Lectures on geometry and physics of Higgs bundles

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1 Classical Integrable systems

1.1 Prelude: symplectic and Poisson geometry

A symplectic structure on a manifold M is a two-form ω which is closed and non-degenerate. By Darboux theorem, on any symplectic manifold (M,ω) there exists local coordinates (p_i, x_i) such that $\omega = \sum dp_j \wedge dx_j$. This means that there is no local invariant in symplectic geometry. Note that this is not the case for Riemannian geometry where the curvature is a local invariant. Then we can define so-called Hamiltonian flow X_f of a function f to be a vector field such that $\omega(X_f, Y) = df(Y)$ for all $Y \in \Gamma(TM)$. While the usual gradient of a function f points in the direction of biggest change of f, the Hamiltonian flow points in the direction where f stays constant (the level set). For example, $\mathbb{S}^2$ with the area form and $h : \mathbb{S}^2 \to \mathbb{R}$ the height function. Then the Hamiltonian flow is the rotation around the z-axis.

It turns out The 1-parameter group of diffeomorphisms of M, obtained by integrating the Hamiltonian flow of a function, preserve the symplectic structure $\phi_t^*\omega = \omega$ (by Cartan magic formula). A diffeomorphism f of M preserving its symplectic structure (i.e. $\phi_t^*\omega = \omega$) is called a symplectomorphism. The special case coming from the time flow of a symplectic gradient is called a Hamiltonian diffeomorphism. Not all symplectomorphisms are Hamiltonian. A vector field X is called symplectic if $\omega(X, .)$ is a closed 1-form, Hamiltonian if $\omega(X, .)$ is exact. A vector field associated to a symplectomorphism (resp. hamiltonian diffeomorphism) is symplectic (resp. hamiltonian).

The space of smooth functions $C^\infty(M)$ comes with an extra structure: a Poisson bracket. Concretely, the symplectic form ω allows to associate to two functions $f, g \in C^\infty(M)$ a function denoted by $\{f, g\}$ defined by $\{f, g\} = \omega(X_f, X_g)$. $\dot{f} = \{H, f\}$

Poisson manifold is the disjoint union of symplectic manifolds (in a unique way), called symplectic leaves.

Example 1.1.1 (Coadjoint orbits). $\mathfrak{g} = \mathfrak{so}(3), \mathfrak{g}^* = (\mathbb{R}^3, \times)$

$$\langle g \cdot \xi, a \rangle = \left\langle \xi, ad_{g^{-1}}(a) \right\rangle$$

The action is called symplectic if all $g \in G$ act by symplectomorphisms, The infinitesimal action of an element $a \in \mathfrak{g}$ gives a vector field X_a.

$$X_a(x) = \frac{d}{dt}\Big|_{t=0} \exp(ta) \cdot x$$

1.1.1 Hamiltonian action

The action is called weakly Hamiltonian if each $g \in G$ acts by a Hamiltonian diffeomorphism. A weakly Hamiltonian action of G on M is called Hamiltonian if there is a Lie algebra homomorphism $\phi : \mathfrak{g} \to C^\infty(M)$ such that $X_{\phi(a)} = X_a$. This is a bit abstract, let's see the algebraic definition.

We can always transform a weakly Hamiltonian action into a Hamiltonian one by replacing the group G by a central extension.

1.2 Hamiltonian reduction and symplectic quotient

1.2.1 Moment map

1.2.2 Symplectic topology of Hitchin moduli and Atiyah Bott reduction

1.2.3 Equivalence to GIT quotient

1.3 Classical mechanics: from Newton to Hamilton

1.3.1 Lagrangian mechanics and Riemannian geometry

Example 1.3.1 (Euler top). As Lagrangian system and Arnold's generalization to arbitrary simple Lie algebras

1.3.2 Hamiltonian mechanics

The basic setting of Hamiltonian classical mechanics is as follows. The phase space of a mechanical system is a symplectic manifold M of dimension $2n$. The dynamics of the system is defined by its Hamiltonian $H \in C^\infty(M)$. Namely, the Hamiltonian flow attached to H is the flow corresponding to the vector field X_H. If y_i are coordinates on M, then the differential equations defining the flow (Hamiltonian equations) are written as

$$\frac{dy_i}{dt} = \{H, y_i\}$$

By Darboux theorem we can locally choose (so-called canonical) coordinates x_j, p_j on M such that the symplectic form is $\omega = \sum dp_j \wedge dx_j$. In canonical coordinates Hamiltonian equations are written as

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = \frac{\partial H}{\partial x_i}$$

In the presence of symmetries, we can utilize them to reduce our complicated differential equations to easier ones, even trivial ones if the symmetries are "large" enough. The mathematical underpinning is the aforementioned Hamiltonian reduction. Namely, let G be an Hamiltonian action on M. Let $\mu : M \to \mathfrak{g}^*$ be the moment map. to add Hamiltonian reduction Heruistics This triggers the definition of an integrable system.

Definition 1.3.2. An integrable system on a symplectic manifold M of dimension $2n$ is a collection of smooth functions $F_1,, \dots, F_n$ on M such that they are in involution (i.e., $\{F_i, F_j\} = 0$), and the differentials dF_i are linearly independent on a dense open set in M.

Remark 1.3.3. The condition that dF_i are linearly independent on a dense open set in M is equivalent to the requirement that F_i are functionally independent, i.e. there does not exist a nonempty open set $U \subseteq M$ such that the points $(F_1(u),...,F_n(u)), u \in U$, are contained in a smooth hypersurface in $\mathbb{R}^n$.

The word "integrable" suggests that the system can be integrated by quadratures, is indeed the case as we will see:

Theorem 1.3.4 (Arnold-Liouville)**.** *Let the Hamiltonian be $H = F_1$. For any given $v \in \mathbb{R}^n$, we define the fiber M_v be the set of points $x \in M$ such that $H_i(x) = v_i$.*

- *The fiber M_v is a smooth manifold.*
- *If M_v is compact and connected, it is diffeomorphic to $\mathbb{T}^n$ with coordinates $\theta_1, \dots \theta_n$. In fact, M_v is a half-dimensional torus on which $\omega|_{TM_v} = 0$, sometimes called a Lagrangian torus.*
- *There is an open neighborhood U of v, and a coordinate transformation $(q_j, p_j) \to (\theta_j, I_j)$ called action-angle coordinates such that Hamiltonian equation is given by*

$$\dot{\theta}_j = \omega_j(v), \dot{I}_j = 0$$

  where $\omega_j(v)$ is some function of v.

Example 1.3.5 (Euler top)**.** We consider a rotating solid body attached to a fixed point. The Euler top corresponds to the case where there is no external force. The equation of motion read $\dot{I} = -\omega \wedge J$ where $J = I.\omega$, $I = diag(I_1, I_2, I_3)$ equivalent to

$$\dot{J} = [IJ, J], -\frac{1}{2}tr(J^2) = ||J||^2$$

$$H(J) = 2(\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3})$$

Since $||J||^2$ and H are conserved, we can express J_2 and J_3 in terms of J_1. The exact calculation is gigantic, and you can try to write it out explicitly if you are so inclined. Anyway, you will finally see that J_1 satisfies an equation of the form

$$\dot{J}_1^2 = a + bJ_1^2 + cJ_1^4$$

which means t is an elliptic integral of J_1 or equivalently J_1 is an elliptic function of t. We can actually "predict" the presence of elliptic curve, without cumbersome calculation, by using spectral curve, as we will see in the next section.

Remark 1.3.6 (Digression on elliptic integrals, elliptic curves and Riemann surfaces). Elliptic integrals came from the computations of arc length. More precisely, they are of the form $\int R(t, \sqrt{P(t)})dt$ where $P(t)$ is a cubic or quartic. Legendre divided them into three classes. Let's consider the following one originated from the calculation of arc length:

$$\alpha(x) = \int_0^x \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}}$$

Since the derivative of $\alpha(x)$ is always greater than zero, we have an inverse function denoted by $x = \varphi(\alpha)$, dubbed an elliptic function. By Abel's addition theorem, we can naturally extend its domain to the real axis and even the whole complex plane. It turns out that $\varphi(\alpha)$ is doubly periodic, with period $2\omega_1$ and $2i\omega_2$ where

$$\omega_1 = \int_0^{1/c} \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}}, \omega_2 = \int_0^{1/e} \frac{dt}{\sqrt{(1+c^2t^2)(1-e^2t^2)}}$$

But here comes an issue with the complex plane. $\varphi(\alpha)$ is multi-valued which can be resolved by defining it on a Riemann surface. In fact, it is a function on an elliptic curve $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_{\mathbb{1}} \oplus \mathbb{Z}\omega_2$. Riemann even generalized the notion of elliptic functions to a larger class of functions on Riemann surfaces called abelian functions. Like the case of elliptic functions, they are inversion to abelian integrals which are of the form $\int R(t,\beta)dt$ where $F(t,\beta) = 0$, R is rational and F is irreducible polynomial in β. In modern language, we can interpret it as the map $\Sigma \to J(\Sigma)$, $p \mapsto \int_{p_0}^p \omega$, p_0 is a fixed point.

Consider an elliptic curve $E : y^2 = x^3 + px + q$ and $\omega = \frac{dx}{y}$ so we are indeed computing elliptic integrals. However, the integral is not so well-defined since integrating different paths between p_0 and p might result in different values. However, it is well-defined modulo the integration around a loop γ passing through both p_0 and p, i.e. $\int_\gamma \omega$ called a period of the elliptic curve which depends only on the homology class of the cycle $[\gamma] \in H_1(E,\mathbb{Z})$. Hence the "elliptic integrals" is a map from E to $\mathbb{C}/\Lambda$ complex number modulo the period lattice $\Lambda = \{\int_\gamma \omega | \gamma \in H_1(E,\mathbb{Z})\}$, it turns out to be an isomorphism.

Moreover, from a lattice Λ we can construct the inverse isomorphism by using the "universal" elliptic function - Weierstrass $\wp$ function and Eisenstein series:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda/\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \ G_k(\Lambda) = \sum_{\omega \in \Lambda/\{0\}} \omega^{-2k}$$

The map is as follows:

$$\mathbb{C}/\Lambda \to E(\Lambda) : y^2 = x^3 - 60G_2 - 140G_3, \ z \mapsto [\wp(z) : \wp'(z) : 1], 0 \mapsto [0:1:0]$$

The Eisenstein series $G_k(z) := G_k(\mathbb{Z}z \oplus \mathbb{Z})$ is a modular form of weight $2k$, they are "functions" on the moduli space of elliptic curves $\Gamma \setminus \mathbb{H}$. The elliptic curves and modular forms play an pivotal role as testing fields in Langlands program as we might see in the next talk.

Let's back to the discussion of integrable systems. The concepts and ideas from integrable systems not only can help us solve actual physical problems, but also bring us the the fertile land at the intersection of representation theory, geometry and analysis in the mathematical world.

Example 1.3.7 (Calogero-Moser system).

Example 1.3.8 (Garnier system).

1.4 Modern aspects of classical integrable systems

1.4.1 Lax pair and r-matrix (unfinished)

From the previous "calculation" about the Euler top, we see the emergence of a new structure called Lax pairs. A Lax pair L, M consists of two matrices, functions on the phase space of the system, such that the Hamiltonian evolution equations may be written as

$$\dot{L} = [M, L]$$

but unfortunately the conserved quantities, TrL^n, either vanish or are functions of J^2, and therefore the Hamiltonian is not included in this set of conserved quantities To cure this problem some modifications are needed. Let us introduce a diagonal matrix: $\mathrm{I} = diag(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$ with $\mathrm{I}_k = \frac{1}{2}(\mathrm{I}_i + \mathrm{I}_j - \mathrm{I}_k)$ We assume that all I_j are different and we set $L(\lambda) = \mathrm{I} + \lambda^{-1}J$, $M(\lambda) = \lambda\mathrm{I} + I$ Note that all the higher traces $Tr(L(\lambda)j)$ for $j > 3$ can be recovered from these two. Another way of saying this is that: The coefficients of η in the characteristic polynomial $det(L(\lambda) - \eta 1)$ are all conserved quantities: Elliptic curve!

1.4.2 Spectral curves

branched cover and genus of spectral curves

Example 1.4.1 (hyperelliptic curve).

digression: topological recursion

1.4.3 Riemann surfaces and integrability

Riemann-Hurwitz and Riemann-Roch
Jacobi variety and theta functions
Spectral curves, eigenbundles and integrability
cf [1]

1.5 Hitchin systems as integrable systems

1.5.1 Moduli space of G-bundles

Grothendieck's results on principal G-bundles on $\mathbb{P}^1$
$Bun_G(X)$ as double quotient and arithmetic manifolds (Shimura varieties)

1.5.2 Higgs bundles and classical Hitchin integrable systems

stable bundles and Higgs fields
The classical Hitchin system for $G = GL_n, SL_n, PGL_n$
proof: step 1: Hamiltonian reduction
step 2: Spectral curves

1.5.3 (Parabolic, twisted) Hitchin systems as universal integrable systems

Principal bundles with parabolic structures (Used in Hyperbolic band theory [7])
Classical Hitchin systems with parabolic structures

Example 1.5.1 (Garnier system)**.**

Twisted classical Hitchin systems (Used in Hyperbolic band theory [7])

Example 1.5.2 (Twisted Garnier system)**.**

Example 1.5.3 (Elliptic Calogero-Moser system)**.** cf. [4]

Almost all classical integrable system can be constructed as Hitchin integrable systems and variants.

2 Quantum mechanics and quantization

2.1 A first glance at quantization

2.1.1 Hochschild cohomology and deformation quantization

intro to quantum mechanics
deformation of associative algebras and Hochschild cohomology cf. [3]
deformation quantization of Possion manifolds and formality (homotopy transfer) cf. [8]

2.1.2 Quantum moment map and quantum Hamiltonian reduction

cf. [3]

2.2 Hyperbolic band theory and Higgs bundles

2.2.1 band theory and eigenbundles

2.2.2 Higgs bundles as crystal moduli of hyperbolic band theory

Example 2.2.1 (hyperelliptic curve and parabolic structure)**.**

2.2.3 Higgs bundles as complex momenta

2.2.4 Boundary-bulk correpsondence

Part III
Langlands program through the lens of quantization and Hitchin fibration

1 Quantization and representation theory

1.1 Geometric quantization and theta correpsondence

1.1.1 Heisenberg group and Weil representation

Harmonic oscillator and its quantization
Heisenberg group, Schrodinger representation, Stone-von Neumann theorem
Symplectic groups, Weil representation and polarization
Spin groups and metaplectic groups: spin vs oscillator
cf. [10]

1.1.2 Geometric quantization of symplectic manifolds

polarization and geometric quantization
Hamiltonian reduction commute with quantization (used in relative Langlands)

Example 1.1.1 (Quantum GIT conjecture)**.** cf. [9]

1.1.3 Howe duality and theta correspondence

representation of symmetric groups and Howe duality
Weil representation and theta correspondence

1.2 Deformation quantization and quantum integrable system

1.2.1 Quantization of classical integrable system

Quantum Hamiltonian reduction

1.2.2 Quantization of Garnier model: Gaudin model

Bethe ansatz and sugawara construction
opers and twisted differential operators

1.2.3 quantization of Hitchin integrable system

higher rank Lie algebra and Feigin–Frenkel theorem
Quantization of Hitchin system for SL_2

2 Langlands program, quantization and Higgs bundles

2.1 Automorphic forms, theta lifting and Langlands functoriality

2.1.1 Elliptic curves, modularity and global Langlands program

2.1.2 Theta lifting and functoriality

reductive dual pair and (global, local) theta correpsondence
Jacquet-Langlands correspondence
Trace formula
automorphic periods in terms of special values of L-function (used in relative Langlands)
cf. [6]

2.1.3 Interlude: theta correspondence, Siegel-Weil formula and BSD conjecture

Sum of two squares and Siegel-Weil formula
generalization of Gross-Zagier formula

2.2 Interlude: Hitchin fibration and fundamental lemma

2.2.1 stalization of trace formula

2.2.2 endoscopy

2.2.3 geometrization

2.3 Geometric Langlands program and Higgs bundles

Bun_G as double quotient and sheaf-function dictionary
Geometric Langlands duality as S-duality

2.4 Interlude: Quantum Langlands program, Quantum q-Langlands program and analytic Langlands program

2.4.1 Quantum Langlands as generic twist of N=4 SYM

quantum affine algebra or Yangian: quantum spin chain and q-opers (Quantum q-langlands)

2.5 analytic Langlands through elliptic curves

cf. [5]

3 Relative Langlands program and quantization

3.1 Hamiltonian G-varities, relative period and L-function

3.2 theta correspondence and hyperspherical variety

3.2.1 Whittaker induction commute with symplectic reduction

3.3 geometric quantization and deformation quantization of hyperspherical variety

3.4 relative Langlands duality as dual hyperspherical variety

cf. [2]

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