拓扑展示解析版

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目录

0.1 Knots and Links

Definition 0.1.1. *knot [3, 2]*

A subset K of a (topological) space X is a *knot* if K is homeomorphic with a sphere S^P.

Definition 0.1.2. *topology [1, 76]*

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:
(1) $\emptyset$ and X are in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Definition 0.1.3. *topological space [1, 76]*

A *topological space* is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Definition 0.1.4. *homeomorphism [1, 105]*

Let X and Y be topological spaces and $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a *homeomorphism*.

Definition 0.1.5. *unit sphere [1, 156]*

Define the *unit sphere* S^{n-1} in $\mathbb{R}^n$ by the equation $S^{n-1} = \{x \mid \| x \| = 1\}$.

Definition 0.1.6. *link [3, 2]*

A subset K of a (topological) space X is a *link* if K is homeomorphic with a disjoint union $S^{P_1} \cup \cdots \cup S^{P_r}$ of one or more spheres.

Definition 0.1.7. *equivalent [3, 2]*

Two links K and K' are *equivalent* if there is a homeomorphism $h : X \to X$ s.t. $h(K) = K'$.

Remark 0.1.1. *[3, 2]*

Unless otherwise stated, we shall always take X to be $\mathbb{R}^n$ or S^n.

Remark 0.1.2. *[3, 3]*

These definitions are not universally accepted.
Knots can be considered as embeddings $K : S^P \to S^n$, i.e., $K : S^P \hookrightarrow S^n$.

Proof 0.1.1.

$\Rightarrow$:
$K \subseteq S^n$ is homeomorphic with a sphere S^P, i.e., $\exists$ a homeomorphism $h : S^P \to K$.
We need to construct an embedding $f : S^P \to S^n$ s.t. $f(S^P) = K$.
Define the inclusion map $i : K \to S^n$, $x \mapsto x$.
Let $f = ih$.
h, i are continuous. $\Rightarrow f$ is continuous.
h, i are injective. $\Rightarrow f$ is injective.
$f(S^P) = K$.
h, i have their continuous inverses. $\Rightarrow f$ has its continuous inverse.
Therefore f is an embedding from S^P to S^n s.t. $f(S^P) = K$.
$\Leftarrow$:
Assume we have a knot that is an embedding $f : S^P \to S^n$.
Let $K = f(S^P)$.
$K \subseteq S^n$, $f : S^P \to K$ is a homeomorphism.
Therefore K is homeomorphic with S^P.
$\square$

Remark 0.1.3. *[3, 3]*

There are also other (stronger) notions of equivalence.
(1) Map equivalence: K and K' considered as maps and require $h \circ K = K'$.
(2) Oriented equivalence: All spaces are endowed with orientations, all of which h is required to preserve, i.e., h is orientation-preserving.

Definition 0.1.8. *orientation [1, 447]*

Given a line segment L in $\mathbb{R}^2$, an *orientation* of L is simply an ordering of its end points. The first, say a, is called the *initial point*, and the second, say b, is called the *final point*, of the oriented line segment.

Definition 0.1.9. *ambient isotopy [3, 3]*

A homotopy $h_t : X \to X$ is called an *ambient isotopy* if h_0 is the identity and each h_t is a homeomorphism.
Note: $h_t(x) = h(x,t)$, $h : X \times I \to X$.

Definition 0.1.10. *join [3, 6]*

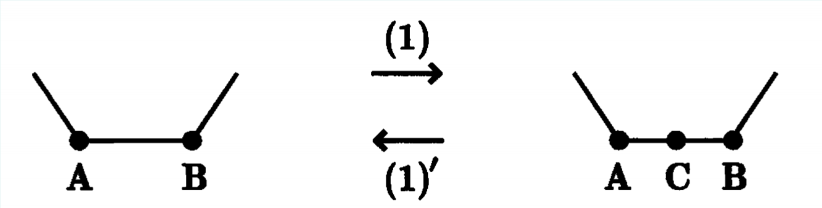
If X, Y are topological spaces, then their *join* is the factor space $X \star Y = (X \times Y \times I)/\sim$, where $\sim$ is the equivalence relation: $(x,y,t) \sim (x',y',t'). \iff \begin{cases} t = t' = 0 \text{ and } x = x'. \\ \text{or} \\ t = t' = 1 \text{ and } y = y'. \end{cases}$

Example 0.1.1. *[3, 6]*

If Y is a point, we have the *cone* $C(X) = X \star \{\text{pt}\}$.

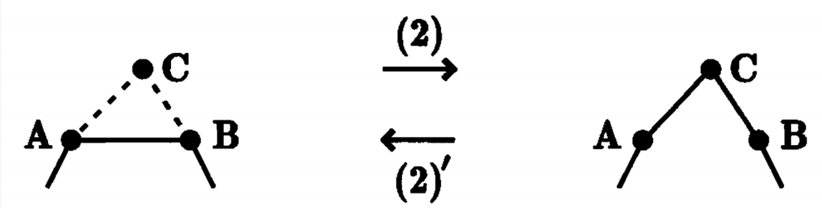
Definition 0.1.11. *elementary knot move [2, 7]*

On a given knot K we may perform the following four operations.
(1) We may divide an edge, AB, in space of K into two edges, AC, CB, by placing a point C on the edge AB.
(1)' [The converse of (1)] If AC and CB are two adjacent edges of K s.t. if C is erased AB becomes a straight line, then we may remove the point C.



(2) Suppose C is a point in space that does not lie on K. If the triangle ABC, formed by AB and C, does not intersect K, with the exception of the edge AB, then we may remove AB and add the two edges AC and CB.

(2)' [The converse of (2)] If there exists in space a triangle ABC that contains two adjacent edges AC and CB of K, and this triangle does not intersect K, except at the edges AC and CB, then we may delete the two edges AC, CB and add the edge AB.

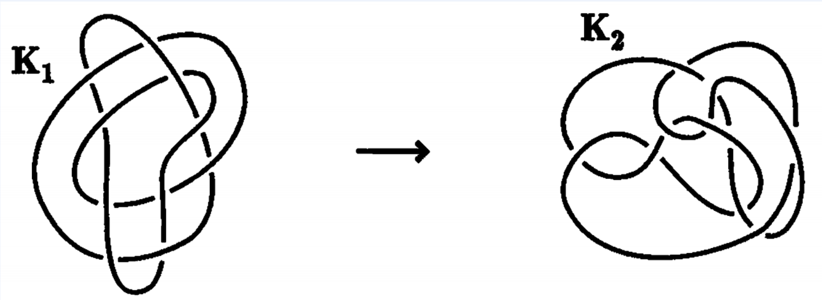


These four operations (1), (1)', (2) and (2)' are called the *elementary knot moves*.

Definition 0.1.12. *equivalent, equal [2, 8]*

A knot K is said to be *equivalent* (or *equal*) to a knot K' if we can obtain K' from K by applying the elementary knot moves a finite number of times.

Example 0.1.2. *Perko's pair [2, 8]*



It is possible to change the knot K_1 into the knot K_2 by performing the elementary knot moves a significant number of times. This was only shown in 1970 by the American lawyer

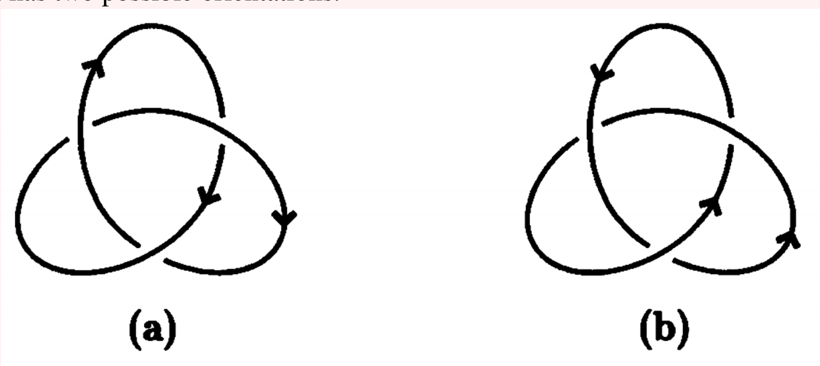
K. A. Perko.

Remark 0.1.4. *[2, 8]*

The elementary knot move (2) allows us to replace an edge AB with the edges AC and CB. Since the points within the triangle ABC do not intersect with the knot itself, intuitively we may rephrase the definition of equivalent:
Two knots are equivalent if in space we can alter one continuously, without causing any self-intersections, until it becomes transformed into the other knot.

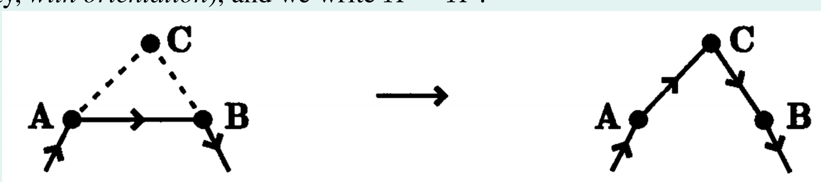
Remark 0.1.5. *[2, 8]*

A knot has no starting point and no endpoint, i.e., it is a simple closed curve (to be precise a closed polygonal curve). Therefore, we can assign an orientation to the curve.
Any knot has two possible orientations.



Definition 0.1.13. *equivalent with orientation [2, 9]*

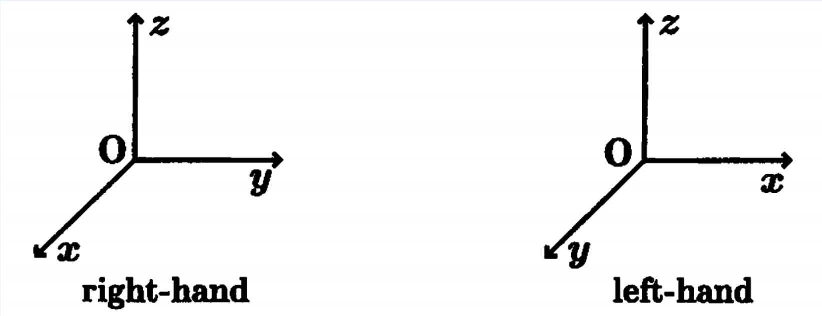
If two oriented knots K and K' can be altered with respect to each other by means of oriented elementary knot moves, then we say K and K' are *equivalent with orientation preserved* (or, for brevity, *with orientation*), and we write $K \cong K'$.



Remark 0.1.6. *[2, 9]*

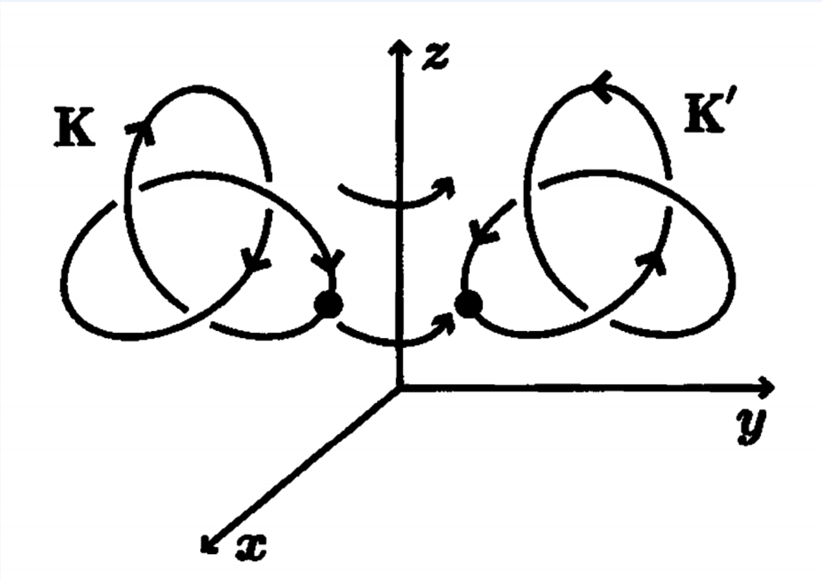
Two knots that are equivalent without an orientation assigned are not necessarily equivalent with orientation when we assign an orientation to the knots.

Example 0.1.3. *[2, 10]*



Example 0.1.4. *[2, 12]*

Consider $\varphi(x,y,z) = (-x,-y,z)$, which is an orientation-preserving auto-homeomorphism.
Since φ maps the oriented left-hand trefoil knot K to K', these two knots are equivalent with orientation. K' is the knot with the "reverse" orientation to K.



Definition 0.1.14. *oriented left-hand knot*

If you use your left thumb to trace the direction of the knot, the natural bending direction of your other fingers will be consistent with the twisting direction of the knot, then the knot is an *oriented left-hand knot*.

Definition 0.1.15. *oriented right-hand knot*

If you use your right thumb to trace the direction of the knot, the natural bending direction of your other fingers will be consistent with the twisting direction of the knot, then the knot is an *oriented right-hand knot*.

Definition 0.1.16. *link [2, 15]*

A *link* is a finite, ordered collection of knots that do not intersect each other.

Definition 0.1.17. *component [2, 15]*

Each knot K_i is said to be a *component* of the link.

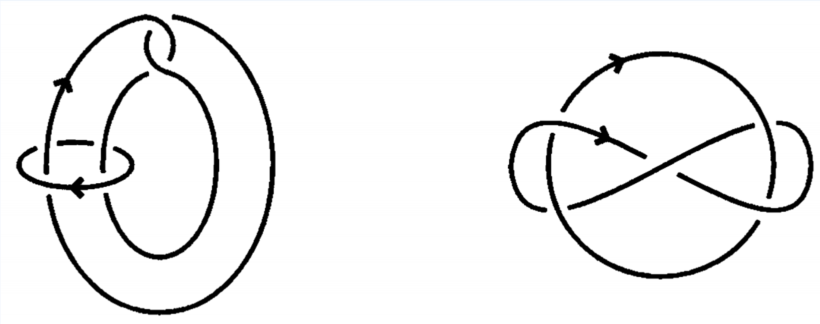
Definition 0.1.18. *equivalent [2, 15]*

Two links $L = \{K_1, K_2, \cdots, K_m\}$ and $L' = \{K'_1, K'_2, \cdots, K'_n\}$ are *equivalent* (or *equal*) if the following two conditions hold:
(1) $m = n$, that is, L and L' each have the same number of components.
(2) We can change L into L' by performing the elementary knot moves a finite number of times. To be exact, using the elementary knot moves we can change K_1 to K'_1, K_2 to K'_2, $\cdots$, K_m to K'_n $(m = n)$. (We should emphasize that the triangle of a given elementary knot move does not intersect with any of the other components.)
We may replace (2) by the following (2)':
(2)' There exists an auto-homeomorphism φ that preserves the orientation of $\mathbb{R}^3$ and maps $\varphi(K_1) = K'_1$, $\varphi(K_2) = K'_2$, $\cdots$, $\varphi(K_m) = K'_n$.

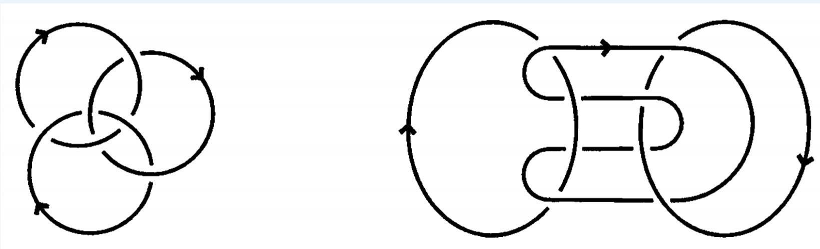
Remark 0.1.7. *[2, 15]*

We may suitably reorder the components.
Usually, therefore, (2)' is replaced by the following (2A):
(2A) There exists an auto-homeomorphism that preserves the orientation of $\mathbb{R}^3$ and maps the collection $K_1 \cup \cdots \cup K_m$ to the collection $K'_1 \cup \cdots \cup K'_n$.

Example 0.1.5. *Whitehead link [2, 17]*



Example 0.1.6. *Borromean rings [2, 17]*



Theorem 0.1.1. *[2, 14]*

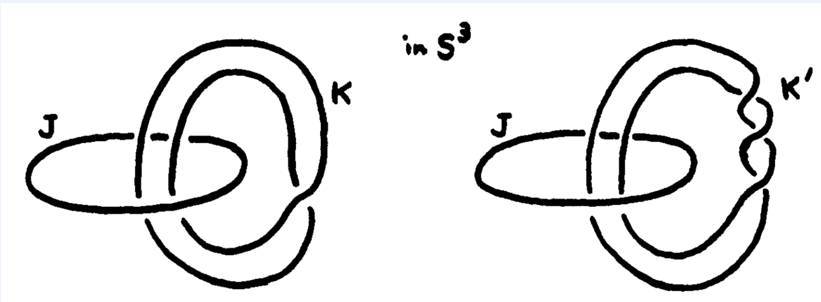
If two knots K_1 and K_2 that lie in S^3 are equivalent, then their complements $S^3 - K_1$ and $S^3 - K_2$ are homeomorphic.

Proof 0.1.2.

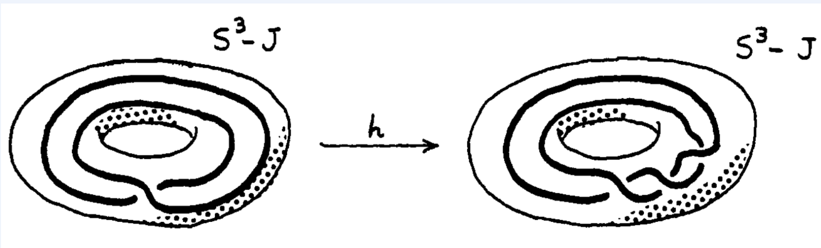
K_1 and K_2 are equivalent, i.e., $\exists$ a homeomorphism $f : S^3 \to S^3$ s.t. $f(K_1) = K_2$.
Consider the restriction $f\mid_{S^3 \setminus K_1} : S^3 \setminus K_1 \to S^3$.

$\forall\, x \in S^3 \setminus K_1$, if $f\mid_{S^3\setminus K_1}(x) = f(x) \in K_2$.
$f(K_1) = K_2$. $\Rightarrow \exists\, y \in K_1$ s.t. $f(y) = f(x)$.
f is a homeomorphism. $\Rightarrow x = y$, which is contradicting.
We have that $f\mid_{S^3\setminus K_1}(S^3 \setminus K_1) \subseteq S^3 \setminus K_2$.
$\forall\, y \in S^3 \setminus K_2$, f is a homeomorphism. $\Rightarrow \exists\, x \in S^3$ s.t. $f(x) = y$.
If $x \in K_1$. $\Rightarrow f(x) = y \in K_2$, which is contradicting. $\Rightarrow x \in S^3 \setminus K_1$.
Then $f\mid_{S^3\setminus K_1}$ is surjective.
f is a homeomorphism. $\Rightarrow f$ is injective. $\Rightarrow f\mid_{S^3\setminus K_1}$ is injective.
f is continuous. $\Rightarrow f\mid_{S^3\setminus K_1}$ is continuous.
f is a homeomorphism. $\Rightarrow f^{-1}$ is continuous. $\Rightarrow f^{-1}\mid_{S^3\setminus K_2}$ is continuous.
Similarly, $f^{-1}\mid_{S^3\setminus K_2}(S^3 \setminus K_2) \subseteq S^3 \setminus K_1$.
Therefore, $f\mid_{S^3\setminus K_1}$ is a homeomorphism from $S^3 \setminus K_1$ to $S^3 \setminus K_2$.
□

Example 0.1.7. *[3, 49]*



Example 0.1.8. *[3, 49]*



Proposition 0.1.1. *[3, 50]*

The integral homology and cohomology groups of the complement of a link in $\mathbb{R}^n$ or S^n are independent of the particular embedding.

Example 0.1.9. *[3, 50]*

With a knot $K^P \subseteq S^n$, $H \star (S^n - K^P) \cong H \star S^{n-P-1}$.

0.2 Knot Group

Definition 0.2.1. *group [3, 51]*

If K^{n-2} is a knot (link) in $\mathbb{R}^n$, the fundamental group $\pi_1(\mathbb{R}^n - K)$ of the complement is called, simply, the *group* of K.

Remark 0.2.1. *[3, 51]*

The group is the same, up to isomorphism, if we consider the knot in S^n rather than $\mathbb{R}^n$.

Proposition 0.2.1. *[3, 51]*

If B is any bounded subset of $\mathbb{R}^n$ s.t. $\mathbb{R}^n - B$ is path-connected and $n \geq 3$, then the inclusion induces an isomorphism $\pi_1(\mathbb{R}^n - B) \underset{i_*}{\rightarrow} \pi_1(S^{n-1} - B)$.

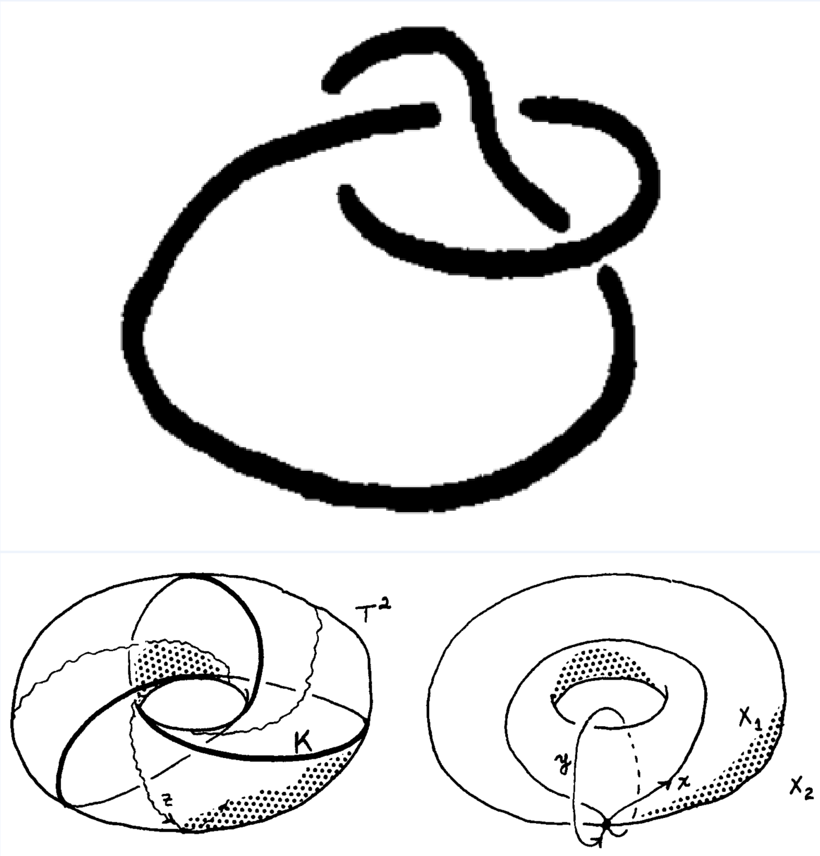
Example 0.2.1. *[3, 51]*

The naturally included $S^{n-2} \subseteq \mathbb{R}^{n-1} \subseteq \mathbb{R}^n \subseteq \mathbb{R}^n + \infty = S^n$ is the *trivial knot* or *unknot* of codimension two.

Proposition 0.2.2. *[3, 51]*

The unknot has group $\pi_1(S^n - S^{n-2}) \cong \mathbb{Z}$.

Example 0.2.2. *trefoil [3, 51]*



This knot has been drawn on the surface of a "standardly embedded" torus T^2 to employ Van Kampen's theorem.

Definition 0.2.2. *lens space [3, 234]*

Choosing fixed longitude and meridian generators l_1 and m_1 for $\pi_1(\partial V_1)$, we may write $h_*(m_2) = pl_1 + qm_1$ where p and q are coprime integers. The resulting M^3 is called the *lens space* of type (p, q) and denoted traditionally by $M^3 = L(p, q)$.
In other words, a 3-manifold is a lens space if and only if it contains a solid torus, the closure of whose complement is also a solid torus.

Lemma 0.2.1. *[3, 52]*

The trefoil is not of trivial knot type.

Proof 0.2.1. *[3, 52]*

Let X_1 and X_2 denote the closed solid tori, as shown, bounded by T^2 with K removed.
Then we have presentation $\pi_1(X_1) = \langle x \mid - \rangle$, $\pi_1(X_2) = \langle y \mid - \rangle$.
Let $X_0 = X_1 \cap X_2 = T_2 - K$.
We have $\pi_1(X_0) \cong \mathbb{Z}$, whose generator z equals x^2 in $\pi_1(X_1)$ and y^3 in $\pi_1(X_2)$.
Thus by Van Kampen, we get that $\pi_1(S^3 - \text{trefoil}) = \langle x, y \mid x^2 = y^3 \rangle$, which is nonabelian.
Therefore the trefoil is not of trivial knot type.
□

Definition 0.2.3. *torus knot [3, 53]*

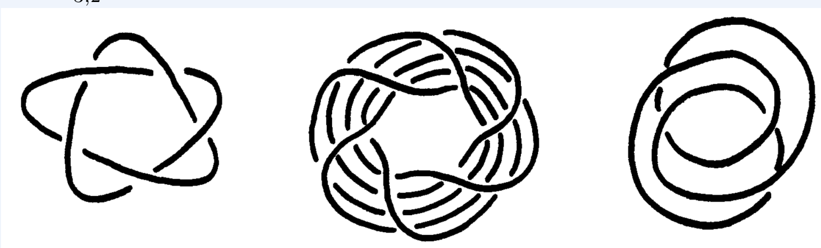
Choosing integers p, q which are relatively prime, the *torus knot* $T_{p,q}$ of type p, q is the knot which wraps around the standard solid torus T in the longitudinal direction p times and in the meridinal direction q times.

Example 0.2.3. *[3, 53]*

The trefoil is $T_{2,3}$.

Example 0.2.4. *[3, 53]*

$T_{2,5}$, $T_{5,6}$ and $T_{3,2}$.



Remark 0.2.2. *[3, 53]*

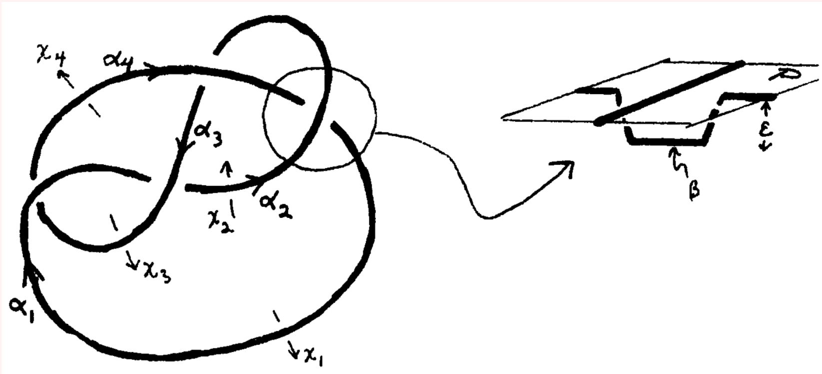
(1) $T_{\pm 1,q}$ and $T_{p,\pm 1}$ are of trivial type.
(2) The type of $T_{p,q}$ is unchanged by changing the sign of p or q, or by interchanging p and q.

Theorem 0.2.1. *[3, 54]*

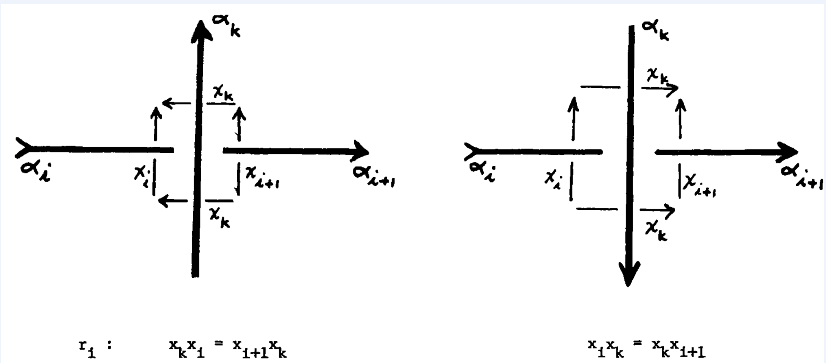
If $1 < p < q$, then the group $G_{p,q}$ determines the pair p, q.

Remark 0.2.3. *the Wirtinger presentation [3, 56]*

It is a presentation of the group of a knot K in $\mathbb{R}^3$, given a "picture" of the knot.
By a picture is a finite number of arcs $\alpha_1, \cdots, \alpha_n$ in a plane P (say, the $x - y$ plane). Each α_i is assumed connected to α_{i-1} and α_{i+1} (mod n) by undercrossing arcs exactly as pictured below. The union of these is the knot K.



Example 0.2.5. *[3, 57]*



Theorem 0.2.2. *[3, 57]*

The group $\pi_1(\mathbb{R}^3 - K)$ is generated by the (homotopy classes of the) x_i and has presentation $\pi_1(\mathbb{R}^3 - K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle$.

Moreover, any one of the r_i may be omitted and the above remains true.

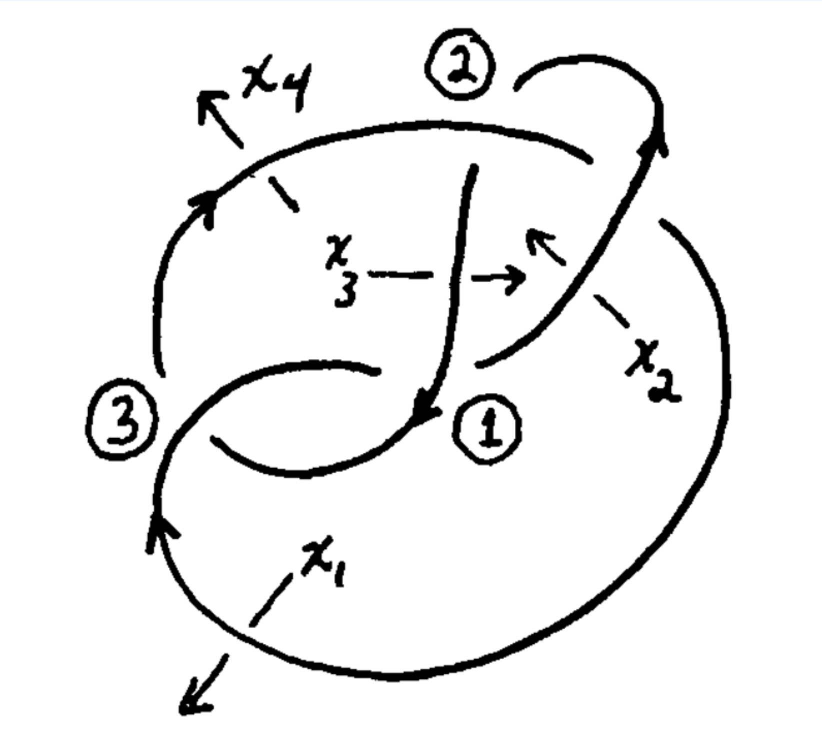
Example 0.2.6. *[3, 58]*

For the knot K below, we have a presentation with generators x_1, x_2, x_3, x_4 and relations

(1) $x_1x_3 = x_3x_2$.

(2) $x_4x_2 = x_3x_4$.

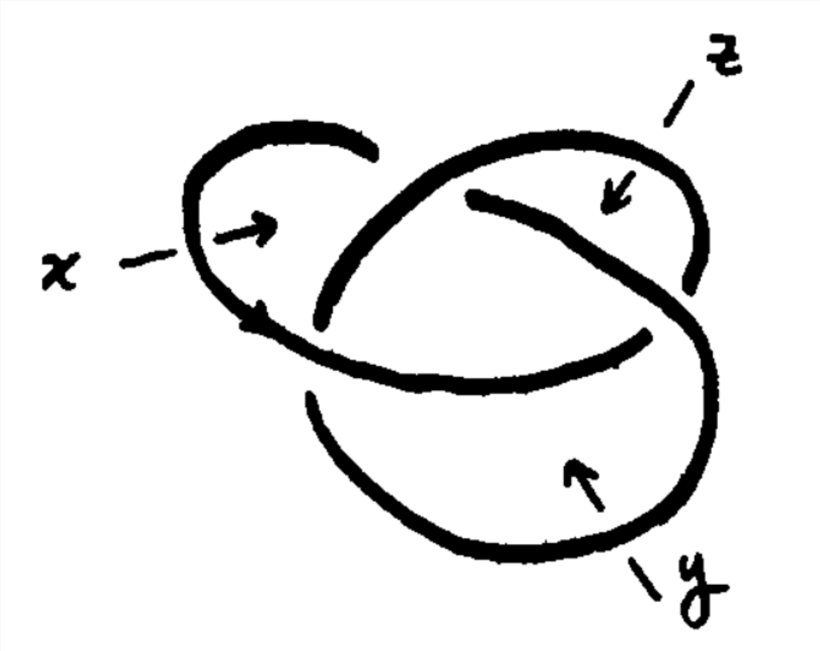
(3) $x_3x_1 = x_1x_4$.



In other words, $\pi_1(\mathbb{R}^3 - K) \cong \langle x_1, x_3 \mid x_1^{-1}x_3x_1x_3^{-1}x_1x_3 = x_3x_1^{-1}x_3x_1\rangle$.

Example 0.2.7. *[3, 60]*

We recompute the group of the trefoil using the Wirtinger method.



We have generators x, y, z and relations

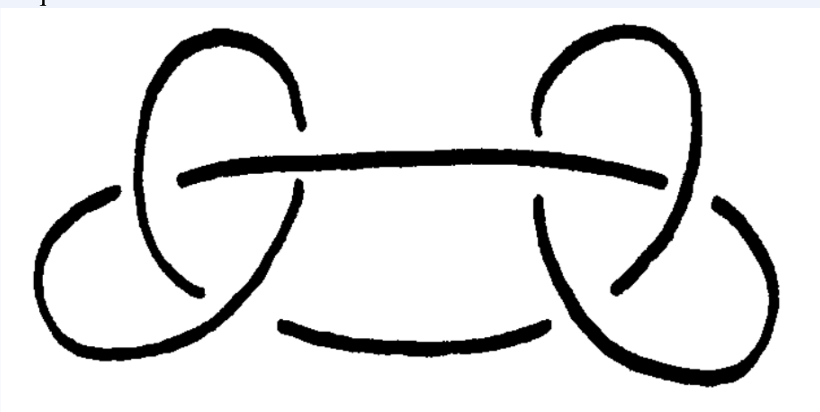
(1) $xz = zy$.

(2) $yx = xz$.

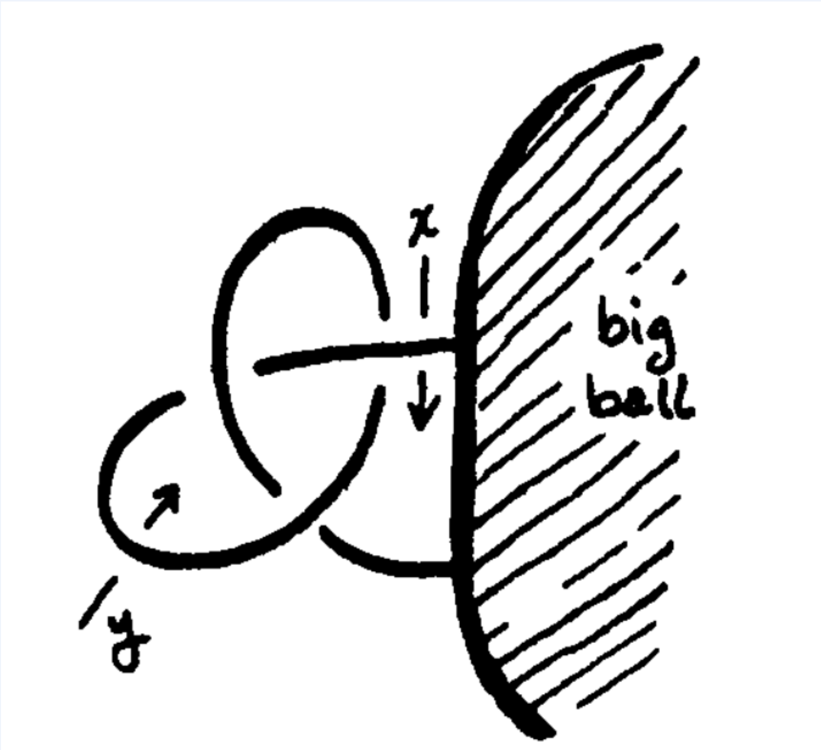
Thus we have another presentation for the trefoil group $G_{2,3} = \langle x, y \mid xyx = yxy \rangle$.

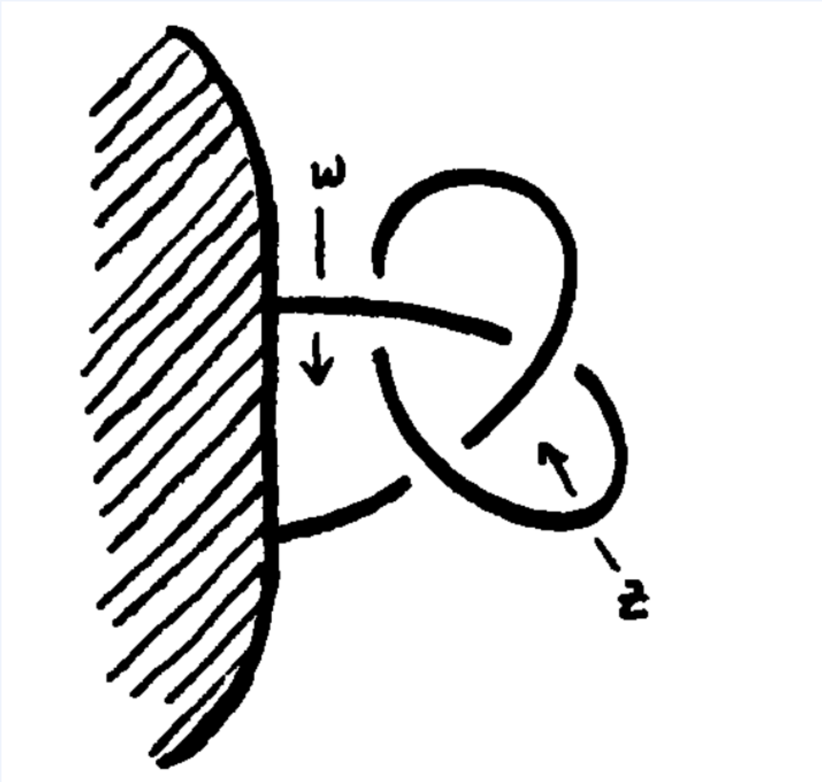
Example 0.2.8. *[3, 61]*

This is the square knot.



Decompose K into two parts.


x
y
big ball



The complements of them have the homotopy type of the complement of the trefoil.
The union of these complements gives the complement of the square knot.
Using Van Kampen's theorem, we see that
$G = \langle x, y, w, z \mid xyx = yxy, wzw = zwz, x = w\rangle = \langle x, y, z \mid xyx = yxy, xzx = zxz\rangle.$

Definition 0.2.4. *regular [3, 63]*

Let K be a polygonal knot in $\mathbb{R}^3$. Let P be any plane and $p : \mathbb{R}^3 \to P$ the orthogonal projection. Say that P is *regular* for K provided that every $p^{-1}(x)$, $x \in P$, intersects K in 0, 1 or 2 points and, if 2, neither of them is a vertex of K.

Definition 0.2.5. *deficiency [3, 64]*

The *deficiency* of a group presentation equals the number of generators minus the number of relations.

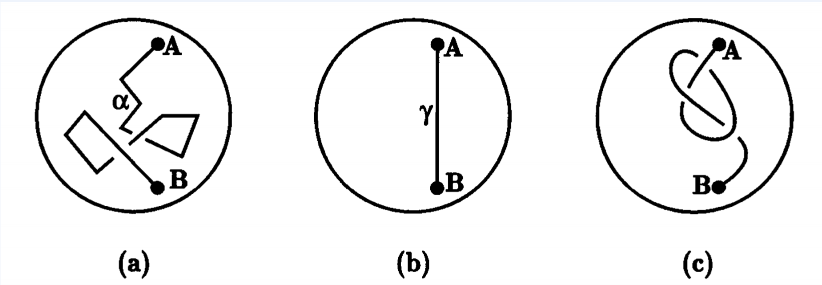
Corollary 0.2.1. *[3, 64]*

Every tame knot group has a finite presentation of deficiency one.

0.3 Knot Decomposition and the Semi-group of a Knot

Example 0.3.1. *[2, 18]*

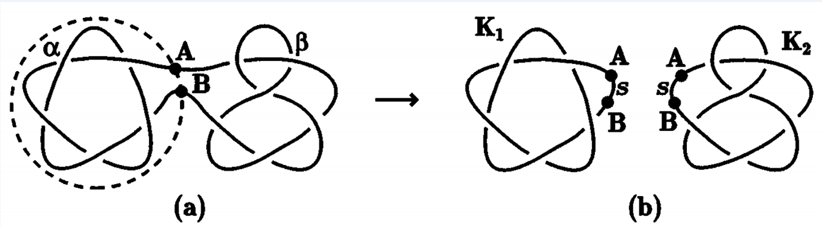
Consider a sphere S in S^3 (or $\mathbb{R}^3$) and the ball that is bounded by S, B^3, i.e., the 3-dimensional ball whose boundary is S.
In the interior of B^3, take a simple curved line α (in fact, a polygonal line) whose endpoints A, B are on the surface S.
If this curve α intersects S only at the points A and B, it is called a (1, 1)-tangle.



(a), (b) are trivial (1, 1)-tangles, while (c) is a nontrivial (1, 1)-tangle.

Example 0.3.2. *[2, 18]*

Suppose K is a knot (or link) in S^3 and there exists a 2-dimensional sphere $\sum$ that intersects (at right angles) K at exactly 2 points A and B.



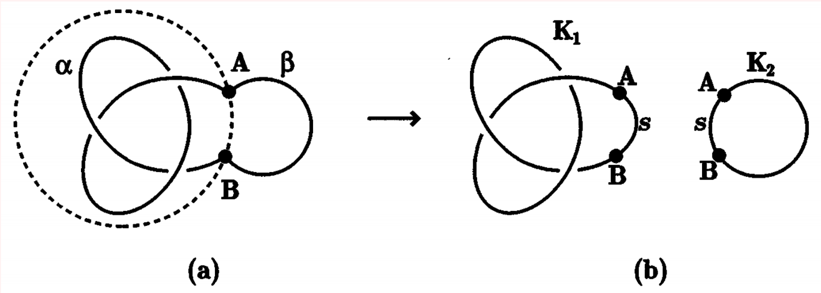
Since K lies in S^3, K is divided by $\sum$ into two (1, 1)-tangles α and β, one of which lies within $\sum$ and the other without.
This gluing process is more easily visualized if we drop down a dimension. For if we take two disks and glue them along their boundaries, in this case a circle, we obtain the 2-dimensional sphere.

Remark 0.3.1. *[2, 19]*

What we have shown is that a knot K can be decomposed into two knots K_1 and K_2.
The choice of s is arbitrary.
Because if we connect A to B by means of some other simple polygonal line that lies on $\sum$, s' for example, we shall once again decompose K into two knots say K_1' and K_2', are equivalent (since we may apply the elementary knot moves to s on $\sum$ to change it into s').

Remark 0.3.2. *[2, 19]*

If one of α or β, say, β, is the trivial (1, 1)-tangle, then K_2' is the trivial knot.
In such cases, K_1 and K_2 are not, strictly speaking, a "true" decomposition of K.



In fact, K and K_1 are equivalent, and so we do not think of K as being decomposed into simpler knots.

Definition 0.3.1. *prime knot [2, 19]*

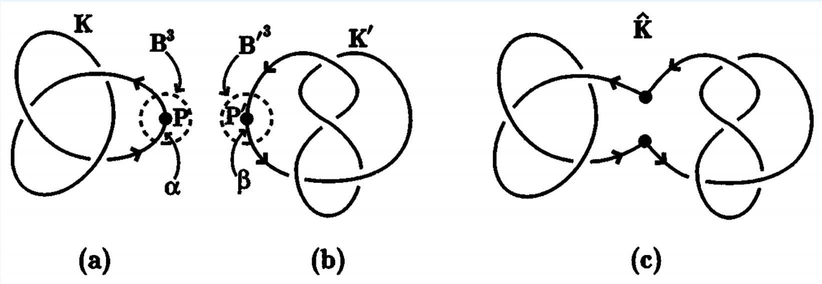
When a true (non-trivial) decomposition cannot be found for K, we say that K is a *prime knot*.

Theorem 0.3.1. *[2, 20]*

(The uniqueness and existence of a decomposition of knots)
(1) Any knot can be decomposed into a finite number of prime knots.
(2) This decomposition, excluding the order, is unique. That is to say, suppose we can decompose K in two ways: $K_1, K_2, \cdots, K_m$ and $K_1', K_2', \cdots, K_n'$. Then $m = n$ and if we suitably choose the subscript numbering of $K_1, K_2, \cdots, K_m$, $K_1 \cong K_1'$, $K_2 \cong K_2'$, $\cdots$, $K_m \cong K_n'$.

A proof of this theorem can be found in Schubert (not the one who write art songs).

Example 0.3.3. *[2, 20]*



Suppose P is a point on an (oriented) knot K in S^3. We may think of P as the centre of a ball, B^3, with a very small radius.
(1) K intersects (at right angles) exactly two points on the surface of boundary sphere of B^3.
(2) In the interior of B^3, the (1, 1)-tangle, α, that is obtained from K is a trivial tangle.
Similarly, to some other knot K' in another 3-dimensional sphere S^3, we may choose a point P' and, as above, obtain from K' a trivial (1, 1)-tangle, β, in some other ball B'^3.
If we now glue these two balls along this sphere, applying a homeomorphism that reverses throughout the orientation of the sphere of oe of these balls, we obtain a 3-dimensional sphere, S^3. In gluing process the end (initial) point of α and the initial (end) point of β are joined.
Therefore, in this 3-dimensional sphere, S^3, a new, single, oriented knot, $\hat{K}$ is formed.

Definition 0.3.2. *sum [2, 21]*

The knot $\hat{K}$ that is formed in the above process is said to be the *sum* of K and K' (or the *connected sum*), and is denoted by $K\#K'$.

Remark 0.3.3. *[2, 21]*

This knot $K\#K'$ is idependent of the points P and P' that were originally chosen. We can therefore say that $K\#K'$ is uniquely determined by K and K'.

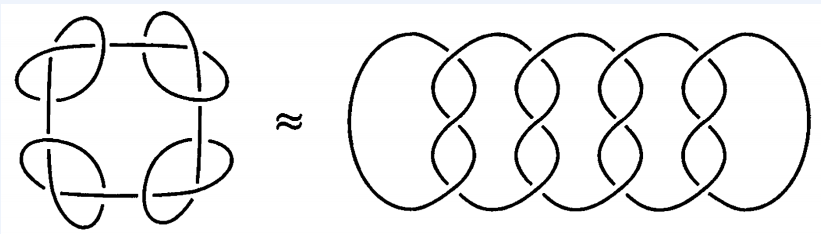
Proposition 0.3.1. *[2, 22]*

The sum of two knots is commutative, i.e., $K_1 \# K_2 \cong K_2 \# K_1$.
More concretely, $K_1 \# K_2$ and $K_2 \# K_1$ are equivalent with orientation.
Also, the associative law holds, $K_1 \#(K_2 \# K_3) \cong (K_1 \# K_2) \# K_3$.

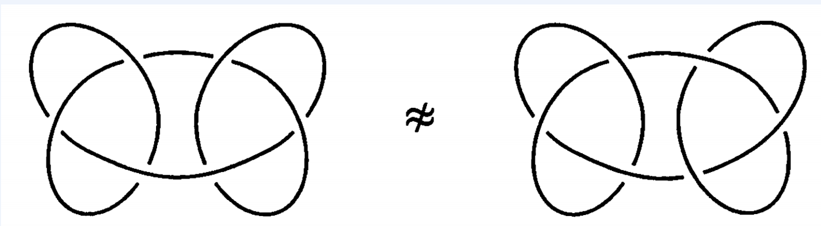
Remark 0.3.4. *[2, 22]*

The above definition of the sum of knots is defined on the set of all (oriented) knots, $\mathcal{A}$.
However, this definition does not make $\mathcal{A}$ a group.
In fact, $\mathcal{A}$ does not possess inverse elements.
For example, suppose K is the trefoil knot, for K it is impossible to find a K' s.t. $K \# K' \cong O$, the trivial knot.
Therefore $\mathcal{A}$ is only a semi-group. This semi-group is called the semi-group formed under the operation of the sum of knots.

Example 0.3.4. *[2, 22]*



Example 0.3.5. *[2, 22]*



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