Hyperbolic spaces and Higgs bundles

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Abstract

This report examines Gothen's article on surfaces and Higgs bundles, with a particular focus on the application of hyperbolic structures and surface group representations. To facilitate understanding, I will first provide an overview of the necessary background in hyperbolic geometry as a preliminary. The primary objective of this report is to explore the relationship between the moduli space of surface group representations and the moduli space of Higgs bundles on PSL(2,ℝ). In this connection, the purpose of the report is to derive the corresponding properties of PSL(2,ℝ)-Higgs bundles.

1 Introduction

Focussing on the hyperbolic case we describe how to obtain all hyperbolic structures on a given topological surface, and how to parametrise them. We will first provide an overview of key concepts in hyperbolic geometry and surface group representations. Then we introduce moduli spaces of some structures about surfaces and explore some of their properties. Finally we introduce Higgs bundles and explain how they relate to hyperbolic surfaces. However, prior to this, it is important to briefly restate the objective of our study.

$$\mathcal{R}(\Gamma_g, PSL(2,\mathbb{R})) \overset{\cong}{\longleftrightarrow} \mathcal{M}(X, PSL(2,\mathbb{R}))$$

where $\mathcal{R}(\Gamma_g, PSL(2,\mathbb{R}))$ is a $PSL(2,\mathbb{R})$-*representation moduli space*, and $\mathcal{M}(X, PSL(2,\mathbb{R}))$ is a $PSL(2,\mathbb{R})$-*Higgs bundle moduli space*.

There are three complete plane geometries of constant curvature: spherical, Euclidean and hyperbolic geometry. We explain how a closed oriented surface can carry a geometry which locally looks like one of these.

We can analyze arc length and isometry groups within the Euclidean plane. The differential arc length, denoted ds, is given by

$$ds^2 = dx^2 + dy^2.$$

If A represents an element of $Isom(\mathbb{E}^2)$, $A : \mathbb{E}^2 \to \mathbb{E}^2$ can be written

$$A : x \mapsto Bx + v,\ B \in O(2),\ v \in \mathbb{R}^2.$$

Then, we can obtain a flat torus $\mathbb{E}^2/\Gamma$ in the Euclidean plane through the action of a group $\Gamma = \langle A, B\rangle \subseteq Isom(\mathbb{E}^2)$ (lattices in $\mathbb{E}^2$), where A and B are translations. We all know that there is a universal covering map from flat torus $\mathbb{E}^2/\Gamma$ to torus $\mathbb{T}^2$.

The *Killing–Hopf theorem* states that complete connected Riemannian manifolds of constant curvature are isometric to a quotient of a sphere, Euclidean space, or hyperbolic space by a group acting freely and properly discontinuously. These manifolds are called *space forms*.

2 Preliminaries

2.1 Three typical models of hyperbolic space

I. Hyperboloid model

Hyperboloid model $\mathbb{I}^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1, x_{n+1} > 0\}$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}$ is the *Lorentzian scalar product.*

We define the *Lorentz group* $O(n, 1)$ as the group of linear isomorphisms f of $\mathbb{R}^{n+1}$ that preserve the Lorantzian scalar product, i.e., $\langle v, w\rangle = \langle f(v), f(w)\rangle$ for any $v, w \in \mathbb{R}^n$. Elements in $O(n, 1)$ preserve $\mathbb{I}^n$ and form the subgroup $O^+(n, 1)$. We find $Isom(\mathbb{I}^2) = O^+(n, 1)$.

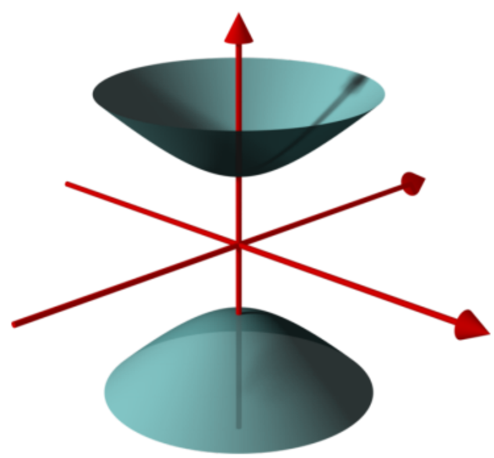

Figure 1: The hyperboloid with two sheets defined by the equation $\langle x, x\rangle = -1$. The model $\mathbb{I}^n$ is the upper sheet.

II. Poincaré disc

Poincaré disc $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

The metric tensor on $\mathbb{D}^n$ is obviously not the Euclidean one of $\mathbb{R}^n$, but instead is induced by a particular diffeomorphism between $\mathbb{I}^n$ and $\mathbb{D}^n$.

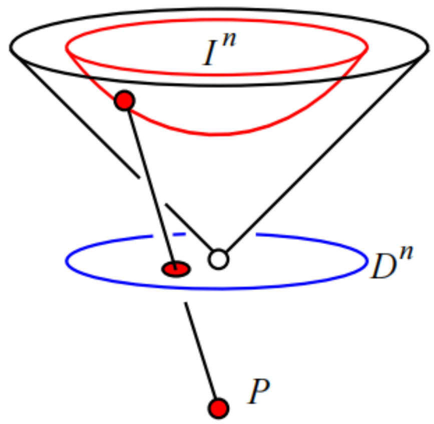

Figure 2: The projection towards $P = (0, \ldots, 0, -1)$ induces a bijection between the hyperboloid model $\mathbb{I}^n$ and the disc model $\mathbb{D}^n$.

The projection p may be written as:

$$p(x_1, \ldots, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{x_{n+1} + 1}$$

and is indeed a diffeomorphism $p : \mathbb{I}^n \to \mathbb{D}^n$ that transports the metric tensor on $\mathbb{I}^n$ to some metric tensor g on $\mathbb{D}^n$.

By computation, we can obtain that the metric tensor g at $x \in \mathbb{D}^n$ is:

$$g_x = \left(\frac{2}{1 - \|x\|^2} \right)^2 \cdot g_x^E$$

where g^E is the Euclidean metric tensor on $\mathbb{D}^n \subset \mathbb{R}^n$.

We can obtain the geodesics of the Poincaré disc from the projection, as shown in the figure.

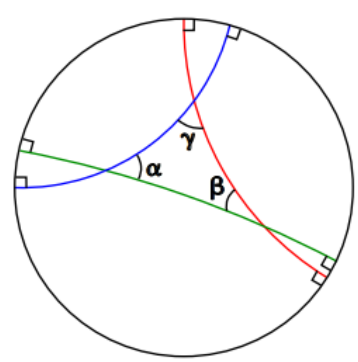

Figure 3: Three lines that determine a hyperbolic triangle in the Poincaré disc. The inner angles α, β, and γ coincide with the Euclidean ones, and we have $\alpha + \beta + \gamma < \pi$.

III. Half-space model

Half-space model $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

To generate the half-space model from the Poincaré disc, we require the following inversion transformation.

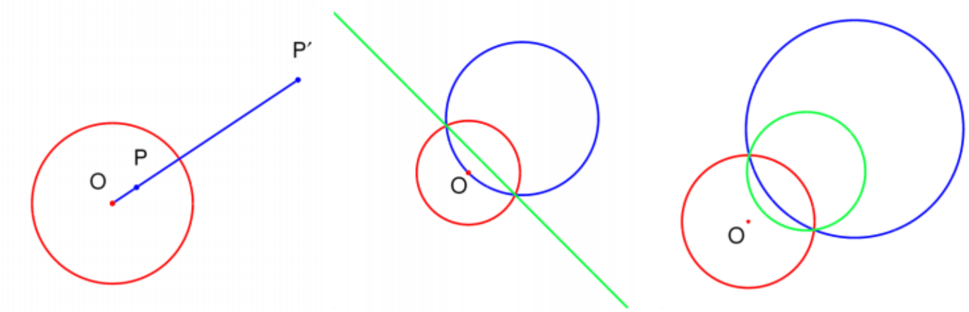

Figure 4: The inversion trough a sphere of center O and radius r moves P to P' 0 so that $|OP| \cdot |OP'| = r^2$ (left). It transforms a k-sphere S (blue) into a k-plane (green) if $O \in S$ (center) or into a k-sphere (green) if $O \notin S$ (right).

The explicit form of the inversion: $\varphi(x_1, \ldots, x_n) = \frac{(x_1, \ldots, x_n)}{\|x\|^2}$, which obtains the half-space model from the Poincaré disc as follows.

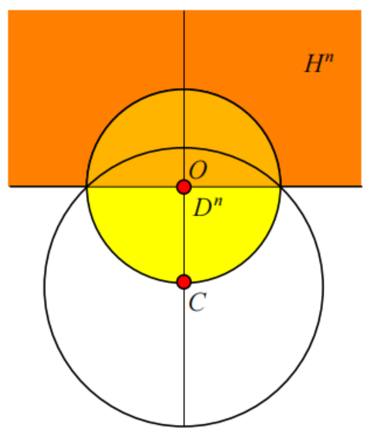

Figure 5: The inversion along the sphere with center $C = (0, \ldots, 0, -1)$ and radius $\sqrt{2}$ transforms the Poincaré disc $\mathbb{D}^n$ into the half-space model $\mathbb{H}^n$. Here $n = 2$.

By computation, we can obtain that the metric tensor g at $x \in \mathbb{H}^n$ is:

$$g_x = \frac{1}{x_n^2} \cdot g^E.$$

$Isom(\mathbb{H}^n) \cong Isom(\mathbb{D}^n) \cong Isom(\mathbb{I}^n) \cong O^+(n,1)$. Additionally, on the conformal models $\mathbb{D}^n$ and $\mathbb{H}^n$, the isometry group is generated by inversions along spheres and reflections along Euclidean planes orthogonal to the boundary.

In the model $\mathbb{H}^2$ geodesics are open arcs of semi-circles orthogonal to the real axis $\mathbb{R} = \{y = 0\} \subset \mathbb{C}$ together with open half-lines orthogonal to $\mathbb{R}$. Note that the hyperbolic plane is complete, so these curves do in fact have infinite hyperbolic length.

2.2 Hyperbolic plane

Now we can consider the case when $n = 2$, in which the hyperbolic plane is the upper half-plane $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. We can derive the arc length from the metric tensor of $\mathbb{H}^2$, i.e.

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

With the hyperboloid model the isometry group $Isom(\mathbb{H}^n)$ is the matrix group $O^+(n,1)$. We now see that in dimensions n = 2 the group $Isom^+(\mathbb{H}^n)$ is also isomorphic to some familiar groups of 2×2 matrices. Next, consider the Möbius transformation.

Orientation preserving isometries can be represented by *Möbius transformations*

$$z \mapsto A \cdot z = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

is a real 2×2-matrix of determinant one. As examples we can take $A = \left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)$ which gives a hyperbolic translation whose axis is the real axis in $\mathbb{H}^2$, and $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ which gives a hyperbolic rotation about $i \in \mathbb{H}^2$ by the angle 2θ.

Note that A and $-A$ define the same Möbius transformation, so the group of orientation preserving isometries is really the quotient group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$.

A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ also determines a *Möbius anti-transformation*

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}.$$

For instance, the (biholomorphic) inversion sending $\mathbb{H}^2$ to $\mathbb{D}^2$ is

$$z \mapsto \frac{\bar{z} + i}{i\bar{z} + 1}.$$

Möbius transformations and anti-transformations form $Conf(\mathbb{H}^2)$. Also, inversions along circles and reflections along lines orthogonal to $\mathbb{R}$ generate $Conf(\mathbb{H}^2)$. Hence, we have $Isom(\mathbb{H})^2 = Conf(\mathbb{H})^2$. In particular, $Isom^+(\mathbb{H}^n) = PSL(2, \mathbb{R})$.

2.3 Hyperbolic surface

Let us review the classification of surfaces; any closed, connected, orientable surface is diffeomorphic to S_g for some $g \geq 0$, where $S_g = \underbrace{T\#\dots\#T}_{g}$, tori $T = \mathbb{S}^1 \times \mathbb{S}^1$.

We extend our investigation to a larger interesting class of surfaces. Let $g, b, p > 0$ be three natural numbers. The *surface of finite type* $S_{g,b,p}$ is the surface obtained from S_g by removing the interior of b disjoint discs and p points. We say that $S_{g,b,p}$ has genus g, has b boundary components, and p punctures. Its Euler characteristic is

$$\chi(S_{g,b,p}) = 2 - 2g - b - p.$$

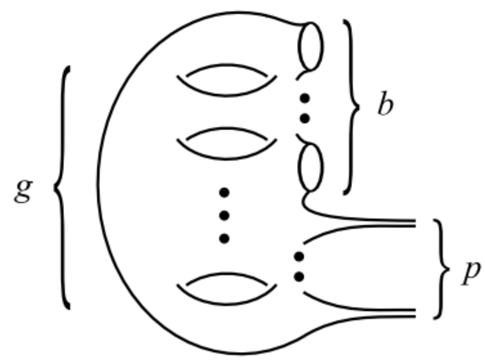

Figure 6: The finite type surface $S_{g,b,p}$.

We now wish to perform geometrization, which involves equipping $S_{g,b,p}$ with a metric, particularly a hyperbolic one for genus $g \geq 2$.

We now construct hyperbolic structures on all the surfaces S_g of genus $g \geq 2$, and more generally on all the surfaces $S_{g,b,p}$ of negative Euler characteristic. We start with a simple block, the *pair-of-pants* $S_{0,3}$, with Euler characteristic -1.

Given three real numbers $a, b, c > 0$ there is (up to isometries) a unique complete finite-volume hyperbolic pair-of-pants with geodesic boundary, with boundary curves of length a, b, and c.

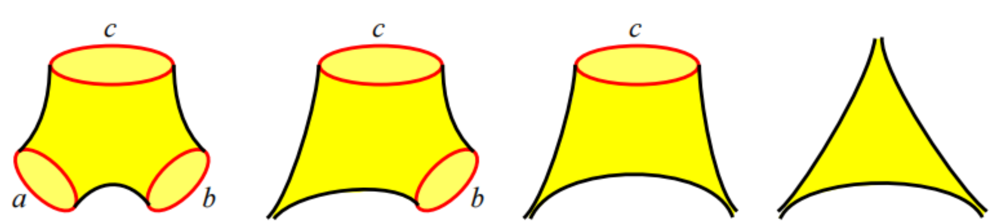

Figure 7: The last three surfaces may be considered as some degenerated hyperbolic pairs-of-pants where one or more boundary lengths a, b, or c are zero, and we get cusps instead of geodesic boundary components there.

We can construct the desired pair of pants by gluing specific hyperbolic hexagons, with the operation as follows:

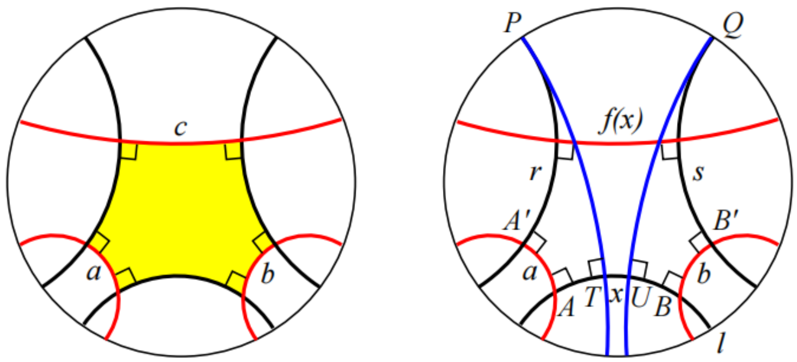

Figure 8: A right-angled hexagon with alternate sides of length a, b and c (left) and its construction (right).

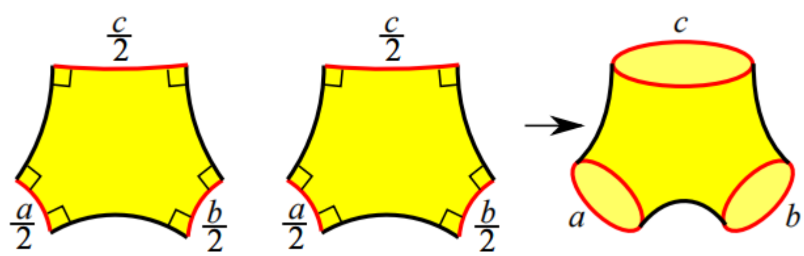

Figure 9: By gluing two identical right-angled hexagons along their black sides we get a hyperbolic pair-of-pants with geodesic boundary.

Two manifolds with hyperbolic structures can be glued along their boundaries, which serve as geodesics, to form a new manifold that naturally inherits a hyperbolic structure. This demonstrates that any pair-of-pants can be endowed with a hyperbolic structure.

The pairs-of-pants can be used as building blocks to construct topologically all finite type surfaces with $\chi < 0$. If $\chi(S_{g,b,p}) < 0$ then $S_{g,b,p}$ decomposes topologically into $-\chi(S_{g,b,p})$ (possibly degenerate) pairs-of-pants, and $S_{g,b,p}$ admits a complete hyperbolic metric with b geodesic boundary components of arbitrary length.

Additionally, we can also geometrise the few orientable surfaces with $\chi > 0$, but since there are no cusps in the elliptic and flat geometries we do not consider surfaces with punctures. The compact orientable surfaces with $\chi > 0$ are the sphere and the disc, and they all have an elliptic metric with geodesic boundary. Those with $\chi = 0$ are the torus and the annulus, and they admit flat metrics with geodesic boundary.

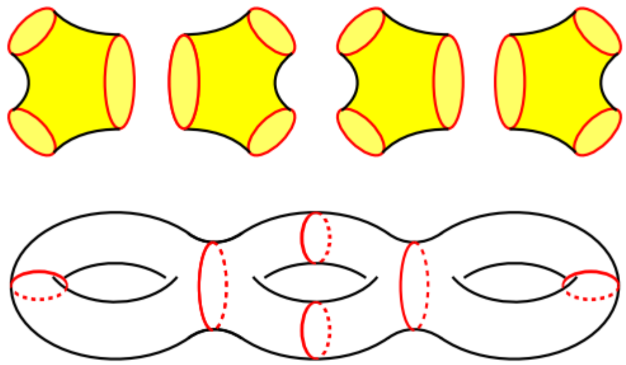

Figure 10: Every surface of finite type with $\chi < 0$ decomposes into pair-of-pants. We show here a decomposition of S_3.

2.4 Surface group representations

As we shall see, a closed orientable topological surface of genus g can be given a hyperbolic structure for any $g \geq 2$. We can emulate the construction of a flat torus to create a hyperbolic 2-torus. Geometrically, it can be generated from a hyperbolic octagon with equal side lengths that are to be glued together. The sum of the interior angles should be 2π.[1]

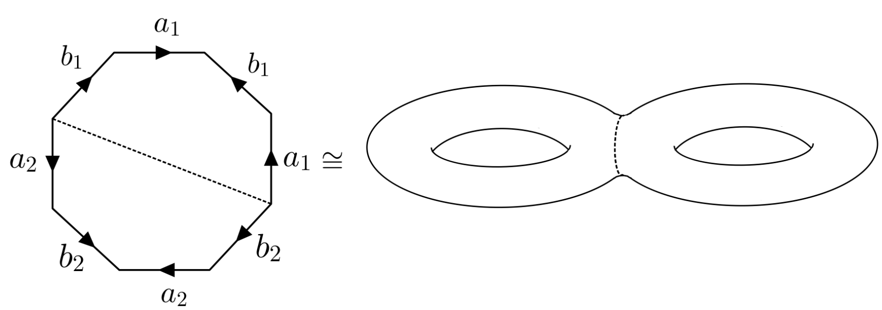

Figure 11: Genus 2 surface from an octagon.

In order to write the surface as $\mathbb{H}^2/\Gamma$ for a suitable subgroup $\Gamma \subset SL(2,\mathbb{R})$ we take hyperbolic translations A_i and B_i in $Isom(\mathbb{H}^2)$ giving the required identifications, and let Γ be the group generated by these translations. The octagon then becomes a fundamental domain for the action of Γ. The condition that the interior angles add up to 2π is equivalent to the identity

$$[A_1, B_1][A_2, B_2] = I$$

in $SL(2,\mathbb{R})$. In general, we let Γ_g be the abstract group

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

[1] In hyperbolic geometry, the sum of the interior angles equals $(n-2)\pi -$ Area.

This group is known as a *surface group*, which can be identified with the fundamental group of a topological surface of genus g.

In order to study all such subgroups we consider representations $\rho\colon \Gamma_g \to SL(2,\mathbb{R})$. We say that ρ is *Fuchsian* if it is injective and its image is discrete, i.e., consists of isolated points.[2] When ρ is Fuchsian it can be proved that the action of Γ_g on $\mathbb{H}^2$ is properly discontinuous. Hence the orbit space

$$S_\rho := \mathbb{H}^2/\rho(\Gamma_g),$$

is a nice hyperbolic surface of genus g with charts coming from $\mathbb{H}^2$. Conversely, the Killing–Hopf Theorem again tells us that any closed orientable hyperbolic surface is of this form.

However, it is certainly not true that any representation $\rho\colon \Gamma_g \to SL(2,\mathbb{R})$ defines a closed hyperbolic surface. For example, the trivial representation clearly does not. This leaves us with the following

Question: Let $\rho\colon \Gamma_g \to SL(2,\mathbb{R})$ be a representation. How can we tell if ρ defines a closed hyperbolic surface?

2.5 Topology and algebra of $SL(2,\mathbb{R})$

In order to answer the question we shall define an invariant of representations $\rho\colon \Gamma \to SL(2,\mathbb{R})$. For that we shall need to understand how the topology and algebra of $SL(2,\mathbb{R})$ interact. In fact $SL(2,\mathbb{R})$ can be identified topologically with the product of a circle and a plane, using polar decomposition of matrices.

The subgroup $SO(2) \subseteq SL(2,\mathbb{R})$ of rotation matrices $E(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ can be identified with a circle.

The map $E\colon \mathbb{R} \to SO(2)$, $\theta \mapsto E(\theta)$ wraps the real line around the circle, and it satisfies $E(0) = I$ and $E(\theta_1 + \theta_2) = E(\theta_1)E(\theta_2)$. In other words, E is a group homomorphism from the additive group $\mathbb{R} \to SO(2)$.

Now, thinking of $SO(2)$ inside $SL(2,\mathbb{R})$, we want to extend this picture and find a group $\widetilde{SL}(2,\mathbb{R})$ containing $\mathbb{R}$, with a surjective group homomorphism $p\colon \widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ which restricts to $E\colon \mathbb{R} \to SO(2)$, i.e., making the diagram

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \widetilde{SL}(2,\mathbb{R}) \\ \big\downarrow E & & \big\downarrow p \\ SO(2) & \longrightarrow & SL(2,\mathbb{R}) \end{array}$$

commutative (the horisontal maps are inclusions). In fact it follows from general theory that such a group exists and is essentially unique; it is known as the *universal covering group* of

[2] Recall that topological notions make sense viewing $SL(2,\mathbb{R}) \subseteq \mathbb{R}^4$. In fact the Implicit Function Theorem applied to this equation shows that $SL(2,\mathbb{R})$ is a 3-dimensional *Lie group*, meaning that it can be covered by local coordinate systems in 3-space and that the group operations are differentiable in these coordinates.

$SL(2,\mathbb{R})$. We shall explain how it can be constructed explicitly, using the action of $SL(2,\mathbb{R})$ on the hyperbolic plane $\mathbb{H}^2$.

So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Write

$$j(A,z) = cz + d$$

for the denominator of $A \cdot z$. Note that

$$j(E(\theta), i) = i\sin\theta + \cos\theta = e^{i\theta},$$

which indicates that this function can used to keep track of the phase θ. For each fixed A, we can consider the holomorphic function

$$\begin{aligned} \mathbb{H}^2 &\to \mathbb{C} \setminus \{0\} \\ z &\mapsto j(A,z) = cz + d. \end{aligned}$$

Observe that $cz + d \neq 0$ for $z \in \mathbb{H}$. Therefore, since $\mathbb{H}^2$ is simply connected, there is a continuous determination of the logarithm of $j(A,z) = cz + d$, i.e., a continuous map $\phi \colon \mathbb{H}^2 \to \mathbb{C}$ such that

$$e^{\phi(z)} = cz + d.$$

We want to make the point that such a ϕ can be explicitly calculated: simply choose a value θ for the argument $\arg(ci + d)$, write $ci + d = re^{i\theta}$ and let $\phi(i) = \log(r) + i\theta$. Then

$$\phi(z) - \phi(i) = \int_\gamma \frac{dz}{z} = \int_0^1 \frac{c(z-i)dt}{c(i + t(z-i)) + d}$$

(here γ parametrises the segment joining $ci + d$ to $cz + d$). Note that ϕ is not unique, but it is uniquely determined by the choice of $\phi(i)$. Thus any two determinations ϕ differ by an integer multiple of $2\pi i$.

Now define $\widetilde{SL}(2,\mathbb{R})$ as the set of pairs (A,ϕ), where $A \in SL(2,\mathbb{R})$ and $\phi \colon \mathbb{H}^2 \to \mathbb{C}$ is any continuous determination of the logarithm of $j(A,z)$. The product on $\widetilde{SL}(2,\mathbb{R})$ is defined by

$$(A_1, \phi_1) \cdot (A_2, \phi_2) = (A_1 A_2, \tilde{\phi}),$$

where

$$\tilde{\phi}(z) := \phi_1(A_2 \cdot z) + \phi_2(z).$$

It is an easy calculation to check that

$$j(A_1 A_2, z) = j(A_1, A_2 \cdot z) j(A_2, z)$$

which implies that indeed

$$e^{\tilde{\phi}(z)} = j(A_1 A_2, z)$$

as required. It is not hard to check that this defines a group structure on $\widetilde{SL}(2,\mathbb{R})$. For example, for $A = I$, the identity matrix, we can take $\phi(z) = 0$ and $(A, 0)$ is the neutral element. Moreover, $(A, \phi)^{-1} = (A^{-1}, \tilde{\phi})$, where

$$\tilde{\phi}(z) = -\phi(A^{-1} \cdot z). \tag{2.1}$$

The projection $p\colon \widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is of course just $(A,\phi) \mapsto A$. The inclusion $\mathbb{R} \hookrightarrow \widetilde{SL}(2,\mathbb{R})$ is given by $\theta \mapsto (E(\theta), \phi_\theta)$, where ϕ_θ is the determination of $\log(j(E(\theta), z))$ which satisfies $\phi_\theta(i) = i\theta$ (recall that $j(E(\theta), i) = e^{i\theta}$).

Clearly $p(A,\phi) = I$ if and only if $A = I$. Moreover, $j(I, z) = 1$, so ϕ is a determination of the logarithm of the constant function $z \mapsto 1 \in C$, i.e., it is a constant $\phi \in \pi\mathbb{Z} \subset \mathbb{R}$.

Thus, the kernel of $p\colon \widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ consists of pairs (I, ϕ), where I is the identity matrix and ϕ is a constant function taking values in $2\pi\mathbb{Z} \subset \mathbb{R}$.

2.6 The Toledo Invariant

Let $\rho\colon \Gamma \to SL(2,\mathbb{R})$ be a representation. We shall associate an integer invariant to ρ. This invariant is known as the Toledo invariant. Write

$$A_i = \rho(a_i), \quad B_i = \rho(b_i)$$

for $i = 1, \ldots, g$. Choose lifts $\tilde{A}_i$ and $\tilde{B}_i$ in $\widetilde{SL}(2,\mathbb{R})$ such that $p(\tilde{A}_i) = A_i$ and $p(\tilde{B}_i) = B_i$, and define the *Toledo invariant* of ρ to be

$$\tau(\rho) = \frac{1}{\pi} \prod_{i=1}^{g} [\tilde{A}_i, \tilde{B}_i].$$

In view of the relation defining Γ_g, the product $\prod_{i=1}^{g} [\tilde{A}_i, \tilde{B}_i]$ is in the kernel of p. Hence, the Toledo invariant is an even integer.[3]

A celebrated inequality due to Milnor states that

$$|\tau(\rho)| \leq 2g - 2$$

for every representation $\rho\colon \Gamma_g \to SL(2,\mathbb{R})$. The following beautiful result shows that representations with maximal Toledo invariant (known as *maximal representations*) have a special geometric significance.

$$\rho\colon \Gamma_g \to SL(2,\mathbb{R}) \textit{ is Fuchsian} \Leftrightarrow |\tau(\rho)| = 2g - 2,$$

which is a consequence of the *Goldman's theorem*.

One might wonder about the significance of the sign of the Toledo invariant. If we conjugate a representation ρ by the outer automorphism of $SL(2,\mathbb{R})$ given by conjugation by a reflection we obtain a representation $\bar{\rho}$ with $\tau(\bar{\rho}) = -\tau(\rho)$. In fact, the hyperbolic surface $S_{\bar{\rho}}$ is obtained from S_ρ by a change of orientation, i.e., by composing all charts with a reflection in $\mathbb{H}^2$.

[3] Odd Toledo invariants arise from representations $\rho\colon \Gamma_g \to PSL(2,\mathbb{R})$ which do not lift to $SL(2,\mathbb{R})$.

3 Main result

3.1 The moduli space of representations

Let us now take a global view and consider all representations of Γ_g in $SL(2,\mathbb{R})$ simultaneously. It is natural to consider $\rho_1 \sim \rho_2$ if they differ by overall conjugation by an element of $SL(2,\mathbb{R})$, corresponding to a change of basis in $\mathbb{R}^2$. It also turns out that two hyperbolic structures on the same topological surface are isometric by an isometry which can be continuously deformed to the identity if and only if the corresponding Fuchsian representations are equivalent in this sense.

Thus the *moduli space of representations* is defined to be the orbit space

$$\mathcal{R}(\Gamma_g, SL(2,\mathbb{R})) = Hom(\Gamma_g, SL(2,\mathbb{R}))/SL(2,\mathbb{R})$$

under the conjugation action.

A homomorphism $\rho\colon \Gamma_g \to SL(2,\mathbb{R})$ is determined by $2g$ matrices

$$A_i = \rho(a_i),\ B_i = \rho(b_i), \quad i = 1,\dots,g$$

satisfying the single relation $\prod[A_i, B_i] = I$. Hence $Hom(\Gamma_g, SL(2,\mathbb{R}))$ can be identified with the subspace of $\mathbb{R}^{6g}$ cut out by the 3 scalar equations given by $\prod[A_i, B_i] = I$. It follows that it is a variety of dimension $6g-3$, which is smooth at the locus of irreducible representations. The conjugation action by $SL(2,\mathbb{R})$ reduces the dimension by 3, and so the moduli space has dimension

$$\dim \mathcal{R}(\Gamma_g, SL(2,\mathbb{R})) = 6g-6.$$

The Toledo invariant separates the moduli space into subspaces

$$\mathcal{R}_d \subseteq \mathcal{R}(\Gamma_g, SL(2,\mathbb{R})) = \bigsqcup_{d=\tau(\rho)} \mathcal{R}_d$$

corresponding to representations with invariant d. Goldman showed that the $\mathcal{R}_d$ are in fact connected components of the moduli space, except in the maximal case $|d| = 2g-2$. It turns out that $\mathcal{R}_{2g-2}$ has 2^{2g} connected components. However, these components get identified after projecting onto

$$\mathcal{R}(\Gamma_g, PSL(2,\mathbb{R})) = \bigsqcup_{d=\tau(\rho)} \mathcal{R}_d^{\mathcal{P}},$$

which thus has just one connected component with Toledo invariant $2g-2$. This is not surprising because, by Goldman's Theorem, the subspace $\mathcal{R}_{2g-2}$ is exactly the locus of Fuchsian representations and, moreover, any two Fuchsian representations into $SL(2,\mathbb{R})$ define the same hyperbolic surface if and only if they coincide after projecting to $PSL(2,\mathbb{R})$. Accordingly, the corresponding connected component $\mathcal{T} = \mathcal{R}_{2g-2}^{\mathcal{P}} \subseteq \mathcal{R}(\Gamma_g, PSL(2,\mathbb{R}))$ is known as the *Fuchsian locus*.

As we have seen, Fuchsian locus parametrises all hyperbolic structures on the topological surface S_g up to a natural equivalence. It is a classical result that the space of such hyperbolic structures can be identified with $\mathbb{R}^{6g-6}$.

3.2 Riemann surfaces and Teichmüller space

If we have a hyperbolic surface $S_g \cong \mathbb{H}^2/\Gamma$ for a Fuchsian representation of Γ, then the local coordinates in $\mathbb{H}^2$ give S_g the structure of a Riemann surface.[4] Generally, hyperbolic surfaces possess a natural Riemann surface structure. We write X_ρ for the Riemann surface constructed from a Fuchsian representation ρ in this way.

From another perspective, the *Uniformisation Theorem* asserts that any Riemann surface can be represented as a hyperbolic surface. This means the space of all Riemann surfaces with the same underlying topological surface of genus g can be identified with the Fuchsian locus $\mathcal{T}$, which is known as *Teichmüller space*

$$\begin{aligned} Teich(S_g) &= \{\text{deformation of complex structure on } X_\rho\}/\sim_{\text{biholomorphic}} \\ &= \{\text{hyperbolic structure on } S_g\}/\sim_{\text{isotopic}} . \end{aligned}$$

We have seen that Teichmüller space is a $(6g-6)$-dimensional connected space. It is a classical result that Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$.[5]

3.3 Higgs bundles

A $PSL(2,\mathbb{R})$*-Higgs bundle* on X consists of three pieces of data

$$(L, \beta, \gamma)$$

where, $L \to X$ is a holomorphic line bundle, and $\beta \in H^0(X; K \otimes L)$ and $\gamma \in H^0(X; K \otimes L^*)$ can be seen as holomorphic differentials which take values in the line bundles L and L^*, respectively.

In a manner analogous to the conjugation action on representations, there is a natural notion of isomorphism of Higgs bundles, and the set of isomorphism classes of $PSL(2,\mathbb{R})$-Higgs bundles forms the *moduli space* $\mathcal{M}(X, PSL(2,\mathbb{R}))$. It is a complex algebraic variety of complex dimension $3g-3$. We note that in order to get a reasonable moduli space it is necessary to restrict to so-called *semistable Higgs bundles*. This is analogous to the way in which one restricts to semisimple representations in the moduli space of representations.[6]

There is a natural field arising from *non-Abelian Hodge theory*, the study of the corresponding moduli spaces of these objects. From Simpson's work on the construction of the moduli spaces, we have the following four moduli spaces:

- **Betti moduli space** $M_B(X,r)$: the moduli space of rank r representations $\pi_1(X) \to GL(r,\mathbb{C})$;
- **de Rham moduli space** $M_{dR}(X,r)$: the moduli space of rank r flat bundles over X;

[4] Indeed the changes of coordinates are Möbius transformations of $\mathbb{H}^2$, which are certainly holomorphic.
[5] Fenchel-Nielsen coordinates identify Teich (S_g) with $\mathbb{R}^{6g-6}$, more precisely with $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$.
[6] Similarly, there is a connection between X_ρ and S_g.

- **Dolbeault moduli space** $M_{\mathrm{Dol}}(X,r)$: the moduli space of semistable rank r Higgs bundles over X with vanishing Chern classes;

- **Hodge moduli space** $M_{\mathrm{Hod}}(X,r)$: the moduli space of semistable rank r λ-flat bundles over X with vanishing Chern classes.

The study of these moduli spaces arising from non-Abelian Hodge theory shows that they are also related. More precisely, the Riemann-Hilbert correspondence implies that $M_B(X,r)$ and $M_{dR}(X,r)$ are analytic isomorphic. Moreover, the Hodge moduli space $M_{Hod}(X,r)$ has a fibration over $\mathbb{C}$ such that the fibers over 0 and 1 are exactly $M_{\mathrm{Dol}}(X,r)$ and $M_{dR}(X,r)$, respectively. The underlying topological spaces of $M_{\mathrm{Dol}}(X,r)$ and $M_{dR}(X,r)$ are homeomorphic, and are C^∞ isomorphic over the stable points.

The *Non-abelian Hodge Theorem* for this situation states that there is a real analytic isomorphism

$$\mathcal{R}(\Gamma_g, PSL(2,\mathbb{R})) \cong \mathcal{M}(X, PSL(2,\mathbb{R}))$$

For fixed d we denote by $\mathcal{M}_d$ the subspace of $PSL(2,\mathbb{R})$-Higgs bundles (L,β,γ) with $\deg(L) = d$. Then we have $\mathcal{R}_d^{\mathcal{P}} \cong \mathcal{M}_d$ under the non-abelian Hodge Theorem. In particular, the Fuchsian locus $\mathcal{T}$ corresponds to $\mathcal{M}_{2g-2}$. Thus,

$$\mathcal{M}(X, PSL(2,\mathbb{R})) = \bigsqcup_{d=deg(L)} \mathcal{M}_d.$$

3.4 Hitchin's parametrisation of Fuchsian locus

Under the non-abelian Hodge isomorphism the image $\mathcal{M}_{2g-2}(X, PSL(2,\mathbb{R}))$ of Ψ corresponds to the Fuchsian locus $\mathcal{R}_{2g-2}^{\mathcal{P}}$ of $\mathcal{R}(\Gamma_g, PSL(2,\mathbb{R}))$. Thus, in particular, Hitchin obtains a parametrisation of Teichmüller space by quadratic differentials.

A particular class of $PSL(2,\mathbb{R})$-Higgs bundles can be obtained by taking $L = K$. Then γ is a section of the line bundle $K \otimes K^*$ which is naturally isomorphic to the trivial line bundle on X. In other words, γ is simply a holomorphic function on X, so we can set $\gamma = 1$ (the constant function). Moreover, β is a section of $K^2 = K \otimes K$. In other words it is a *quadratic differential*, so it can locally be written as $\beta(z) = b(z)(dz)^2$, where $b(z)$ satisfies an appropriate transformation rule under changes of coordinates. The vector space $H^0(X, K^2)$ of quadratic differentials on X has complex dimension $3g-3$ which equals the dimension of the moduli space $\mathcal{M}(X, PSL(2,\mathbb{R}))$. This construction defines a map

$$\begin{aligned} \Psi\colon H^0(X,K^2) &\to \mathcal{M}(X, PSL(2,\mathbb{R})), \\ \beta &\mapsto (K,\beta,1). \end{aligned}$$

The semistability condition alluded to earlier implies that all Higgs bundles in $\mathcal{M}_{2g-2}$ arise in this way. Hence Ψ is an isomorphism onto its image $\mathcal{M}_{2g-2}$.

From the non-abelian Hodge Theorem we already knew that $\mathcal{M}_{2g-2} \cong \mathcal{T}$ is a connected component. But the Higgs bundle construction gives an alternative proof. Using gauge theoretic methods Hitchin also shows that $\mathcal{M}_{2g-2}$ parametrises all hyperbolic metrics on the topological surface underlying X.

Hitchin generalised the construction of the map Ψ to a map

$$\Psi\colon \bigoplus_i H^0(X, K^{d_i}) \to \mathcal{M}(X, G)$$

whose image is again a connected component of the moduli space $\mathcal{M}(X, G)$ of G-Higgs bundles for any simple split real Lie group G, known as *Hitchin component*. The domain of Ψ is a direct sum of spaces of higher holomorphic differentials on X; the integers d_i are determined by the Lie group G.

Similar constructions of special connected components have later been given for Hermitian groups G of non-compact tube type, such as $SU(p, p)$. In this case the domain of the map Ψ turns out to be a moduli space $\mathcal{M}_{K^2}(X, G')$ of so-called K^2-twisted G'-Higgs bundles, for a certain real Lie group G' associated to G, known as *Cayley components*.

4 Conclusion

Both Hitchin components and Cayley components are special because they are not (as all other known components of the moduli space) detected by standard topological invariants of the underlying bundles and the Higgs fields satisfy a certain non-degeneracy condition.

Recently both of these constructions have been unified and generalised. Conjecturally the generalised Cayley components obtained by this construction account for all special connected components of the moduli space and thus opens the door to a complete determination of this important topological invariant.

One important piece of supporting evidence for this conjecture comes from the area of *Higher Teichmüller Theory*. Higher Teichmüller theory has developed in parallel with the Higgs bundle story just described, and there has been a rich cross-fertilisation of ideas between the two areas. Briefly, a higher Teichmüller space is a connected component of the moduli space of representations, which consists exclusively of discrete and injective representations, like the Fuchsian locus in the $PSL(2, \mathbb{R})$-case. It turns out that the generalised Cayley components are indeed higher Teichmüller spaces, and it is expected that all higher Teichmüller spaces are thus obtained.

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