Notes for 2024 autumn seminar in geometry and topology

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1 Preliminaries

This part mainly follows Daniel Huybrechts's Complex geometry: an introduction and Claire Voision's Hodge theory and complex algebraic geometry I.

1.1 Complex analysis and complex geometry

1.2 Sheaf and cohomology

2 Definition of Higgs bundles

2.1 Definition

Let C be a compact Riemann surface, a Higgs bundles on C is a pair (E , Φ),where E is a holomorphic vector bundle on C and Φ is a section of $End(E)\otimes\Omega^1(C)$,that is, an element in H^0(C, $End(E)\otimes\Omega^1(C)$).

3 More discussions about vector bundles

This part mainly refer to Moscow lecture 2: Algebraic curve: Towards moduli space.

3.1 Degree of a vector bundle

3.1.1 Degree of holomorphic vector bundles

We had learned from algebraic curve course that for any meromorphic function on a compact Riemann surface which is not constantly zero, the number of its zeros counted with multiplicity minus the number of its poles counted with multiplicity is always zero, independent of the choice of meromorphic function.

Using the language of vector bundle, we can describe the fact in this way: any meromorphic section of the trivial (holomorphic) line bundle on a compact Riemann surface has its zeros (counted with multiplicity) minus poles (counted with multiplicity) equal zero. Here a meromorphic section means a holomorphic section of the holomorphic bundle restricted on the Whole Riemann surface with finite points cut. Locally it is just a meromorphic function on an open set of the Riemann surface, thus it has well defined zeros and poles.

Is there similar property for general holomorphic line bundles on a Riemann surface? That is, for a given holomorphic line bundle and take any of its meromorphic section, whether it is true that the number of its zeros (counted with multiplicity) and its poles (counted with multiplicity) is a constant independent of the choice of meromorphic sections? This is true, since for any two meromorphic sections on one holomorphic line bundle, we can define their ratio which is a meromorphic function f and then the difference of the constant counted respectively by their own zeros and poles is the number of zeros of f (counted with multiplicity) minus the number of poles of f (counted with multiplicity), which is just zero.

This constant relies only on the Riemann surface and the holomorphic line bundle. We call it the "degree" of the holomorphic line bundle on the Riemann surface. For holomorphic vector bundles of rank n ($n \geq 1$), we also define its degree to be the corresponding holomorphic line bundle of the skew symmetry n forms on it.

3.1.2 Example: tautological bundle on $\mathbf{CP}^1$

CP^1 is the one dimension complex projective space and O(-1) is:

{ (l, x) $\in$CP^1 $\times$C^2 :x is a point on line l}

here we take the view that CP^1 is all lines in C^2through the origin.Then we define a meromorphic section onO(-1) :

s: CP^1 $\rightarrow$O(-1)

$$[z_1,\ z_2] \rightarrow (1, \mathrm{z}_2/z_1)$$

It has no zeros but a simple pole at [0,1] , so it has degree -1.

3.1.3 Degree of smooth or topological vector bundle

Consider a vector field V(x) in an open set U of R^n which has discrete zeros. Fix a point x in U, one can take a closed ball B in U centered at x such that vector field does not vanish anywhere in B-x. Then the restriction of $V(x)/\,|V(x)|$ on ∂B defines a continuous map from ∂B to a unit sphere in R^n. The mapping degree of this map is stable when we choose smaller and smaller B. Therefore we can define this "eventual" degree to be the "index" of the vector field V at x.

Since this index is defined by local information of the vector field, and a section of a rank n vector bundle over a n-dimensional manifold is locally in the same situation as above, we can define the index of a section of a rank n vector bundle over a n-dimensional manifold M at a point x in M.

Hopf-Poincaré theorem states that, for a orientable closed manifold M of dimension n and a rank n vector bundle on it, any section with discrete zeros has the same sum of index on M. We define the sum as the degree of this vector bundle.

Now consider a Riemann surface C, which is a one-dimensional complex manifold, First we consider a smooth/topological rank 1 complex vector bundle E on it. Let s be a smooth/continuous section of this bundle. Since C is a two-dimensional manifold and E is a rank 2 vector bundle on it (over R), there is a well defined degree of E in the above sense. For a rank n complex vector bundle E' on C, we define the degree of E' to be the degree of the one-dimensional complex vector bundle $\wedge^n$E'.

Note that the definition of degree of vector bundle over a compact Riemann surface C is compatible since by local normal form a meromorphic section s of a holomorphic vector bundle on C is locally a map z $\to$z^n, where n is the order of s as well as the mapping degree between unit circles in C .Here we choose the orientation compatible with the orientation of the Riemann surface and complex plane.

3.2 Picard group and classification of line bundles

3.2.1 Divisors and Picard group

A divisor on a Compact Riemann surface is an integral valued function supported on a finite set. For a meromorphic function f on the Riemann surface, we can define a divisor Div(f), which takes the value of the order of f at each point. Divisors of this kind are called principal divisors. Two divisors are called linear equivalent if they differ by a principal divisor.

It's easily checked that the set of all divisors on a Riemann surface C forms an abelian group under addition and the set of principal divisors forms a subgroup. The quotient group is called the Picard group, denoted as Pic(C), of the Riemann surface, whose elements are the linear equivalence class of divisors on it.

For a divisor on a Riemann surface C, we define its degree to be the sum of all nonzero values it takes. A principal divisor must have degree zero and hence there is also a well defined concept of degree on any linear equivalence class, that is, on any element in Picard group of the Riemann surface. This gives a set-theoretic decomposition of Pic(C):

Pic(C)=...$Pic_{-1}(C)$⊔Pic$_0$(C)⊔Pic$_1$(C)⊔...

where Pic_i(C) means the linear equivalence classes with degree i, and notice that Pic_0(C) is indeed a group. And for any d $\in$Z,Pic$_d$(C) can be represented as D+Pic_0(C), where D is an arbitrary element in Pic_d(C).

3.2.2 Divisors of line bundles

For a holomorphic line bundle on a compact Riemann surface C, any of its meromorphic sections gives a divisor on C, as in the case of a meromorphic function. Since any two meromorphic sections of one holomorphic line bundle differ by a factor of meromorphic, their corresponding divisors are linearly equivalent. So we can associate to any holomorphic line bundle on C an element in Pic(C).

We can define a group structure on the set of all isomorphism classes of holomorphic line bundles on C to make the map to Pic(C) a group homomorphism. Note these facts:
(i)tensor product of a pair of holomorphic line bundles gives a holomorphic line bundle
(ii)this product is associative and commutative
(iii)the tensor product of trivial line bundle and any line bundle E is isomorphic to E
(iv)the tensor product of any line bundle E and its dual bundle E* is isomorphic to trivial line bundle
Moreover, (v)suppose s_i is a meromorphic section of line bundle E_i, (i=1, 2), there is a meromorphic section $s_1 \otimes s_2$ of $E_1 \otimes E_2$ and Div($s_1 \otimes s_2$) = Div(s_1) + Div(s_2).
These facts imply a group structure on the set of isomorphism class of holomorphic line bundles on a Riemann C and a group homomorphism to Pic(C).

This homomorphism is injective. Suppose E is a line bundle on a compact Riemann surface C mapping to identity in Pic(C). Take a meromorphic section s on E and then we have a meromorphic function f on C such that Div(s)=Div(f). Thus s/f is also a meromorphic section on E and Div(s/f)=0. This means that s/f is a holomorphic and nowhere vanishing section of E, which provides E with a global frame. Therefore E must be a trivial line bundle on C.

In fact, this homomorphism is also surjective. That is, for any linear equivalence class of divisors on a compact Riemann surface C, we can construct a holomorphic line bundle on C such that the meromorphic section divisors of it coincides with the given linear equivalence class of divisors. (cf. theorem 8.1 of Moscow lecture 2: Algebraic curves: Towards Moduli space)

In summary, there is a group isomorphism between the group of isomorphism classes of line bundles on a compact Riemann surface C with tensor product and its Picard group Pic(C). The group structure of Pic(C) and help us classify and calculate the isomorphism classes of line bundles on C. (for example, we can generate them if we know the line bundles corresponding to the generators of Pic(C))

For example, Σ_0=CP^1 is the Riemann sphere and any divisors on it with degree zero is a principal divisor. So Pic(Σ_0) is isomorphic with addition group (Z,+). We have construct a line bundle on Σ_0 in Example 1.1.1, whose degree is -1. Thus it generate all isomorphism class of line bundles on Σ_0 by tensoring and taking dual. But for general Riemann surface of genus g ¿ 1, Pic_0 is not trivial.

For general compact Riemann surface C, we have a description of its Picard group Pic(C) using the Jacobian of C. We have discussed above that Pic(C) is essentially $Z\times$Pic$_0$(C). And for Riemann surface of genus g, denoted as Σ_g, $Pic_0(\Sigma_g)$ is isomorphic with $Jac_0(\Sigma_g)$ which is isomorphic with $C^g/\mathrm{Z}^{2g}. In particular, we see that \Sigma_0 = CP^1$ is of genus 0 and its Picard group is isomorphic with addition group $(Z, +)$.

4 Moduli space of holomorphic vector bundles and Higgs bundles on a Riemann surface

Now we are going to consider the classification problem of general holomorphic vector bundles on a compact Riemann surface up to isomorphism. We will construct a space that parametrizes the isomorphism classes, that is, their moduli space. Then we will consider analogous problem of Higgs bundles on a Riemann surface, which are holomorphic vector bundles with another more structure (a field called Higgs field) attached.

This part mainly refer to the notes of Dr. John Benjamin McCarthy, Introduction to Higgs Bundles.

4.1 Classification of smooth complex vector bundles

Consider a Riemann surface of genus g, denoted as Σ_g. Then up to smooth/topological isomorphism, complex vector bundles on Σ_g is classified by its rank and degree.

This results follows from this theorem:

4.1.1 Theorem

Let $E \to M$ be a real vector bundle over a manifold. If rank E >dim M, there is a real vector bundle E' such that rank E'= dim M and E $\cong E' \oplus \mathbf{1}_M^{rankE-dimM}$, where **1** means trivial rank one real vector bundle on M.

Therefore, smoothly/topologically, a complex vector bundle E on a Riemann surface C can be decomposed into the direct sum of a complex line bundle and a set of trivial complex line bundles. The second component is determined by rank E and the first component is determined by deg E.

4.2 Classification of holomorphic vector bundles

Since isomorphism in holomorphic sense implies isomorphism in smooth sense, and the isomorphism in smooth sense is characterized by the rank and degree of vector bundles, we focus on the following problem: how to classify holomorphic vector bundles of given rank and degree over a fixed Riemann surface up to isomorphism in holomorphic sense?

The answer to this question is no longer discrete as it was in the smooth classification case. Instead, it is a huge moduli space, which we denote by $\mathcal{N}_{n,d}^g$. Here n and d stands for the rank and degree of vector bundle and g is the genus of the base Riemann surface.

Although there are different complex structure of Riemann surfaces of genus g, it turns out magically that $\mathcal{N}^g_{n,d}$ only depends on g.

4.2.1 Exhausting all possible isomorphism class of holomorphic vector bundles of rank n and degree d over a Riemann surface of genus g; Dolbeault operator

Now we are going to figure out, for a (in fact the only one up to isomorphism) complex smooth vector bundle of rank n and degree d over a Riemann surface of genus g, what are all possible structure of holomorphic vector bundles we can equip to it.

Given any holomorphic vector bundle E over complex manifold X, there is a differential operator on X that picks out holomorphic sections of E from the set of smooth sections of E. Suppose in local chart, a smooth section can be written as s_U=Σs^i $\otimes$e$_i$, then Dolbeault operator can be defined locally as:

$$\bar{\partial}_\epsilon s_U = \Sigma\bar{\partial}(\mathrm{s}^i) \otimes \mathrm{e}_i$$

Notes that this is well defined globally since E is a holomorphic vector bundle and the result is a section in $\Omega^{0,1}$(X) $\otimes\Gamma(X,E)$,or denoted as $\Omega^{0,1}$(E).A smooth is in the kernel of this operator if and only if it is a holomorphic section of E. In addition, for any smooth function f on X,$\bar{\partial}_\epsilon(fs) = \bar{\partial}(f)s + f\bar{\partial}_\epsilon(s)$ and $\bar{\partial}_\epsilon{}^2 = 0$.

These properties of a Dolbeault operator on a holomorphic vector bundle motivates the definition of a general Dolbeault operator on a complex smooth vector bundle on X:

Definition: a Dolbeault operator on a smooth complex vector bundle E over a complex manifold X is a C-linear operator

$$\bar{\partial}_\epsilon : \Gamma(X,E) \rightarrow \Omega(X)^{0,1} \otimes \Gamma(X,E)$$

which satisfies following properties: for any smooth function f on X and smooth section s of E, $\bar{\partial}_\epsilon(fs) = \bar{\partial}(f)s + f\bar{\partial}_\epsilon(s)$ and $\bar{\partial}_\epsilon{}^2 = 0$.

By an application of **Newlander-Nirenberg theorem**, for a Dolbeault operator $\bar{\partial}_{\epsilon]}$ on the smooth complex vector bundle E on complex manifold X, there is a unique holomorphic structure on E such that $\bar{\partial}_\epsilon$ is the natural Dolbeault operator associated to this holomorphic vector bundle, and the set of holomorphic sections are the kernel of this operator.

Let Dol(E) denote the set of all Dolbeault operators defined on the fixed smooth complex vector bundle on Riemann surface Σ_g. Then there is a bijection between Dol(E) and the set of holomorphic structures on E. But we still need to take a quotient to reduce to isomorphism class.

4.2.2 Quotient, classification up to isomorphism

For a complex smooth vector bundle E on Riemann surface Σ_g, we denote $\mathcal{G}$C its automorphism group, called the **Gauge group** of E. Gauge of E naturally acts on Dol(E) by conjugation action. Two Dolbeault are called equivalent if they in the same orbit under this action.

Dolbeault operators in Dol(E) corresponding to isomorphic holomorphic structures on E if and only if they are equivalent.

Dolbeault operators are like connections with only (0,1)-part. In particular, Dol(E) is an affine space modeled on the infinite dimensional vector space $\Omega^{0,1}(\text{End(E)}) = \Omega^{0,1}(\Sigma_g) \otimes \Gamma(\Sigma_g, E)$.

We can put a reasonable topology on this affine space Dol(E), then Dol(E)/$\mathcal{G}_C$ is a space which is in bijection with holomorphic structures on E up to isomorphism. But this space is bad-behaved. For example, it's not Hausdorff.

4.2.3 Geometric Invariant Theory, GIT

4.2.4 Moduli space of holomorphic vector bundles

To get a well-behaved moduli space, we need to contract the range of holomorphic vector bundles we consider on E.

Definition: A holomorphic vector bundle E on Σ_g is called (semi-)stable, if for any non-zero holomorphic proper sub-bundle F $\subset E$, we have deg F / rank F $<(\leq)$ deg E / rank E.

We call deg F / rank F the slope of a vector bundle F, denoted as $\mu(F)$.

We denote the set of all stable and semi-stable holomorphic vector bundles in Dol(E) as $Dol(E)^s$ and $Dol(E)^{ss}$.

Then geometric invariant theory tells us $Dol(E)^s / \mathcal{G}_C$ is Hausdorff and modified quotient $Dol(E)^{ss} // \mathcal{G}_C$ is also Hausdorff.

Notes that when (n,d)=1, i.e. rank and degree are relatively prime, $Dol(E)^s$ and $Dol(E)^{ss}$ coincide.

Definition: The moduli space of stable holomorphic vector bundles of rank n and degree d over a Riemann surface Σ_g is $Dol(E)^s / \mathcal{G}_C$, denoted as $\mathcal{N}^g_{n,d}$.

About the structure of $\mathcal{N}^g_{n,d}$, we have the following theorems:

Theorem (Mumford, Narasimhan-Seshadri, Atiyah-Bott, Ramanan, others..). When (n, d) = 1, $\mathcal{N}^g_{n,d}$ is a non-singular, projective complex algebraic variety, and a fine moduli space for the classification problem we are considering (i.e. there is a universal bundle over $\mathcal{N}^g_{n,d} \times \Sigma_g$ which restricts to each given holomorphic vector bundle E on each slice $\{[\text{E}]\} \times \Sigma_g$).

Theorem (Narasimhan-Seshadri '65, Donaldson '82). The following three spaces are isomorphic:

1. $\mathcal{N}^g_{n,d}$

2. Moduli space of projectively flat irreducible connections on the underlying smooth bundle E $\to \Sigma_g$

3. The character variety $Hom^{irr}_d(\hat{\pi}(\Sigma_g), U(n)))/U(n)$ classifying irreducible projective unitary representations of the fundamental group of Σ_g (of a certain type).

The equivalence (1) $\iff$ (3) was the original theorem of Narasimhan-Seshadri, and uses algebraic geometry and representation theory. The equivalence (2) $\iff$ (3) is given by taking the holonomy of the connection, and in the other direction by constructing the associated bundle to the universal $\pi_1(\Sigma_g)-$bundle over Σ_g given by its universal cover . The equivalence(1)

$\Longleftrightarrow$ (2) was proven by Donaldson, who used gauge theory techniques.

It was proven by Goldman that the representation variety is symplectic, and by Atiyah-Bott that the moduli space of flat connections is symplectic (this structure is called the Atiyah-Bott symplectic form). These two symplectic structures agree. Since $\mathcal{N}^g_{n,d}$ is naturally a complex manifold, it turns out that through the Narasimhan-Seshadri theorem it is a compact *Kähler* manifold (at least when (n, d) = 1).

The dimension of moduli space is given by

$$dim_R \mathcal{N}^g_{n,d}\text{=2+2}n^2\text{(g-1)}$$

4.3 Higgs bundles

For the purpose of classifying Higgs bundles on a Riemann surface of genus g and whose holomorphic vector bundles are of degree d and rank n using Dolbeault, we need to rephrase the definition:

Definition: Σ_g is a Riemann surface of genus g, E is a smooth complex vector bundle over Σ_g of degree d and rank n. Then a Higgs bundle over Σ_g of degree d and rank n is a pair $(\bar{\partial}_\epsilon, \Phi), where\ \bar{\partial}_\epsilon$ is a Dolbeault operator on E and Φ is a holomorphic (with respect to $\bar{\partial}_\epsilon$) section of $\Omega(\Sigma_g)^1$⊗End(E), that is $\bar{\partial}_\epsilon(\Phi) = 0$.

We denote by $\mathcal{B}$ the set of all Higgs bundles on (Σ_g, E).

$$\mathcal{B}=\{(\bar{\partial}_\epsilon, \Phi) \in Dol(E) \times \Omega^{1,0}(\text{End(E)}) \text{: } \bar{\partial}_\epsilon(\Phi) = 0\}$$

If everything is set up right, $\mathcal{B}$ is a infinite-dimensional orbifold and there is an action of $\mathcal{G}_C$ on $\mathcal{B}$ by conjugation on both Dolbeault operator and Higgs field:

$$\text{g}(\bar{\partial}_\epsilon, \Phi) = (g\bar{\partial}_\epsilon g^{-1}, \text{ g}\Phi\text{g}^{-1})$$

Similar to the case of holomorphic vector bundle, if we want to get well-behaved moduli space, we need to consider stable and semi-stable Higgs bundle:

Definition: We call a Higgs bundle (E, Φ)$(semi-)stable$, if for any proper, non-zero, $\Phi - invariant$ sub-bundle F of E, we have $\mu(F) < (\leq) \mu(E)$.

Definition: The moduli space of stable Higgs bundles of rank n and degree d over a Riemann surface Σ_g is $\mathcal{B}^s/\mathcal{G}_C$, which we denote by $\mathcal{M}^g_{n,d}$. Where $\mathcal{B}^s$ is the set of stable Higgs bundles in $\mathcal{B}$.

For the structure of $\mathcal{M}^g_{n,d}$, we have the theorem by Hitchin, Simpson, Donaldson-Corlette:

Theorem: The following are isomorphic:

(1) $\mathcal{M}^g_{n,d}$

(2) The moduli space of irreducible projectively flat connections on smooth complex vector bundle E over Σ_g

(3) The character variety $Hom^{irr}_d(\hat{\pi}(\Sigma_g), GL(n,\text{C})))/GL(n,\text{C})$ classifying complex representations of the fundamental group (of a certain type).

Hitchin proved (1) $\iff$ (2), in the case of n = 2, d = 1 (and fixed determinant bundle). Work of Donaldson-Corlette on harmonic representations gives (2) $\iff$ (3), and Carlos Simpson proved the rest of the theorem for any (n, d).

The dimension of $\mathcal{M}^g_{n,d}$ is dim_R=4+4n^2(g-1), which is twice the dimension of $\mathcal{N}^g_{n,d}$. This can be explained in the following way: given a stable vector bundle E, standard moduli space voodoo tells you that the tangent space to $\mathcal{N}^g_{n,d}$ at [E] is given by cohomology group $H^1(\Sigma_g$, End(E)). By Serre's theorem we have:

$T^*_{[E]}\mathrm{N}^g_{n,d} \cong \mathrm{H}^1(\Sigma_g, \mathrm{End(E)})^* \cong \mathrm{H}^0(\Sigma_g, \mathrm{End(E)} \otimes \Omega(\Sigma_g))$

which is precisely the space of compatible Higgs fields for E. Namely, there is an inclusion

$$T^*\mathcal{N}^g_{n,d} \subset \mathrm{M}^g_{n,d}$$

4.3.1 Example

In the rank 1 case, there are no Higgs bundles that aren't arising from stable vector bundles (of course everything is stable), so there is an isomorphism

$\mathcal{M}^g_{n,d} \cong \mathrm{T}^*\mathcal{N}^g_{n,d} \cong \mathrm{T}^* Jac_d(\Sigma_g) \cong (\mathrm{C}^g/\mathrm{Z}^{2g} \times \mathrm{R}^{2g})$

In this case the complex structures and geometric structures on $\mathcal{M}^g_{n,d}$ are all very explicit.