) = p_1, u(0,1) = p_1, u(-1,0) = p_3, \text{where } p_1, p_2 \text{ and } p_3 \text{ are three distinct points in } \Gamma.$$

Then the original problem can be reduced to proving the following statement:

Definition 5.5 (The equivalent version of the Plateau's Problem)**.** Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve and let $D \subset \mathbb{R}^2$ be the open unit disk. Let $\mathcal{F}_\Gamma^*$ be the set of maps $u : \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$. $\quad \left(\iint_D \|\nabla u\|^2 dxdy < \infty\right)$
- $u : \partial D \to \Gamma$ is monotone and onto.
- $u(1,0) = p_1, u(0,1) = p_1, u(-1,0) = p_3$,where p_1, p_2 and p_3 are three distinct points in Γ.

Let $\mathcal{E}_\Gamma^* := \inf_{u \in \mathcal{F}_\Gamma^*} E(u)$. Then there exists a map $w \in \mathcal{F}_\Gamma^*$ such that

$$\mathrm{E}(w) = \mathcal{E}_\Gamma^*$$

Last Part of the proof The rest of the part needs some background knowledge in Functional Analysis and Geometric Analysis.

Preparations

Definition 5.6 (Sobolev spaces)**.** Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, then the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \left\{u \in L^p(\Omega) : D^{\boldsymbol{\alpha}} u \in L^p(\Omega) \forall \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \leq k\right\}.$$

This space is equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\boldsymbol{\alpha}| \leq k} \left\|D^{\boldsymbol{\alpha}} u\right\|_{L^p(\Omega)}.$$

Basically, the Sobolev Space is just the more general version of $C^k(\Omega)$, where the $'C'$ here represents the convergence of the weak derivative with order less than k in L^p space. In this case, consider that this space gives us the availability to make every integral, such as $\int_\Omega \nabla u$, sense.

Let (T, d) be a compact metric space, where d is a metric on T, and let $\mathcal{C}(T, \mathbb{R})$ denote the collection of continuous functions $f : T \to \mathbb{R}$, where the metric is the usual supremum norm, that is,

$$\|f - g\|_\infty = \sup_{t \in T} |f(t) - g(t)| \text{ for } f, g \in \mathcal{C}(T, \mathbb{R}).$$

Let $\mathcal{F} \subset \mathcal{C}(T, \mathbb{R})$. Then the modulus of continuity of f is

$$\omega_f(\delta) := \sup_{s,t \in T} \{|f(t) - f(s)| : d(s,t) < \delta\}.$$

A function f is uniformly continuous if and only if $\lim_{\delta \to 0} \omega_f(\delta) = 0$. Making this uniform over a collection of functions yields the following definition.

Definition 5.7 (Uniformly Equicontinuous). A collection of functions $\mathcal{F}$ is **uniformly equicontinuous** on T if

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{F}} \omega_f(\delta) = 0.$$

Equivalently, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{d(s,t)<\delta} |f(s) - f(t)| \leq \epsilon \quad \text{for all } f \in \mathcal{F}.$$

It turns out that the Definition above is precisely what is needed to show that collections of continuous functions are compact.

Theorem 5.8 (Arzelà-Ascoli). *Let (T, d) be a compact metric space. Then $\mathcal{F} \subset \mathcal{C}(T, \mathbb{R})$ is relatively compact if and only if it is uniformly equicontinuous and for some $t_0 \in T$ we have* $\sup_{f \in \mathcal{F}} |f(t_0)| < \infty$.

An equivalent statement to the theorem, which is what we in fact prove, is that the two statements

1. Any sequence $\{f_k\} \subset \mathcal{F} \subset \mathcal{C}(T, d)$ has a convergent subsequence in the supremum norm
2. The collection $\mathcal{F}$ is uniformly equicontinous and pointwise bounded, i.e. $\sup_{f \in \mathcal{F}} |f(t)| < \infty$ for all $t \in T$ are equivalent. (In statement (i), that $\sup_{f \in \mathcal{F}} |f(t_0)| < \infty$ is equivalent to $\sup_{f \in \mathcal{F}} |f(t)| < \infty$ for all $t \in T$ is a consequence of our calculation (1).)

Given $k > 0$, consider the space $\mathcal{F}^*_\Gamma(K)$ of maps $u \in C^0(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$, with the following properties:

- $E(u) \leq k$ **(Give an upper bound k to the Dirichlet energy)**
- $u|_{\partial\mathbb{D}}$ is monotone and onto Γ
- $u(1,0) = p_1, u(0,1) = p_1, u(-1,0) = p_3$,where p_1, p_2 and p_3 are three distinct points in Γ.

Then:

1. $\mathcal{F}^*_\Gamma(K)$ is equicontinuous on ∂D, this requires the Courant-Lebesgue Lemma, which will be skipped here.
2. By the Arzelá-Ascoli Theorem above, $\mathcal{F}^*_\Gamma(K)$ is compact in the topology of uniform convergence on ∂D.
3. The compactness gives us the ability to select a convergent subsequence.

Definition 5.9 (Dirichlet Problem). Let $w \in C^0(\bar{D}) \cap W^{1,2}(D)$, then there is a unique solution

$$u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$$

to the problem

$$\Delta u = 0, \quad u|_{\partial D} = w|_{\partial D}\,.$$

In fact, u minimizes the Dirichlet energy in the space of maps that are equal to w on ∂D, in particular $E(u) \leq E(w)$.

This gives us the chance to use the harmonic function, and the maximum principle, which we learnt or will learn in the PDE and complex analysis course.

Solution to the equivalent version of Plateau's Problem

1. Let w_k be a sequence of maps in $\mathcal{F}_\Gamma^*$ with $\lim_{k\to\infty} E(w_k) = \mathcal{E}_\Gamma^* \quad (\Longrightarrow E(w_k) \leq 2\mathcal{E}_\Gamma^*)$
2. Using Solution to the Dirichlet Problem, let $u_k \in \mathcal{F}_\Gamma^*$ be the harmonic map with $u_k|_{\partial D} = w_k|_{\partial D}$. In particular, $E(u_k) \leq E(w_k) \leq 2\mathcal{E}_\Gamma^*$.
3. Using compactness of $\mathcal{F}_\Gamma^*(2\mathcal{E}_\Gamma^*)$, the family of maps $u_k|_{\partial D} : \partial D \to \Gamma$ is equicontinuous, that is, up to a subsequence, $u_k|_{\partial D}$ converges uniformly to $\gamma : \partial D \to \Gamma$ and γ is monotone and onto.
4. In sum, $u_k \in \mathcal{F}_\Gamma^*$ is a sequence of harmonic maps with

$$\lim_{k\to\infty} E(u_k) = \mathcal{E}_\Gamma^* := \inf_{u\in\mathcal{F}_\Gamma^*} E(u)$$

and $u_k|_{\partial D}$ converges uniformly to $\gamma : \partial D \to \Gamma$, with γ monotone and onto.

5. Because u_k is the solution to some Dirichlet problem, so they are all harmonic, so every $(u_j - u_k)$ are harmonic, and these give us the capability of using the maximum principle:

$$\sup_D |u_j - u_k| = \max_{\partial D} |u_j - u_k| \,(\to 0).$$

6. Therefore u_k converges uniformly to $\bar{u} : D \to \mathbb{R}^3$ with $\bar{u}|_{\partial D} = \gamma$.
7. By Rellich Compactness Theorem, up to subsequences, we can suppose

$$u_l \xrightarrow{L^2} u \quad \text{and} \quad \nabla u_l \xrightarrow{L^2} \nabla u,$$

in the weak L^2-sense. We have

$$E(u) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 \leq \liminf_{l\to\infty} \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u_l|^2 = E_\Gamma$$

and

$$E_\Gamma = A_\Gamma \leq \text{Area}(u) \leq E(u) \leq E_\Gamma.$$

Then u minimizes both area and energy. Moreover, u is almost conformal and harmonic.

6 Further Learning

Douglas only solves the existence of one surface, but there might be many minimal surfaces spanned by one Γ.

Raising the problem to surfaces in higher dimensions, or the cases when the mean curvature is constant, not just zero, is also interesting.

Definition 6.1 (High dimensional Plateau's problem)**.** Find and study n-dimensional surfaces of minimal area or n-dimensional minimal surfaces in $\mathbb{R}^{n+N}$ with a prescribed $(n-1)$-dimensional boundary.

We provide a brief formalization here and focus on how to describe minimality.

For an embedded submanifold $\Sigma \subset (M, g)$ a vector field along Σ is a smooth map $X : \Sigma \to TM$ with $X(p) \in T_pM$ for all $p \in \Sigma$.[9]

Definition 6.2 (Divergence)**.** We define the divergence of X along Σ by

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{k} g\left(D_{e_i} X, e_i\right) = \operatorname{tr}_{T\Sigma} DX$$

where $e_1, \ldots, e_k \in T_p\Sigma$ is an orthonormal basis.

There are still many interesting topics and problems to be discovered and solved.

A How to print them

My original thought was to construct them using the software Mathematica, but I found it is not user-friendly for beginners, so I tried to find STL-type documents that are already created.