A Brief Introduction to Eisenstein Series and Differential Operators on Modular Forms

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Abstract

This is a note aimed to introduce differential operators on modular forms on a seminar about modular forms. Since Eisenstein Series, which are significant examples of Modular Forms, play an important role in this topic , we will give some knowledge about them. Most of the content of the note comes from Zagier modular form.

Contents

1 Something About Eisenstein Series

1.1 Two Natural Ways to Introduce the Eisenstein Series

There are two natural ways to introduce the Eisenstein series. For the first, we observe that the characteristic transformation equation

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \tag{1}$$

of a modular form can be written in the form $f|_k\gamma = f$ for $\gamma \in \Gamma$, where $f|_k\gamma = f : \mathfrak{H} \to \mathbb{C}$ is defined by

$$(f|_k g)(z) = (cz+d)^{-k} f(\frac{az+b}{cz+d}) \quad (z \in \mathbb{C}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R})) \tag{2}$$

One checks easily that for fixed k $\in \mathbb{Z}$, the map f $\mapsto (f|_k g)$ defines an operation of the group $\mathbf{SL}(2,\mathbb{R})$ $(i.e., f|_k(g_1g_2) = (f|_kg_1)|_k(g_2)$ for all $g_1, g_2 \in \mathbf{SL}(2,\mathbb{R}))$ on the vector space of holomorphic functions in $\mathfrak{H}$ having subexponential or polynomial growth. The space $M_k(\Gamma)$ of holomorphic modular forms of weight k on a group $\Gamma \subset \mathbf{SL}(2,\mathbb{R})$ is then simply the subspace of this vector space fixed by Γ.

If we have a linear action v $\mapsto$ v|g of a finite group G on a vector space V , then an obvious way to construct a G-invariant vector in V is to start with $v_0 \in V$ and form the sum $v = \sum_{g\in G} v_0|g$. If the vector v_0 is invariant under some subgroup $G_0 \subset$ G, then the vector $v_0|g$ depends only on the coset $G_0g \in G_0\backslash G$ and we can form instead the smaller sum $v = \sum_{g\in G_0\backslash G} v_0|g$, which again is G-invariant. In the context when G = $\Gamma \subset \mathbf{SL}(2,\mathbb{R})$ is a Fuchsian group (acting by $|_k$) and v_0 a rational function, the modular forms obtained in this way are called *Poincaré series*. An especially easy case is that when v_0 is the constant function 1 and $\Gamma_0 = \Gamma_\infty$, the stabilizer of the cusp at infinity. In this case the series is called an *Eisenstein series*. Let us look at this series more carefully when $\Gamma = \Gamma_1$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\in \mathbf{SL}(2,\mathbb{R})$ sends ∞ to a/c, and hence belongs to the stabilizer of ∞ if and only if c = 0. In Γ_1 these are the matrices $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with n $\in \mathbb{Z}$, i.e. the matrices T^n. We can assume that k is even (since there are no modular forms of odd weight on Γ_1 and hence work with $\overline{\Gamma_1} = \mathbf{PSL}(2,\mathbb{Z})$, in which case the stabilizer $\overline{\Gamma_\infty}$ is the infinite cyclic group generated by T. If we multiply an arbitrary matrix$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the left by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, then the resulting matrix γ' $= \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$ has the same bottom row as γ. Conversely, if $\gamma' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Gamma_1$ has the same bottom row as γ, then from(a'−a)d−(b'−b)c = det(γ) - det(γ') = 0 and (c,d) = 1 we see that a' − a = nc, b' − b = nd for some n $\in \mathbb{Z}$, i.e., $\gamma' = T^n\gamma$. Since every coprime pair of integers occurs as the bottom row of a matrix in $\mathbf{SL}(2,\mathbb{Z})$, these considerations give the formula

$$E_k(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_1} 1|_k\gamma = \sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma_1}} 1|_k\gamma = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(cz+d)^k} \tag{3}$$

for the Eisenstein series (the factor $\frac{1}{2}$ arises because (c d) and (−c − d) give the same element of $\Gamma_1\backslash\overline{\Gamma_1}$). It is easy to see that this sum is absolutely convergent for k $>$ 2 (the number of pairs (c, d) with N $\le$ |cz + d| $<$ N + 1 is the number of lattice points in an annulus of area $\pi(N+1)^2 - \pi N^2$ and hence is O(N), so the series is majorized by $\sum_{N=1}^{\infty} N^{1-k}$), and this absolute convergence guarantees the modularity (and, since it is locally uniform in z, also the holomorphy) of the sum. The function $E_k(z)$ is therefore a modular form of weight k for all even k ≥ 4. It is also clear that it is non-zero, since for $\Im(z) \to \infty$ all the terms in (3) except (c d) = ($\pm$1 0) tend to 0, the convergence of the series being sufficiently uniform that their sum also goes to 0 , so $E_k(z) = 1 + o(1) \neq 0$.

The second natural way of introducing the Eisenstein series comes from the interpretation of modular forms by lattice. Remember that we can identify solutions of the transformation equation (1) with functions $\Lambda \subset \mathbb{C}$ satisfying the homogeneity condition $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$ under homotheties $\Lambda \mapsto \lambda\Lambda$. An obvious way to produce such a homogeneous function if the series converges is to form the sum $G_k(\Lambda) = \frac{1}{2}\sum_{\lambda\in\Lambda\backslash 0} \lambda^{-k}$ of the (−k)th powers of the non-zero elements of Λ n In terms of z $\in \mathfrak{H}$ and its associated lattice $\Gamma_z = \mathbb{Z}.z + \mathbb{Z}.1$, this becomes

$$G_k(z) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} \tag{4}$$

where the sum is again absolutely and locally uniformly convergent for k $\ge$ 2, guaranteeing that $G_k \in M_k(\Gamma_1)$. The modularity can also be seen directly by noting that $(G_k|_k\gamma)(z) = \sum_{m,n}(m'z+n')^{-k}$ where $(m'z+n') = (m,n)\gamma$ runs over the non-zero vectors of $\mathbb{Z}^2\backslash(0,0)$ as (m,n) does.

In fact, the two functions (3) and (4) are proportional, as is easily seen: any non-zero vector (m,n) $\in \mathbb{Z}^2$ can be written uniquely as r(c, d) with r (the greatest common divisor of m and n) a positive integer and c and d coprime integers, so

$$G_k(z) = \zeta(k) E_k(z) \tag{5}$$

where $\zeta(k) = \sum_{r \geq 1} \frac{1}{r^k}$.is the value at k of the Riemann zeta function. It may therefore seem pointless to have introduced both definitions. But in fact, this is not the case. First of all, each definition gives a distinct point of view and has advantages in certain settings which are encountered at later points in the theory: the E_k definition is better in contexts like the famous Rankin-Selberg method where one integrates the product of the Eisenstein series with another modular form over a fundamental domain, while the G_k definition is better for analytic calculations and for the Fourier development. Moreover, if one passes to other groups, then there are σ Eisenstein series of each type, where σ is the number of cusps, and, although they span the same vector space, they are not individually proportional. In fact, we will actually want to introduce a third normalization

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) \tag{6}$$

because, as we will see below, it has Fourier coefficients which are rational numbers.

1.2 Fourier Expansions of Eisenstein Series

Recall that any modular form on Γ_1 has a Fourier expansion of the form$\sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$. The coefficients a_n often contain interesting arithmetic information, and it is this that makes modular forms important for classical number theory. For the Eisenstein series, normalized by (6), the coefficients are given by:

Proposition 1.1. *The Fourier expansion of the Eisenstein series* $\mathbb{G}_k(z)(k even, k \geq 2)$ *is*

$$\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{7}$$

where B_k *is the kth Bernoulli number and where* $\sigma_{k-1}(n)$ *for* $n \in \mathbb{N}$ *denotes the sum of the (k-1)st powers of the positive divisors of n. We recall that the Bernoulli numbers are defined by* *the generating function* $\sum_{k=0}^{\infty} B_k x^k/k! = x/(e^x - 1)$

Proof. A well known and easily proved identity of Euler states that

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \frac{\pi}{\tan \pi z} \quad (z \in \mathbb{C}\backslash\mathbb{Z}) \tag{8}$$

where the sum on the left, which is not absolutely convergent, is to be interpreted as a Cauchy principal value. The function on the right is periodic of period 1 and its Fourier expansion for z $\in \mathfrak{H}$ is given by $\frac{\pi}{\tan \pi z} = \pi \frac{\cos \pi z}{\sin \pi z} = \pi i \frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -2\pi i(\frac{1}{2} + \sum_{r=1}^{\infty} q^r)$, where $q = e^{2\pi i z}$. Substitute this into (8), differentiate k−1 times and divide by $(-1)^{k-1}(k-1)!$ to get $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}(\frac{\pi}{\tan \pi z}) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r (k \geq 2, z \in \mathfrak{H})$, an identity known as Lipschitz's formula. Now the Fourier expansion of G_k (k $\geq$ 2 even) is obtained immediately by splitting up the sum in (4) into the terms with m = 0 and those with m $\neq$ 0:

$$\begin{aligned} G_k(z) &= \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^k} + \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{1}{(mz+n)^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{k-1} q^{mr} \\ &= \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right), \end{aligned} \tag{9}$$

where in the last line we have used Euler's evaluation of ζ(k) (k $\geq$ 0 even) in terms of Bernoulli numbers. The result follows. □

The first three examples of (1.1) *are the expansions*

$$\begin{aligned} \mathbb{G}_4(z) &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \ldots, \\ \mathbb{G}_6(z) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \ldots, \\ \mathbb{G}_8(z) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \ldots. \end{aligned}$$

The other two normalizations of these functions are given by

$$G_4z = \frac{16\pi^4}{3!}\mathbb{G}_4(z) = \frac{\pi^4}{90}E_4(z), \quad E_4(z) = 1 + 240q + 2160q^2 + \dots,$$
$$G_6z = \frac{64\pi^6}{5!}\mathbb{G}_6(z) = \frac{\pi^6}{945}E_6(z), \quad E_6(z) = 1 - 504q - 16632q^2 - \dots,$$
$$G_8z = \frac{256\pi^8}{7!}\mathbb{G}_8(z) = \frac{\pi^8}{9450}E_8(z), \quad E_8(z) = 1 + 480q + 61920q^2 + \dots.$$

Remark. *We have discussed only Eisenstein series on the full modular group in detail, but there are also various kinds of Eisenstein series for subgroups* $\Gamma \subset \Gamma_1$. *We give one example. Recall that a Dirichlet character modulo* $N \in \mathbb{N}$ *is a homomorphism* $\chi : (\mathbb{Z}/N\mathbb{Z})* \to \mathbb{C}*$*, extended to a map* $\chi : \mathbb{Z} \to \mathbb{C}$ *(traditionally denoted by the same letter) by setting* χ*(n) equal to* χ*(n modN) if (n,N) = 1 and to 0 otherwise. If* χ *is a non-trivial Dirichlet character and k a positive integer with* $\chi(1) = (1)^k$*, then there is an Eisenstein series having the Fourier expansion*

$$\mathbb{G}_{k,\chi}(z) = c_k(\chi) + \sum_{n=1}^{\infty}(\sum_{d|n}\chi(d)d^{k-1})q^n$$

which is a "modular form of weight k and character χ *on* Γ_0*(N)."(This means that* $\mathbb{G}_{k,\chi}(\frac{az+b}{cz+d}) = \chi(a)(cz+d)^k\mathbb{G}_{k,\chi}(z)$ *for any z* $\in \mathfrak{H}$ *and any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{Z})$ *with c*$\equiv 0(modN)$*.) Here* $c_k(\chi) \in \overline{\mathbb{Q}}$ *is a suitable constant, given explicitly by* $c_k(\chi) = \frac{1}{2}L(1-k,\chi)$*, where L(s,* χ*) is the analytic continuation of the Dirichlet series* $\sum_{n=1}^{\infty}\chi(n)n^{-s}$.

The simplest example, for N = 4 and $\chi = \chi^{-4}$ *the Dirichlet character modulo 4 given by*

$$\chi_{-4}(n) = \begin{cases} +1 & if \quad n \equiv 1 \quad (mod4), \\ -1 & if \quad n \equiv 3 \quad (mod4), \\ 0 & if \quad n \quad is \quad even \end{cases}$$

and k=1, is the series

$$\mathbb{G}_{1,\chi_{-4}}(z) = c_1(\chi_{-4}) + \sum_{n=1}^{\infty}(\sum_{d|n}\chi_{-4}(d))q^n = \frac{1}{4} + q + q^2 + q^4 + 2q^5 + q^8 + \dots \qquad (10)$$

(The fact that L(0,χ_{-4}*) = 2*$c_1(\chi_{-4} = \frac{1}{2})$ *is equivalent via the functional equation of* $L(s,\chi_{-4})$ *to Leibnitz's famous formula: L(1,*$\chi_{-4}) = 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$*).*

1.3 The Eisenstein Series of Weight 2

In above discussion we restricted ourselves to the case when k ≥ 2, since then the series (3)and (4) are absolutely convergent and therefore define modular forms of weight k. But the final formula (7) for the Fourier expansion of $\mathbb{G}_k(z)$ converges rapidly and defines a holomorphic function of z also for k = 2, so in this weight we can simply define the Eisenstein series $\mathbb{G}_2$, G_2 and $_E2$ by equations (7), (6), and (5), respectively, i.e.,

$$\begin{aligned}
\mathbb{G}_2(z) &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \dots, \\
G_2(z) &= -4\pi^2 (G)_2(z), E_2(z) = \frac{6}{\pi^2} G_2(z) = 1 - 24q - 72q^2 - \dots
\end{aligned} \tag{11}$$

Moreover, the same proof as for (1.1) still shows that $G_2(z)$ is given by the expression (10), if we agree to carry out the summation over n first and then over m:

$$G_2(z) = \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \frac{1}{2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2}. \tag{12}$$

The only difference is that, because of the non-absolute convergence of the double series, we can no longer interchange the order of summation to get the modular transformation equation $G_2(1/z) = z^2 G_2(z)$. (The equation $G_2(z+1) = G_2(z)$, of course, still holds just as for higher weights.) Nevertheless, the function $G_2(z)$ and its multiples $E_2(z)$ and $G_2(z)$ do have some modular properties and, as we will see later, these are important for many applications.

Proposition 1.2. *For z* $\in \mathfrak{H}$ *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{Z})$ *we have*

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c (cz+d). \tag{13}$$

Proof. There are many ways to prove this. We sketch one, due to Hecke, since the method is useful in many other situations. The series (4) for k = 2 does not converge absolutely, but it is just at the edge of convergence, since $\sum_{m,n} |mz+n|$ converges for any real number λ ≥ 2. We therefore modify the sum slightly by introducing

$$G_{2,\epsilon}(z) = \frac{1}{2} \sum_{m,n}' \frac{1}{(mz+n)^2 |mz+n|^{2\epsilon}} \quad (z \in \mathfrak{H}, \epsilon > 0). \tag{14}$$

(Here $\sum$' means that the value (m, n) = (0, 0) is to be omitted from the summation.) The new series converges absolutely and transforms by $G_{2,\epsilon}\frac{az+b}{cz+d} = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z)$. We claim that $\lim_{\epsilon\to+0} G_{2,}(z)$ exists and equals $G_2(z)\frac{\pi}{2y}$, where y = $\Im(z)$. It follows that each of the three non-holomorphic functions

$$G_2^*(z) = G_2(z) - \frac{\pi}{2y}, \quad E_2^*(z) = E_2(z) - \frac{3}{\pi y}, \quad \mathbb{G}_2^*(z) = \mathbb{G}_2(z) - \frac{1}{8\pi y} \tag{15}$$

transforms like a modular form of weight 2, and from this one easily deduces the transformation equation (2.1) and its analogues for E_2 and $_G2$. To prove the claim, we define a function I_ϵ by

$$I_\epsilon(z) = \int_{-\infty}^{\infty} \frac{dt}{(z+t)^2|z+t|^{2\epsilon}} \quad (z \in \mathfrak{H}, \epsilon > -\frac{1}{2}). \tag{16}$$

Then for $\epsilon > 0$ we can write

$$\begin{aligned} G_{2,\epsilon} - \sum_{m=1}^{\infty} I_\epsilon(mz) &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} \\ &+ \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} [\frac{1}{(mz+n)^2|mz+n|^{2\epsilon}} - \int_n^{n+1} \frac{dt}{(mz+t)^2|mz+t|^{2\epsilon}}]. \end{aligned} \tag{17}$$

Both sums on the right converge absolutely and locally uniformly for $\epsilon > \frac{1}{2}$ (the second one because the expression in square brackets is $O(|mz+n|^{-3-2\epsilon})$ by the mean-value theorem, which tells us that f(t) − f(n) for any differentiable function f is bounded in $n \le t \le n+1$ by $max_{n\le t\le n+1}|f'(u)|$), so the limit of the expression on the right as $\epsilon \to 0$ exists and can be obtained simply by putting $\epsilon = 0$ in each term, where it reduces to $G_2(z)$ by (12). On the other hand, for $\epsilon > -\frac{1}{2}$ we have

$$\begin{aligned} I_\epsilon(x+iy) &= \int_{-\infty}^{\infty} \frac{dt}{(x+t+iy)^2((x+t)^2+y^2)^\epsilon} \\ &= \int_{-\infty}^{\infty} \frac{dt}{(t+iy)^2(t^2+y^2)^\epsilon} = \frac{(I(\epsilon)}{y^(1+2\epsilon)}, \end{aligned}$$

where $(I(\epsilon) = \int_{-\infty}^{\infty}(t+i)^{-2}(t^2+1)^{-\epsilon}dt$, so $\sum_{m=1}^{\infty} I_\epsilon(mz) = I(\epsilon)\zeta(1+2\epsilon)/y^{1+2\epsilon}$ for $\epsilon > 0$. Finally, we have I(0) = 0 (obvious),

$$I'(0) = -\int_{-\infty}^{\infty} \frac{log(t^2+1)}{(t+i)^2}dt = (\frac{1+log(t^2+1)}{t+i} - \tan^{-1}t)|_{-\infty}^{\infty} = -\pi$$

and $\zeta(1+2\epsilon) = \frac{1}{2\epsilon} + O(1)$, so the product $I(\epsilon)\zeta(1+2\epsilon)/y^{1+2\epsilon}$ tends to $-\pi/2y$ as $\epsilon \to 0$. The claim follows. □

Remark. *The transformation equation (12) says that G_2 is an example of what is called a quasimodular form, while the functions G_2^*, E_2^* and $\mathbb{G}_2^*$ defined in (15) are so-called almost holomorphic modular forms of weight 2.*

2 Modular Forms and Differential Operators

The starting point for this section is the observation that the derivative of a modular form is not modular, but nearly is. Specifically, if f is a modular form of weight k with the Fourier expansion $\sum_{n=0}^{\infty} a_n q^n$, then by differentiating (1) we see that the derivative

$$Df = f' := \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq} = \sum_{n=1}^{\infty} n a_n q^n \tag{18}$$

(where the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients) satisfies

$$f'(\frac{az+b}{cz+d}) = (cz+d)^{k+2} f'(z) + \frac{k}{2\pi i} c(cz+d)^{k+1} f(z). \tag{19}$$

If we had only the first term, then f' would be a modular form of weight k+2. The presence of the second term, far from being a problem, makes the theory much richer. To deal with it, we will:

1) modify the differentiation operator so that it preserves modularity;

2) make combinations of derivatives of modular forms which are again modular;

3) relax the notion of modularity to include functions satisfying equations like (52);

4) differentiate with respect to t(z) rather than z itself, where t(z) is a modular function.

These four approaches are discussed in the four subsections in the textbook, we choose section one, which is the most accessible to discuss on this seminar.

2.1 Derivatives of Modular Forms

As already stated, the first approach is to introduce modifications of the operator $\mathcal{D}$ which do preserve modularity. There are two ways to do this, one holomorphic and one not. We begin with the holomorphic one. Comparing the transformation equation (19) with equations (11) and , we find that for any modular form $f \in M_k(\Gamma_1)$ the function

$$\mathfrak{V} f := f' - \frac{k}{12} E_2 f, \tag{20}$$

sometimes called the*Serre derivative*, belongs to $M_{k+2}(\Gamma_1)$. (We will often drop the subscript k, since it must always be the weight of the form to which the operator is applied.) A first consequence of this basic fact is the following. We introduce the ring $\widetilde{M}_*(\Gamma_1) := M_*(\Gamma 1)[E_2] = \mathbb{C}[E_2, E_4, E_6]$, called the ring of quasimodular forms on $\mathbf{SL}(2, \mathbb{Z})$. (An intrinsic definition of the elements of this ring, and a definition for other groups $\Gamma \subset G$, will be given in the next subsection in the textbook, not covered by this note unfortunately.) Then we have:

Proposition 2.1. *The ring $\widetilde{M}_*(\Gamma_1)$ closed under differentiation. Specifically, we have*

$$E_2' = \frac{E_2^2 - E_4}{12}, E_4' = \frac{E_2 E_4 - E_6}{3}, E_6' = \frac{E_2 E_6 - E_4^2}{2}. \tag{21}$$

Proof. Clearly$\mathfrak{V} E_4$ and $\mathfrak{V} E_6$, being holomorphic modular forms of weight 6 and 8 on $_1$, respectively, must be proportional to E_6 and E_4^2 , and by looking at the first terms in their Fourier expansion we find that the factors are −1/3 and −1/2. Similarly, by differentiating (11) we find the analogue of (20) for E_2, namely that the function $E_2' - \frac{1}{12} E_2^2$ belongs to $M_4(\Gamma)$. It must thereforebe a multiple of E_4, and by looking at the first term in the Fourier expansion one sees that the factor is −1/12. □

An immediate consequence of Proposition (2.1) is the following:

Proposition 2.2. *Any modular form or quasi-modular form on $_1$ satisfies a non-linear third order differential equation with constant coefficients.*

Proof. Since the ring $\widetilde{M}_*(\Gamma_1)$ has transcendence degree 3 and is closed under differentiation, the four functions f, f', f'' and f''' are algebraically dependent for any $f \in \widetilde{M}_*(\Gamma_1)$. □

We now turn to the second modification of the differentiation operator which preserves modularity, this time, however, at the expense of sacrificing holomorphy. For $f \in M_k(\Gamma)$ (we now no longer require that Γ be the full modular group Γ_1) we define

$$\partial_k f(z) = f'(z) - \frac{k}{4\pi y} f(z), \tag{22}$$

where y denotes the imaginary part of z. Clearly this is no longer holomorphic, but from the calculation

$$\frac{1}{\gamma z} = \frac{|cz+d|^2}{y} = \frac{(cz+d)^2}{y} - 2ic(cz+d) \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{R})) \tag{23}$$

and (19) one easily sees that it transforms like a modular form of weight k+2, i.e., that $(\partial_k f)|_{k+2}\gamma = \partial_k f$ for all $\gamma \in \Gamma$. Moreover, this remains true even if f is modular but not holomorphic, if we interpret f' as $\frac{1}{2\pi i}\frac{\partial f}{\partial z}$. This means that we can apply $\partial = \partial_k$ repeatedly to get non-holomorphic modular forms $\partial^n f$ of weight k + 2n for all n ≥ 0. (Here, as with $\mathfrak{V}_k$, we can drop the subscript k because ∂_k will only be applied to forms of weight k; this is convenient because we can then write $\partial^n f$ instead of the more correct $\partial_{k+2n-2} \dots \partial_{k+2}\partial_k f$.) For example, for $f \in M_k(\Gamma)$ we find

$$\begin{aligned} \partial^2 f &= (\frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k+2}{4\pi y})(f' - \frac{k}{4\pi y} f) \\ &= f'' - \frac{k}{4\pi y} f' - \frac{k}{16\pi^2 y^2} f - \frac{k+2}{4\pi y} f' + \frac{k(k+2)}{16\pi^2 y^2} f \\ &= f'' - \frac{k+1}{2\pi y} f' + \frac{k(k+1)}{16\pi^2 y^2} f \end{aligned} \tag{24}$$

and more generally, as one sees by an easy induction,

$$\partial^n f = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(k+r)_{n-r}}{(4\pi y)^{n-r}} D^r f, \tag{25}$$

The inversion of (25) is

$$D^n f = \sum_{r=0}^{n} \binom{n}{r} \frac{(k+r)_{n-r}}{(4\pi y)^{n-r}} \partial^r f \tag{26}$$

and describes the decomposition of the holomorphic but non-modular form $f^(n) = D^n f$ into non-holomorphic but modular pieces: the function $y^{rn}\partial^r f$ is multiplied by $(cz+d)^{k+n+r}(c\overline{z} + d)^{nr}$ when z is replaced by $\frac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Formula (25) has a consequence which will be important in Singular Moduli. The usual way to write down modular forms is via their Fourier expansions, i.e., as power series in the quantity $q = e^{2\pi iz}$ which is a local coordinate at infinity for the modular curve $\Gamma\backslash\mathfrak{H}$. But since modular forms are holomorphic functions in the upper half-plane, they also have Taylor series expansions in the neighborhood of any point $z = x + iy \in \mathfrak{H}$. The "straight" Taylor series expansion, giving f(z + ω) as a power series in w, converges only in the disk $|\omega|$ < y centered at z and tangent to the real line, which is unnatural since the domain of holomorphy of f is the whole upper half-plane, not just this disk. Instead, we should remember that we can map $\mathfrak{H}$ isomorphically to the unit disk, with z mapping to 0, by sending z' $\in \mathfrak{H}$ to $\omega = \frac{z'-z}{z'-\bar{z}}$. The inverse of this map is given by $z' = \frac{z-\bar{z}\omega}{1-\omega}$, and then if f is a modular form of weight k we should also include the automorphy factor $(1-\omega)^k$ corresponding to this fractional linear transformation (even though it belongs to $\mathbf{PSL}(2,\mathbb{C})$ and not Γ). The most natural way to study f near z is therefore to expand$(1-\omega)^k f(\frac{z-\bar{z}\omega}{1-\omega})$in powers of ω. The following proposition describes the coefficients of this expansion in terms of the operator (27).

Proposition 2.3. *Let f be a modular form of weight k and z = x+iy a point of $\mathfrak{H}$. Then*

$$(1-\omega)^{-k} f(\frac{z-\overline{z}\omega}{1-\omega}) = \sum_{n=0}^{\infty} \partial^n f(z) \frac{(4\pi y\omega)^n}{n!} \quad (|\omega| < 1). \tag{27}$$

Proof. From the usual Taylor expansion, we find

$$\begin{aligned}(1-\omega)^{-k} f(\frac{z-\overline{z}\omega}{1-\omega}) &= (1-\omega)^{-k} f(z + \frac{2iy\omega}{1-\omega}) \\ &= (1-\omega)^{-k} \sum_{r=0}^{\infty} \frac{D^r f(z)}{r!} (\frac{-4\pi y\omega}{1-\omega})^r,\end{aligned}$$

and now expanding $(1-\omega)_{-k-r}$ by the binomial theorem and using (25)) we obtain (27). □

Proposition (2.3) is useful because the expansion (27), after some renormalizing, often has algebraic coefficients that contain interesting arithmetic information.