A Brief Introduction to Poincaré Homology Spheres

Topological Object and 3D-Printed Model

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Outline

1. Mathematical Stuff

1.1 From Invariants to Homology Group

In basic topology, we learn about *topology invariants*, the properties of a space invariant under homeomorphisms, such as:

- the dimension d
- Euler characteristic[1] χ
- genus of a surface[2] g
- homotopy group (sequence) π_n

[1] $\chi = V - E + F$.

[2] $g = \frac{\chi - 2 + b}{2}$, b is the number of boundaries of a space.

1.1 From Invariants to Homology Group

However, homotopy groups are hard to compute in general.

And homotopy groups sometimes are not detailed enough, as isomorphic homotopy groups are *necessary* but *not sufficient* to determine two spaces that are homeomorphic.

Namely, homotopy equivalence is *weaker* than homeomorphism.

1.1 From Invariants to Homology Group

We may construct some counterexamples, like lens spaces[1] $L(5;1)$ and $L(5;2)$[2] whose homotopy groups are given by

$$\pi_n(L(p;q)) \cong \begin{cases} \mathbb{Z}/p, & n = 1 \\ \pi_n(S^3), & n > 1 \end{cases}$$

[1] The three-dimensional lens spaces $L(p;q)$ are quotients of S^3 by free $\mathbb{Z}/p$-actions: let p and q be coprime integers and consider S^3 as the unit sphere in $\mathbb{C}^2$, then the $\mathbb{Z}/p$-actions are generated by the homeomorphism $(z_1, z_2) \mapsto \left(e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2\right)$.

[2] J. W. Alexander showed this example in 1919.

1.1 From Invariants to Homology Group

So, we want to inspect more invariants of a space, especially by fancy and powerful algebra. One of them is *(singular) homology groups*.

In general, homology groups also involves using some Abelian groups to study holes in topological spaces.

Here we will quickly appreciate how to define such a group, with *simplex*, *complex*, *chain group* and *boundary operator*.

Unfamiliarity with this part will not hinder much the following discussion of Poincaré homology spheres.

1.1 From Invariants to Homology Group

To do algebra on a shape, we may first *triangulate* it, or build spaces using *simplices*.

Standard n-simplex Δ^n The convex hull of $n + 1$ *affinely independent* points $v_0, ..., v_n$ in $\mathbb{R}^m$ $(m \geq n)$.

- 0-simplex: A point.
- 1-simplex: A line segment.
- 2-simplex: A triangle (including interior).
- 3-simplex: A tetrahedron.

1.1 From Invariants to Homology Group

Simplicial complex K is a collection of standard simplices such that every face of a simplex in K is also in K. The intersection of any two simplices is a face of both.

Singular n-simplex σ A continuous map $\sigma : \Delta^n \to X$ from the standard n-simplex in a topological space X.[1]

[1] This need not be injective, and there can be non-equivalent singular simplices with the same image in X.

1.1 From Invariants to Homology Group

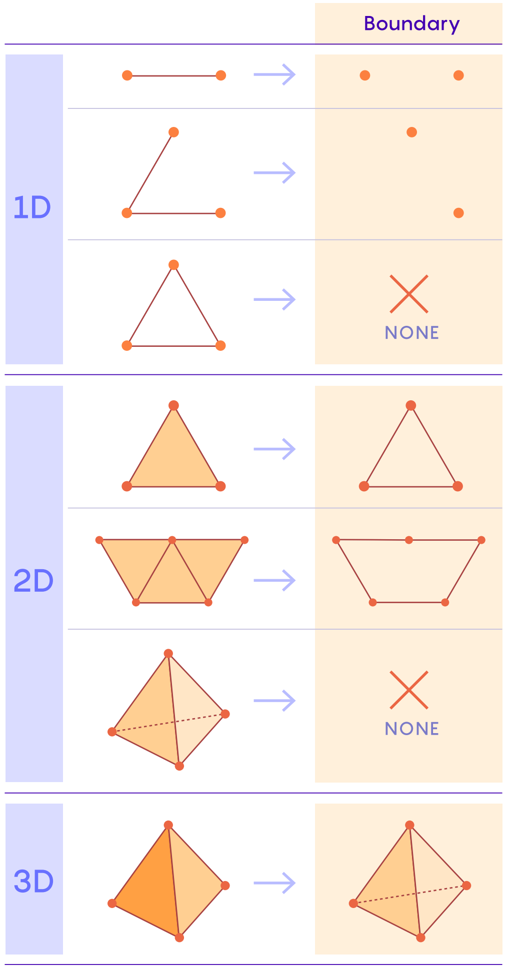

Figure 1: Simplices and boundaries

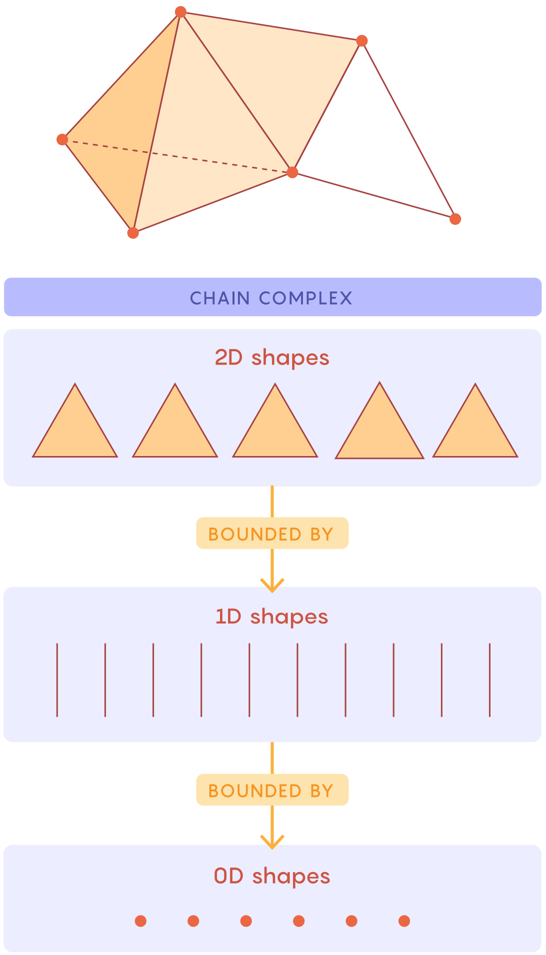

Figure 2: Chain complex

1.1 From Invariants to Homology Group

Singular n-chains group $C_n(X)$ is the free abelian group generated by basis the set of singular n-simplices of X.

Elements of $C_n(X)$ are called singular n-chains and are finite formal sums

$$\sum_i c_i \sigma_i$$

where $c_i \in \mathbb{Z}$.

We are concerned with the “holes”, or “boundaries”, of space.

The boundary operator ∂_n is the key of homology. For singular n-chains groups, it maps from $C_n(X)$ to $C_{n-1}(X)$,

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \mid_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

The symbol $\hat{v}_i$ indicates that vertex v_i is removed.

Example of 2-simplex Let $\sigma = [v_0, v_1, v_2]$, a filled triangle. $\partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. This represents the oriented edges of the triangle.

1.1 From Invariants to Homology Group

Property of boundary operator The most crucial property of the boundary operator is that “the boundary of a boundary is zero.”

$$\partial_{n-1} \circ \partial_n = 0$$

To prove this, compute $\partial_{n-1}(\partial_n(\sigma))$ and observe that terms of σ appears canceling out.

1.1 From Invariants to Homology Group

Singular Homology Group Since $\partial^2 = 0$, we know that the image of ∂_{n+1} is contained in the kernel of ∂_n.

- **Cycles Z_n** Elements in $\ker \partial_n$ (chains with no boundary, i.e., loops or closed surfaces).
- **Boundaries B_n** Elements in $\operatorname{im} \partial_{n+1}$ (chains that are the boundary of something higher-dimensional).

The n-th Homology Group is the quotient

$$H_n(X) = Z_n / B_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

1.1 From Invariants to Homology Group

The intuition is that H_n measures n-dimensional cycles that are not the boundary of an $(n+1)$-dimensional object. These are the “holes.”

Also, there is a similar path to define the simplicial homology group H_n^{Δ} starting by standard simplices. We omit this here.

In fact, for spaces on which both simplicial and singular homology groups can be calculated (i.e. can be triangulated), the two are equivalent $H_n(X) \cong H_n^{\Delta}(K)$.

1.1 From Invariants to Homology Group

Unfortunately, homology equivalence is even *weaker* than homotopy equivalence. We will see this by the example of Poincaré Homology Sphere.

1.2 Poincaré Homology Spheres

In 1900, Poincaré asked if homology was enough to *distinguish the 3-sphere* S^3 from other 3-manifolds.

The answer is no. Poincaré discovered a counterexample in 1904, now called the Poincaré Homology Sphere.

There are several ways to construct the Poincaré Homology Sphere. The most well-known is via the dodecahedron.

1.2 Poincaré Homology Spheres

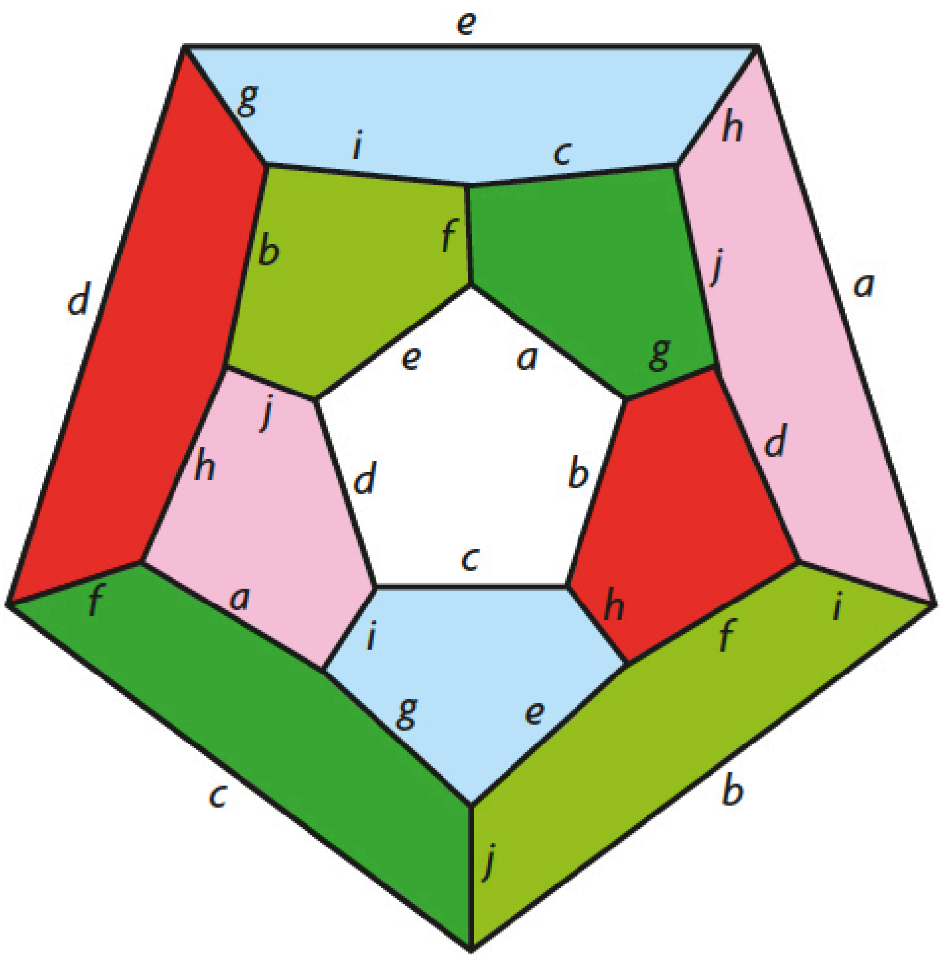

Figure 3: The Poincaré homology sphere shown as a dodecahedral space

1.2 Poincaré Homology Spheres

- Start with a regular *solid* dodecahedron.
- Identify opposite faces. Unlike the 3-torus (cube with opposite faces identified directly), identify opposite pentagonal faces with a twist of $\pi/5$ radians.
- There are 5 nonequivalent vertices and 10 nonequivalent edges, each formed by identifying 3 equivalent edges.
- This creates a closed, compact 3-manifold, denoted as Σ, whose Euler characteristic $\chi(\Sigma) = -5 + 10 - 6 + 1 = 0$.

Why is this a counterexample? We look at its algebra.

The Fundamental Group $\pi_1(\Sigma)$ The fundamental group of this space is non-trivial. It is isomorphic to the *Binary Icosahedral Group*,[1][2][3] a group of order 120.

$$\pi_1(\Sigma) \cong I^*$$

To show this, observe the relations.[4]

[1] The group I^* has a presentation $\langle s, t \mid (st)^2 = s^3 = t^5 \rangle$.

[2] Explicitly, the group I^* is given as the union of 120 even permutations of the following vectors in total: $(\pm 1, 0, 0, 0)$, $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$, $\left(0, \pm\frac{1}{2}, \pm\frac{1}{2\varphi}, \pm\frac{\varphi}{2}\right)$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

[3] The group is perfect, meaning it equals its own commutator subgroup $I^* = [I^*, I^*]$.

[4] Seifert and Threlfall "A textbook of topology", pages 223-225.

1.2 Poincaré Homology Spheres

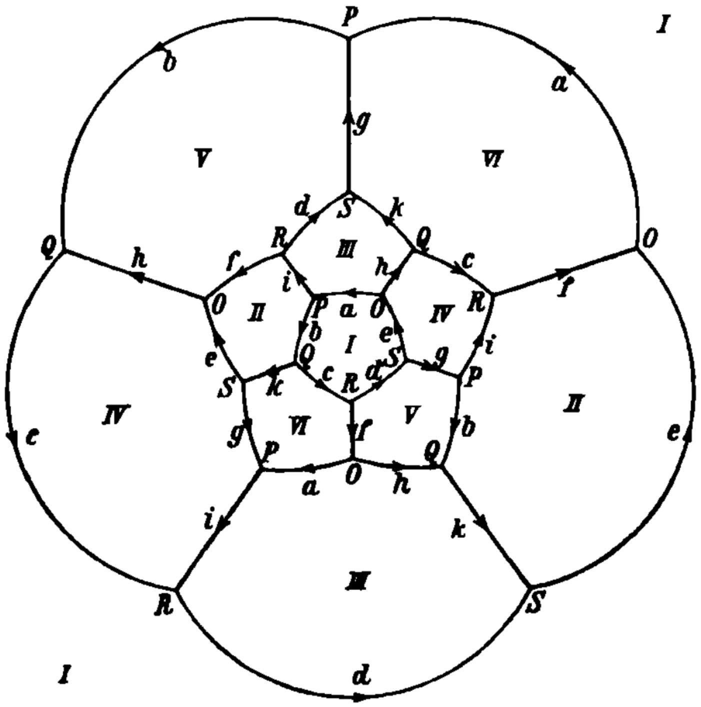

Figure 4: Another dodecahedral illustration

The closed paths give

$$\begin{cases} A{=}aa^{-1}, & B{=}abh^{-1} \\ C{=}hcf, & D{=}f^{-1}d(d^{-1}f) \\ E{=}(f^{-1}d)e, & F{=}f^{-1}f \\ G{=}(f^{-1}d)ga^{-1}, & H{=}hh^{-1} \\ J{=}aif, & K{=}hk(d^{-1}f) \end{cases}$$

Then these become

$$A = D = F = H = \mathbf{1} \quad \text{(I)}$$

and the remaining relations become trivial.

1.2 Poincaré Homology Spheres

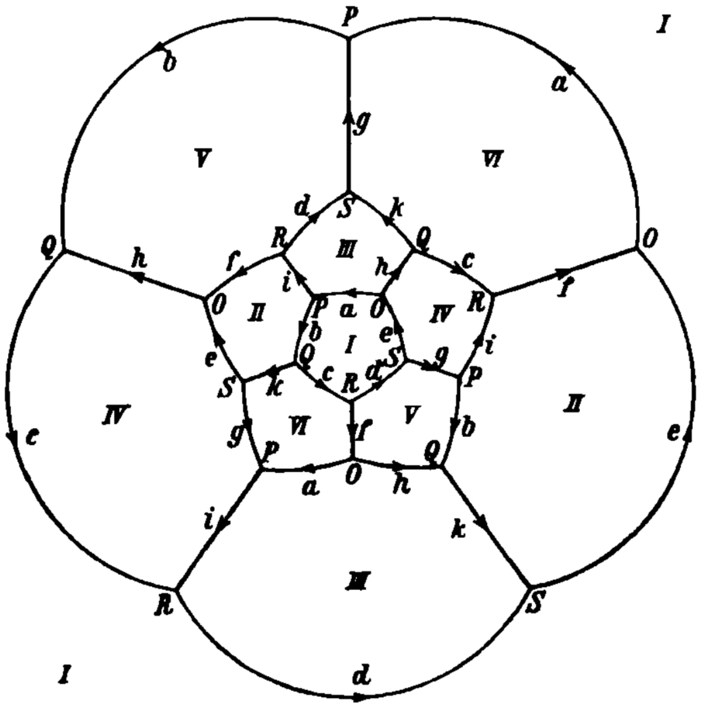

Figure 5: Another dodecahedral illustration

The pentagons give

$$\begin{cases} ABCDE = \mathbf{1} \\ BKEF^{-1}J^{-1} = \mathbf{1} \\ AJDK^{-1}H^{-1} = \mathbf{1} \\ CJ^{-1}G^{-1}EH = \mathbf{1} \\ BH^{-1}F^{-1}DG = \mathbf{1} \\ AG^{-1}K^{-1}CF = \mathbf{1} \end{cases}$$

Then these become, by reducing G and K,

$$\begin{cases} BCE = \mathbf{1} \\ BJEJ^{-1} = \mathbf{1} \\ CJ^{-1}BE = \mathbf{1} \\ BJ^{-1}C = \mathbf{1} \end{cases} \qquad \text{(II)}$$

1.2 Poincaré Homology Spheres

Set (II) into (I) to get

$$B^2C^{-1}B^{-3}C^{-1} = \mathbf{1}$$

Introducing a new generator U by $C^{-1} = B^{-1}U$, this becomes

$$B^2B^{-1}UB^{-3}B^{-1}U = \mathbf{1} \quad \text{or} \quad B^4 = UBU, U^2 = BUB$$

This is exactly the presentation of the binary icosahedral group by

$$B^5 = (BU)^2 = U^3$$

1.2 Poincaré Homology Spheres

The Homology Groups $H_n(\Sigma)$ The homology groups of Σ are the same as those of the 3-sphere S^3.

$$\begin{cases} H_0(\Sigma) = \mathbb{Z} \\ H_1(\Sigma) = \mathbf{0} \\ H_2(\Sigma) = \mathbf{0} \\ H_3(\Sigma) = \mathbb{Z} \\ H_k(\Sigma) = \mathbf{0}, \quad k > 3 \end{cases}$$

Σ has the homology of a sphere, but $\pi_1(\Sigma) \neq 0$, it is not a sphere.

The following shows how to get these results.

1.2 Poincaré Homology Spheres

- Nonempty and path connected spaces have an integral 0-th singular homology group $H_0(\Sigma) = \mathbb{Z}$.
- Since Σ is a 3-dimensional manifold, all homology groups in dimensions higher than 3 are trivial $H_k(\Sigma) = \mathbf{0}, \quad k > 3$.

Then, $H_1(\Sigma)$, $H_2(\Sigma)$, and $H_3(\Sigma)$ need more algebra topology content that there wasn't time to go into detail.

1.2 Poincaré Homology Spheres

- By Hurewicz theorem, the first homology group of a path-connected space is the abelianization of the fundamental group

$$H_1(\Sigma) \cong \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)] = \pi_1(\Sigma)/\pi_1(\Sigma) = \mathbf{0}$$

or explicitly

$$\begin{aligned} H_1(\Sigma) \cong \pi_1(\Sigma)_{ab} &= \left(\langle s, t \mid (st)^2 = s^3 = t^5 \rangle\right)_{ab} \\ &= \langle s, t \mid 2(s+t) = 3s = 5t \rangle \\ &= \langle t \mid s = 2t, 3s = 5t \rangle \\ &= \langle t \mid t = 0 \rangle \\ &= \mathbf{0} \end{aligned}$$

1.2 Poincaré Homology Spheres

- By Poincaré Duality and cohomology for the closed, orientable 3-manifold Σ,

$$H_3(\Sigma) \cong H^{3-3}(\Sigma) = H^0(\Sigma)$$

  Nonempty and path connected spaces have an integral 0-th singular cohomology group $H^0(\Sigma) = \mathbb{Z}$.

- By Poincaré Duality and cohomology for the closed, orientable 3-manifold Σ,

$$H_2(\Sigma) \cong H^{3-2}(\Sigma) = H^1(\Sigma)$$

1.2 Poincaré Homology Spheres

$H^1(\Sigma)$ relates to $H_1(\Sigma)$. The Universal Coefficient Theorem gives the cohomology group $H^1(\Sigma) \cong \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}) \oplus \mathrm{Ext}(H_0(\Sigma), \mathbb{Z}) = \mathbf{0} \oplus \mathbf{0} = \mathbf{0}$.

2. 3D Printing for Academic Purposes

2.1 Quick Start

Why 3D Print Mathematical Objects?

- Visualize and explore abstract concepts
- Share research with audiences
- Showcase personal taste as a decorative item

Figure 6: 3D printed Klein bottle

2.1 Quick Start

A Complete 3D Printing Process

Idea / Object / Code of Equations

↓ Modeling Software

3D Model File (e.g. STL[1])

↓ Slicer Software

Computer Numerical Control File (e.g. G-code[2])

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[1] STereoLithography or Standard Tessellation Language

2.1 Quick Start

↓ 3D Printer

Printed Object

↓ Commons Tools

Post-Processed Object

Total time: 2 hours to several days depending on size and complexity

[2] Geometric Code

2.1 Quick Start

Options for a Topological Model

Parametric Modeling

- Python/MATLAB/whatever scripts
- SolidWorks

Slicing

- Slicer of your 3D-printer brand (Bambu Studio, etc.)

3D Printer

- Fused Deposition Modeling
- StereoLithography Apparatus

Direct Modeling

- Blender
- Rhinoceros 3D
- Scan/Measure a real object

Post-Process

- Support removal
- Build up
- Sanding
- Draw/Paint

2.1 Quick Start

Key to Advancement

There are hundreds of parameters for 3D print, but for a specific model, only several key constraints matter most, and the others can be left to default.

For a math-demo models, consider

- **Material**: price affordable, proper printing temperature, bright colors
- **Overhangs**: $\leq$ 55-65° without supports, and < 5-10mm horizontal bridges

2.1 Quick Start

- **Support**: mathematical objects sometimes have bizarre geometries needing support, the interface between support and model ≥ 1 layer height for easy removal
- **Layer Height and Size**: typical size $\geq$ 10 cm, layer height $\geq$ 0.2mm since these models is not for extreme smoothness or mechanical assembly precision.

See Also

- **thingiverse.com**: Free pre-made models
- **wiki.bambulab.com**: Troubleshooting guide, learn more about Bambu Lab printer and the software

2.2 Our Printed Instances

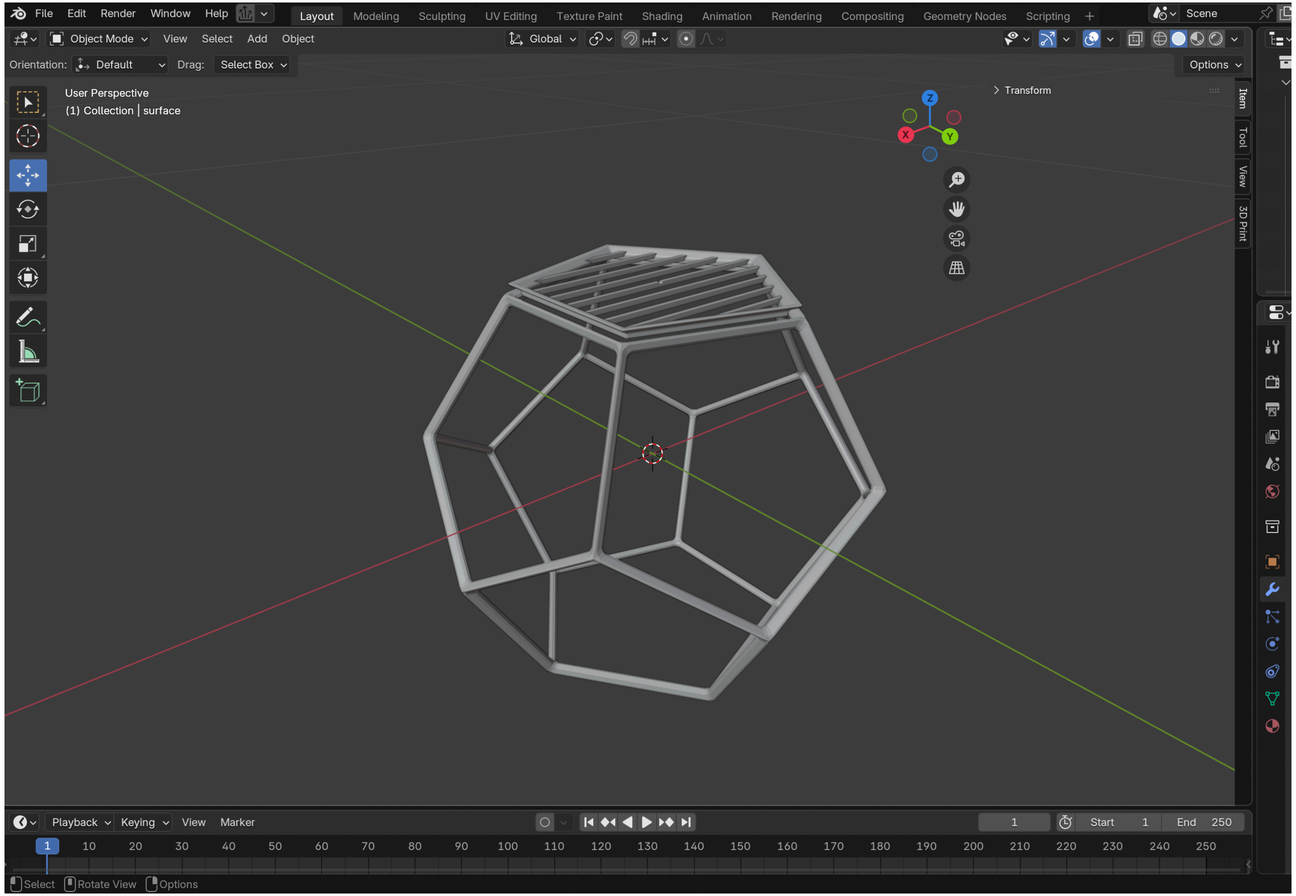

Figure 7: Handmade dodecahedron frame and faces model for Poincaré homology sphere in Blender

2.2 Our Printed Instances

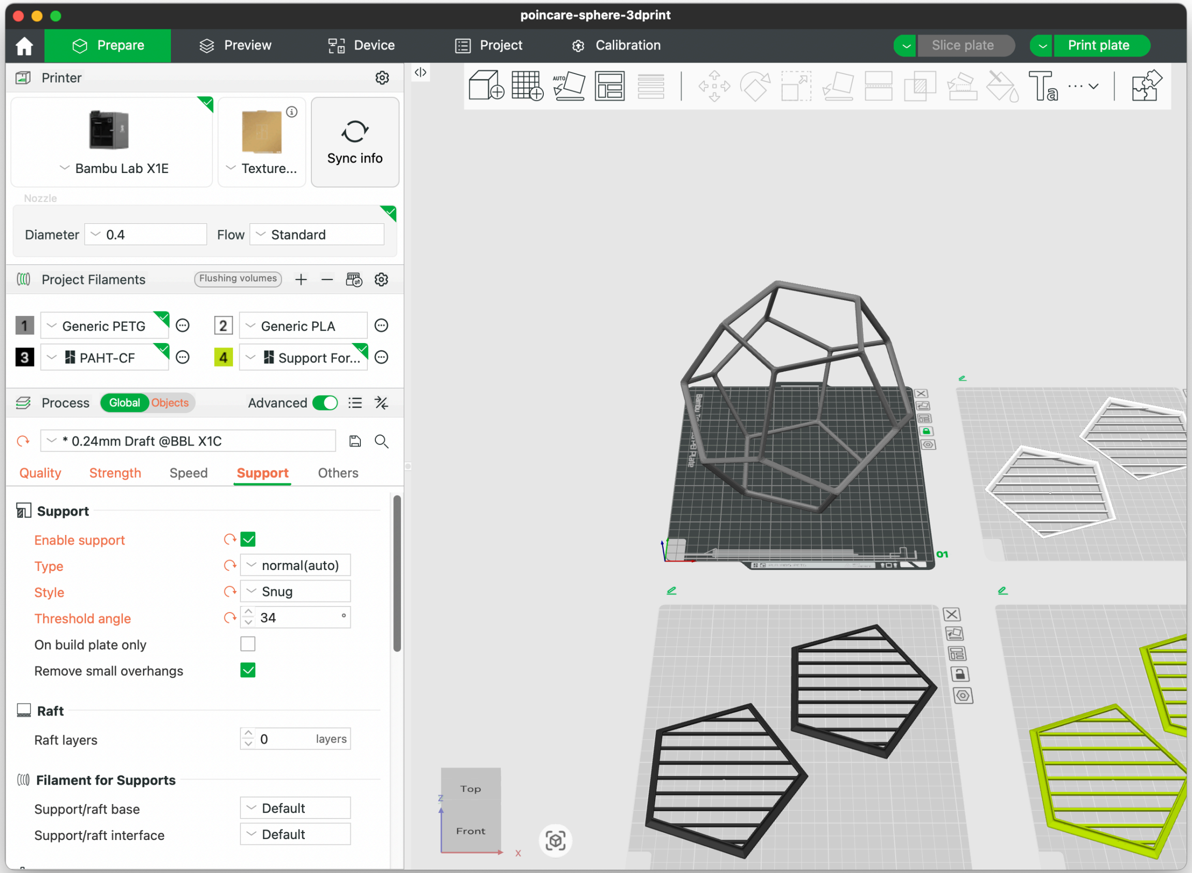

Figure 8: Parts of model to print in BambuStudio

2.2 Our Printed Instances

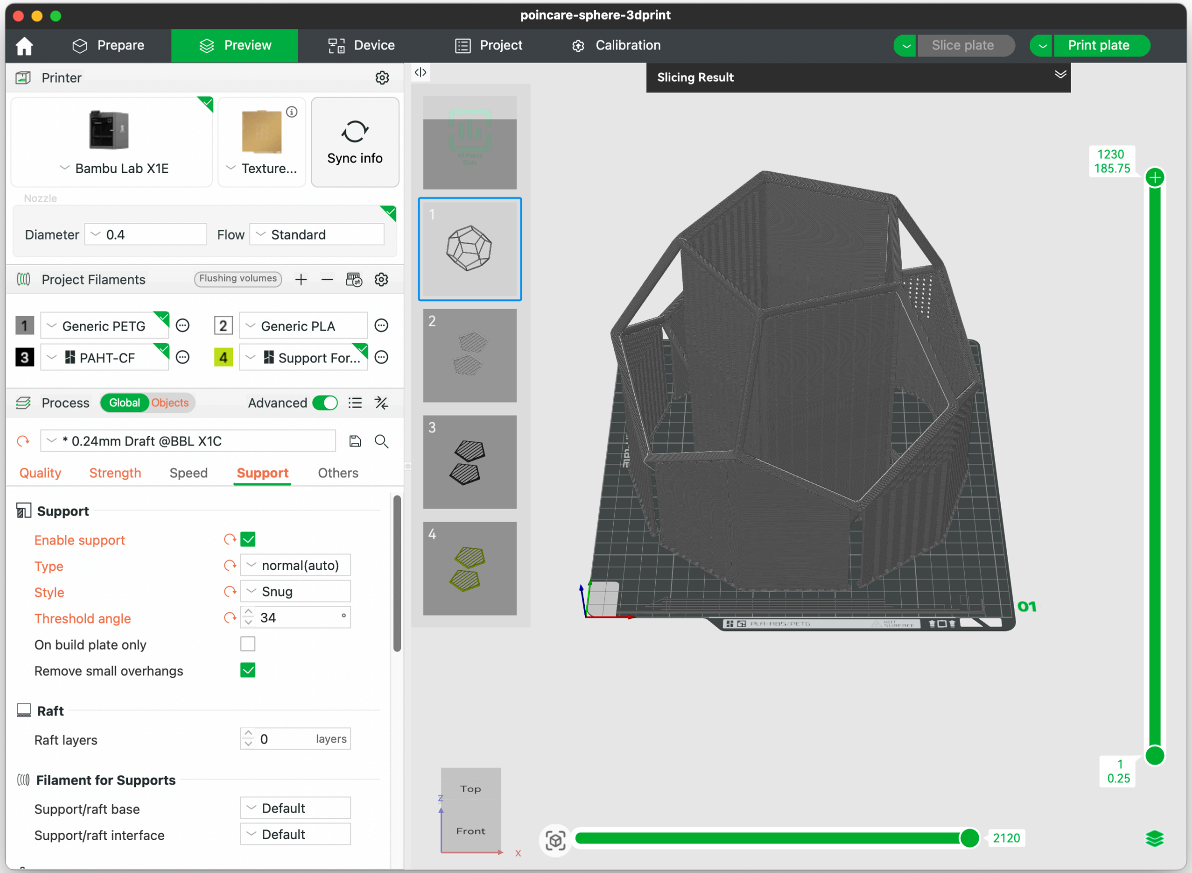

Figure 9: Sliced Poincaré homology sphere model frame in BambuStudio

2.2 Our Printed Instances

Then, we printed the frame and faces of the Poincaré homology sphere model, using filaments of different materials and colors.

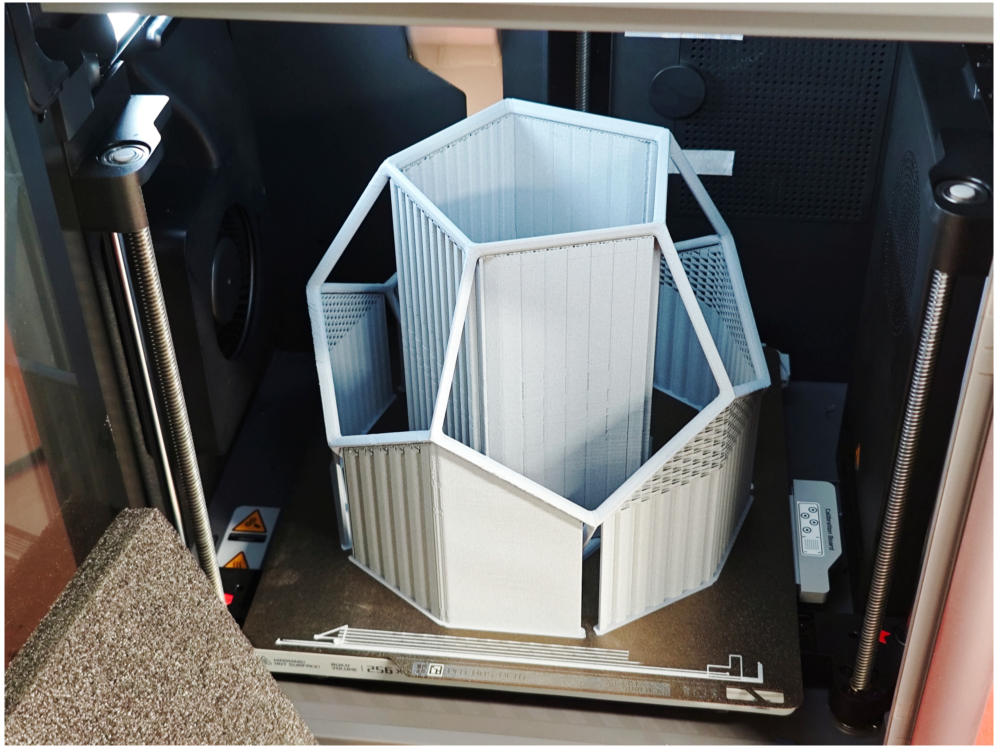

Figure 10: Just printed Poincaré homology sphere model frame

2.2 Our Printed Instances

Figure 11: Poincaré homology sphere model frame

2.2 Our Printed Instances

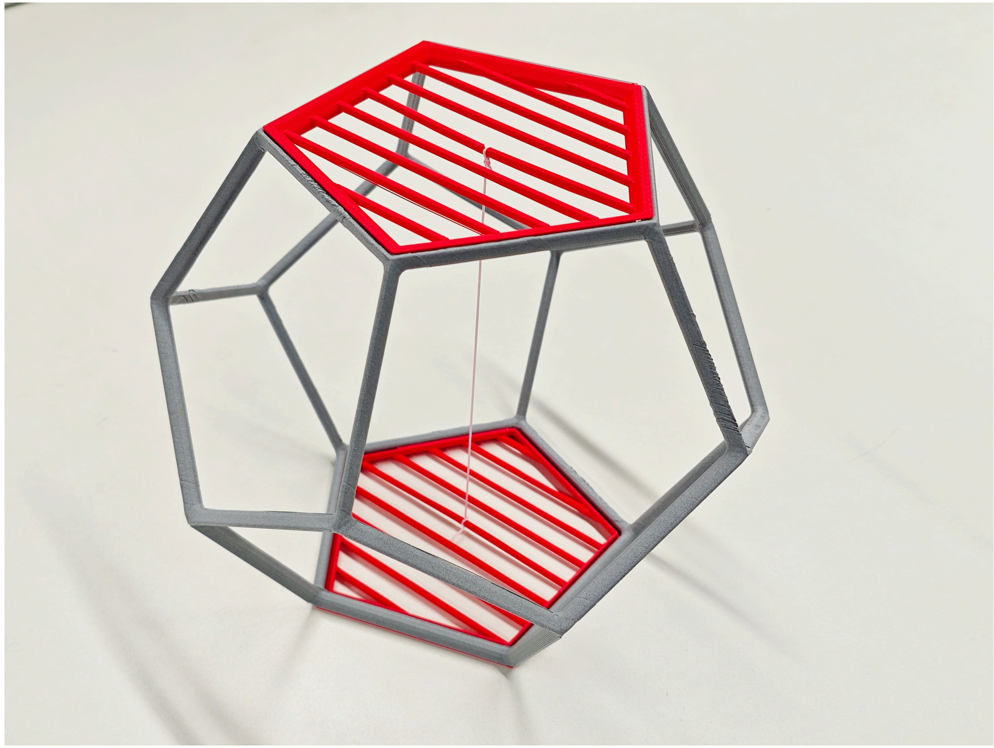

Figure 12: Poincaré homology sphere model building process

2.2 Our Printed Instances

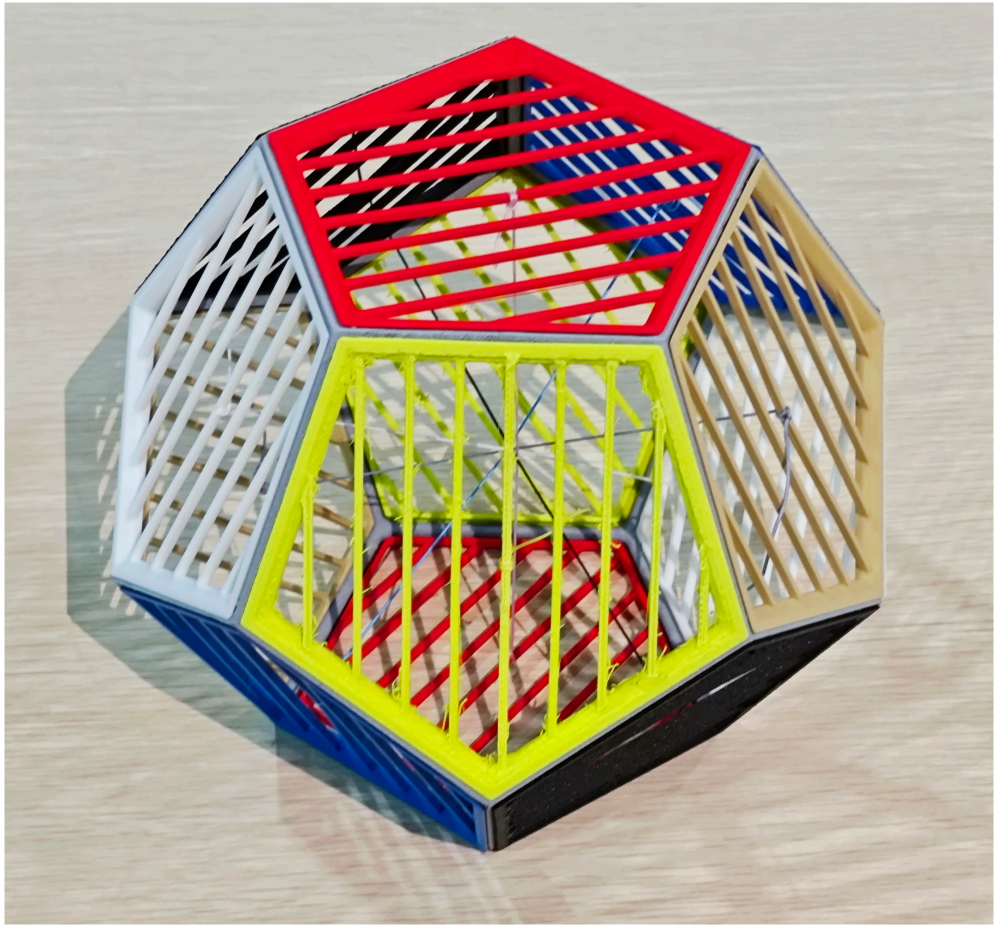

Figure 13: Poincaré homology sphere model completed