# Introduction to Jacobian varieties

October 16, 2023

## 1 Introduction

This is a brief note introducing Jacobian varieties of algebraic curves. Main references are [Mil21], [XZ12] and chapter 12 of [MEK18]

## 2 Explanation for some notations

### 2.1 algebraic curve

All algebraic curves mentioned in this note except those in the last section refer to a smooth proper curve over $\mathbf{C}$, i.e., a compact Riemann surface.

### 2.2 Picard group

Unlike in most books and notes, we use $Pic(C)$ to denote the group of Cartier divisors(or just Weyl divisors because they are the same for algebraic curves) on $C$. And we use $Lic(C)$ to denote the group of principal divisors on $C$.

## 3 Abel-Jacobi map

### 3.1 Definition of Abel-Jacobi map

For n algebraic curve $C$ with given point $x_0$, we try to define a map $u_{x_0}$ from $C$ to the space of linear functionals on the space of holomorphic 1-forms on $C$ by

$$u_{x_0}(x)(w) = \int_{x_0}^{x} w$$

for any holomorphic form $w$. However, this integral might not be well defined as it depend on the path chosen. So we consider $\Lambda$ to be the lattice generated by all loops in $C$(or equivalently, $H_1(C, \mathbf{Z})$). Then we can defines a holomorphic map $u_{x_0} : C \longrightarrow \Omega(C)^*/\Lambda$.
We can extend this map to divisors on $C$ by mapping $\sum_{i=1}^{d} n_i p_i$ to $\sum_{i=1}^{d} u_{x_0}(p_i)$. Note that when we restrict the map on the divisors of degree 0 which is denoted by $Pic^0(C)$, it is indepedent of the choice of $x_0$.
We call $\Omega(C)^*/\Lambda$ the Jacobian variety of $C$ and denote it by $J(C)$ and we call the map $u : Pic^0(C) \longrightarrow J(C)$ the Abei-Jacobi map.
There are some natural questions. First of all one knows $J(C)$ is an commutative group. One may ask whether $\Omega$ is nondegenerate, i.e. whether $J(C)$ is a complex torus? The answer is yes by De-Ram theory and a little hodge theory. One also question the Kernel of $u$ and whether $u$ is surjective. The latter is called the Jacobi inversion theorem.

## 3.2 the kernel of Abel Jacobian map

**Theorem 3.1.** *The kernel of Abel-Jacobi map is exactly $(C)$, the group of principal divisors on $C$.*
*The proof of this theorem requires mainly techniques in complex analysis and Riemann bilinear relations which will be introduced in the following talks of our seminor. So it is not going to be included in this notes. People interested in this section may refer to[MEK18].*

## 3.3 Jacobi's inversion problem

**Theorem 3.2.** *The Abel-Jacobi map is surjective.*

### 3.3.1 reduction of the problem

*By Riemann-Roch, one see that for an algebraic curve $C$ of genus $g$, any divisor on $C$ of degree $\geq g$ is linear equivalent to an effective divisor on $C$.So one has a bijection between $Pic^0(C)/Li(C)$ and the effective divisors of degree $g$ module linear equivalence.*
*Now we try to establish the structure of an algebraic varietiy on effective divisors of degree $g$.*

#### 3.3.2 symmetric power of varieties

*First we look at two easy examples, we define the $n^{th}$ symmetric power of $\mathbf{A}^1$ and $\mathbf{P}^1$. The $n^{th}$ symmetric power of $\mathbf{A}^1$ is just isomorphic to $\mathbf{A}^n$, defined through a ramified covering of $\mathbf{A}^n$ to itself defined by mapping $(x_1, x_2, ..., x_n)$ to*

$$(e_1(x_1, x_2, ..., x_n), e_2(x_1, x_2, ..., x_n), ..., e_n(x_1, x_2, ..., x_n))$$

*, where*

$$e_k(x_1, x_2, ..., x_n) = \sum_{1 \le x_{i_1} < x_{i_2} < ... < x_{i_k} \le n} \prod_{j=1}^{k} x_{i_j}$$

*is the $i^{th}$ symmetric polynomial of $x_1$, $x_2$, ...,$x_n$. In fact this established a homeomorphism of topological space between $\mathbf{A}^n/S_n$ and $\mathbf{A}^n$.(Why? Hint: Consider the $n^{th}$ symmetric power of $\mathbf{P}^1$ which is in fact $\mathbf{P}^n$.)*

**Remark 3.3.** In fact, the symmetric power of $\mathbf{A}^n$ is not as easy, for example, we consider the $2^{th}$ symmetric power of $\mathbf{A}^2$. We consider the morphism

$$\phi : \quad A^4 \longrightarrow A^5$$

$$(x_1, x_2, y_1, y_2) \mapsto (x_1 + x_2, y_1 + y_2, x_1x_2, y_1y_2, x_1y_2 + x_2y_1)$$

.

One can show the image of $\phi$ is in fact a singular closed subvariety of $\mathbf{A}^5$ defined by the ideal generated by $t_1t_2 - t_3 - t_4 - t_5$ in $\mathbf{C}[t_1, t_2, t_3, t_4, t_5]$.
However, for an algebraic curve, its $d^{th}$ symmetric product is an nonsingular projective variety.

**Definition 3.4.** Now we define the $d^{th}$ symmetric power of any affine variety $SpecA$. Consider the ring $A^{\otimes d}$. Consider the canonical action of $S_d$ on $A^{\otimes d}$, there is an invariant subring $(A^{\otimes d})^{S_d}$. Then we let the $d^{th}$ symmetric power of $SpecA$ be $Spec(A^{\otimes d})^{S_d}$. For an arbitrary variety $V$, we just patch the symmetric sum of affine opens of $V$ together to get $V^{(d)}$, which holds an morphism $\pi : V^d \longrightarrow V^{(d)}$ which is a quotient map of topology spaces and has the following universal property:
For any symmetric morphism $\phi : V^d \longrightarrow T$, $\phi$ factors through $d$ with a unique morphism $\gamma$ such that $\phi = \gamma \circ \pi$.

**Proposition 3.5.** *The $d^{th}$ symmetric product of an algebraic curve is nonsingular. Sketch of proof: Consider the completion of local ring at any point and use Cohen's stucture theorem.*

**Proposition 3.6.** *The $d^{th}$ symmetric product of an algebraic curve is projective. Sketch of proof: Consider its homogeneous coordinate ring.*

### 3.3.3 sketch of proof

*First, we note that for an algebraic curve $C$ of genus $g$, the effective divisors of degree $g$ has a the structure $C^{(g)}$, the $g^{th}$ symmetric power of $C$. Then through the Abel-Jacobi map $C^{(g)}$ maps holomorphically to $J(C)$. As both $C^{(g)}$ and $J(C)$ are projective varieties, by GAGA we know that any holomorphic map between them is algebraic.*

*We can even consider the differential of this map. To show the surjectivity of the map, it suffices to show the surjectivity of differential map at some point. When $g = 1$ it is obvious, we only consider when $g \geq 2$.*

*Let $D = \sum_1^g p_j$ be a effective divisor of degree $g$ on $C$.We take the local coordinate $(z_1, z_2, ..., z_g)$, at the point $D$ of $C^{(g)}$. We take a basis $\omega_1, \omega_2, ..., \omega_g$ of $\Omega(C)$, where $\omega_i = f_{ij} dz_j$ in an neigborhood of $p_j$ and $f_{ij}$ is holomorphic.*

*Then we know the Jacobian matrix of the map is $(f_{ij}(p_j))_{g \times g}$. This matrix is related to the Bill-Noether matrix.*

*We want to show the matrix $(f_{ij}(p_j))_{g \times g}$ is rank full at generic points of $C^{(g)}$. We can consider the map $\Phi$:*

$$C \longrightarrow \mathbf{P}^{g-1}$$

$$p \mapsto (\omega_1(p) : \omega_2(p) : \cdots : \omega_g(p))$$

*since the canonical bundle is base point free. Since $C$ is projective, the map $\Phi$ is algebraic and the image of $\Phi$ is a closed subvariety of $\mathbf{P}^{g-1}$. Also note that the image of $\Phi$ can not be contained in any hyperplane of $\mathbf{P}^{g-1}$ since $\omega_1, \omega_2, ..., \omega_g$ are linearly independent. Note that for $p \in C$, $\Phi(p)$ is contained in some hyperplane $H$, means $\omega(p) = 0$ for some $\omega \in \Omega(C)$. Since any $w \in \Omega(C)$ is of degree $2g-2$, there is at most $2g-2$ points whose image is contained in a given hyperplane. Therefore there is at most $\binom{g+2g-3}{2g-3}$ points in $C^{(g)}$ where every column of the Jacobian matrix is in the same hyperplane. Note that the points such that the Jacobian matrix is not rank full form a closed subscheme of $C^{(g)}$, and it is equppied with a quasi-finite morphism(in fact finite because it is proper) to the variety representing the hyperplanes of $P^{g-1}$, which is isomorphic to $P^{g-1}$ itself which is of dimension lower than that of $C^(g)$. So* *it is a subscheme of degree lower than that of $C^{(g)}$. So $(f_{ij}(p_j))_{g\times g}$ is rank full at generic points of $C^{(g)}$.*

**Remark 3.7.** In fact showing the surjectivity does not quite require the construction of the structure as an algebraic variety for $C^{(g)}$, we can just consider the map from $C^g$. However we can further show $C^{(g)}$ is birational equvalent to $J(C)$. And further more $C^{(g)}$ and $J(C)$ are both moduli spaces of some moduli problems.

# 4 Jacobian varieties are projective varieties

**Theorem 4.1.** *A complex torus $E = V/\Lambda$ is a projective variety if and only if it has a positive definite Hermitian form which takes interger values on $\Lambda$.*

**Theorem 4.2.** *The Jacobian of an algebraic curve is an projective variety.*

*Sketch of proof: Such a Hermitian form on $\Omega(C)^*$ can be obtained from the intersection pairing:*

$$H_1(C, \mathbf{Z}) \times H_1(C, \mathbf{Z}) \longrightarrow \mathbf{Z}$$

.

# 5 Algebraic definition of Jacobian varieties

*Given any field $k$ and an complete nonsingular curve $C$.*
*For any noetherian $k$-Scheme $T$, we consider $C \times_{spec(k)} T$,and the natural projection $q : C \times_{spec(k)} T \longrightarrow T$ and $p : C \times_{spec(k)} T \longrightarrow C$.*
*For a Cartier divisor $D$ on $C \times_{spec(k)} T$ , $D$ is flat over $T$ if and only if the intersection divisor $D_t = D.C_t$ is defined for all closed points $t \in T$. Moreover if $T$ is connected, then $degD_t$ is independent of $t$. (See HartshorneII.9 or AVs by Milne)*
*If the flatness condition above is satisfied, we say the divisors $D_t|t \in T$ form an algebraic family of divisors of $C$ parametrized by $T$.*
*For any $k$-Scheme $T$, let*

$$Pic^0(T) = \{\mathcal{L} \in Pic(C \times_{spec(k)} T)|deg\mathcal{L}_t = 0 \ for \ all \ t \in T/q^*Pic(T)\}.$$

*In fact $Pic^0(T)$ is a contravariant functor from the category of noetherian $k$-scheme to sets. And it is representable.(see theorem 1.1[Mil21]) We define the scheme repre-*

*senting this functor to be* $J(C)$.
*Moreover,* $C^{(d)}$ *represents the functor* $Div_C^d$ :

$$T \longrightarrow$$

$$\{D|D \; is \; aneffective \; Cartier \; divisor \; on \; X \times_{Spec \; k} T \; flat \; over \; T$$
$$and \; degD_t = d \; for \; all \; t \in T\}$$

*(see theorem3.13[Mil21])*

# References


[MEK18] *Victor V. Prasolov Maxim E. Kazaryan, Sergei K. Lando.* Algebraic curves towards moduli spaces. *Springer, 2018.*

[Mil21] *J.S. Milne. Jacobian varieties.* personal website, *2021.*

[XZ12] *Akhill Mathew Xinwen Zhu. Algebraic geometry.* website, *2012.*