complement 为什么奇数 weight modular form 不存在 (即 p 2次型)

i.e. Let k be an odd number

$$f\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^{2k}}f(z)$$

① $f(z)=f(z+1)$ ✓

② $f(z+1)=f\left(\frac{-1\cdot z - 1}{0\cdot z - 1}\right)=-f(z)\Rightarrow f(z)=0$.

Then we start with Hecke Operator

~~E∈~~

First define on free ab gp

E be a set X_E: free ab gp generated by E

$$X_E:\ \bigoplus \mathbb{Z}e_i\quad e_i\in E$$

A correspondence on E is a hom T of X_E into itself

we write

$$T: X_E \longrightarrow X_E$$

$$x \longrightarrow \sum_{y\in E} n_y(x)\,y$$

Here $n_y(x)\in\mathbb{Z}$. (为 x 在 X_E 中系数)

F be a numerical valued function on E

F can be extended to a function on X_E naturally

$F(e_1, e_2, e_3)$ $\quad F(ae_1+be_2+ce_3)=a+2b+3c$

$(1, 2, 3)$ $\quad \in X_E \quad \in X_E$

define transform of F by T, denoted TF

$$TF(x)=F\circ T=\sum_{y\in E} n_y(x)\,F(y)$$

well defined by linearity

Then we introduce $T(n)$ i.e. $T(n)$ on 集 R

(transform a lattice to a sum of $\boxed{\text{lattice}}$)

sub-lattice of index n.

~~$F\in$ 集~~

$\Gamma \in X_{\mathbb{R}}$

$T_{(n)}\Gamma = \sum_{(\Gamma:\Gamma')=n} \Gamma'$ if $T \in R$

Remark what is Γ' such that Γ' is a lattice of index n.

First look at $n\Gamma$ geometrically

$n\Gamma[w_1, w_2] = \Gamma(nw_1, nw_2)$

then we claim $\Gamma' \supseteq n\Gamma$

$\left(\bar{x} \in \Gamma/\Gamma' \quad n\bar{x} = \bar{0} \Rightarrow \underbrace{x \in \Gamma}_{\text{any } x.} \quad nx \in \Gamma'\right)$

then we prove the one to one correspondence

between subgp of Γ containing $n\Gamma$ and subgps of $\Gamma/n\Gamma$

$\Gamma \supseteq \Gamma' \supseteq n\Gamma$

$\Gamma' \leq \Gamma \to \Gamma'/n\Gamma \leq \Gamma/n\Gamma \cong (\mathbb{Z}/n\mathbb{Z})^2$

preserves the index of subgp

Subgp of $(\mathbb{Z}/n\mathbb{Z})^2$ of order n correspond to the sublattices

Γ' of Γ of index n.

$\Gamma/n\Gamma = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$

n prime. subgp of order n in $(\mathbb{Z}/n\mathbb{Z})^2 = n+1$

$(1,0)\ (1,1)\ \ldots\ (1, n-1)\ (0, n-1)$ $\underline{n+1 \text{ direction}}$ in $\mathbb{F}_n$

$\frac{1^2+1}{l-1} = l+1$

then we use this to find Γ'

Another way to present it

$\Gamma = \Gamma(w_1, w_2)$

$\Gamma' = \Gamma(aw_1 + bw_2, dw_2)$ $\quad ad = n \quad a \geq 1 \quad 0 \leq b < d$

(proved latter)

So for $n = 3$

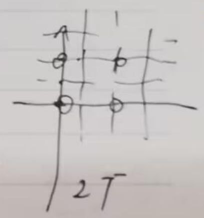
$T(3)\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

homothety operator R_λ $(\lambda \in \mathbb{C}^*)$

$R_\lambda \Gamma = (\lambda\Gamma)$ if $\Gamma \in R$ $\quad \Gamma(w_1, w_2) \to \Gamma(\lambda w_1, \lambda w_2)$

R_λ, T_m: endomorphism of $X_{\mathbb{R}}$

$R_\lambda \Gamma = \lambda\Gamma \quad \lambda^{-1}\Gamma \to \lambda\lambda^{-1}\Gamma$

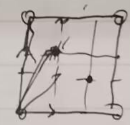


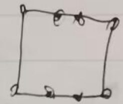
2Γ

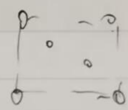
什么是 Γ' $\quad \Gamma' < \Gamma$

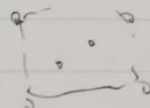
$\Gamma'(aw_1 + bw_2, cw_1 + dw_2)$

with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = n$









property:

(1) $R_\lambda R_\mu = R_{\lambda\mu}$

(2) $R_\lambda T(n) = T_n R_\lambda$ trivial

(3) $T(m) T(n) = T(mn)$ $(m,n)=1$

(4) $T(p^n) T(p) = T(p^{n+1}) + p\, T(p^{n-1}) R_p$.

pf of (3) Γ'' be a sublattice of index mn.

$\exists$ a unique sublattice Γ' of Γ containing Γ'' such that

$(\Gamma:\Gamma')=n$ $(\Gamma':\Gamma'')=m$

$\Gamma/\Gamma'' \cong \mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$

$\Gamma/\Gamma'' \xrightarrow{\varphi} \mathbb{Z}_m \times \mathbb{Z}_n$.

Another view

$T(mn)\Gamma = \sum \Gamma''$ ← easy

$T(m) \circ T(n) \Gamma = \sum \Gamma''$ sum over pairs Γ', Γ'' $(\Gamma:\Gamma')=n$ $(\Gamma':\Gamma'')=m$

反过来 Γ'' index mn 有 Γ' $\Gamma \supset \Gamma' \supset \Gamma''$ Γ' index n.

because Γ/Γ'' has a unique sub gp of order m

↓ 对应到

~~都对应到~~ $\Gamma' \subset \Gamma$ $(\Gamma:\Gamma')=m$

$T(p^n) T(p) \Gamma = \sum \Gamma''$

sum over (Γ', Γ'') with $(\Gamma, \Gamma')=p$ $(\Gamma', \Gamma'')=p^n$

$T(p^{n+1})\Gamma = \sum \Gamma''$ - -

$p R(p) \circ T(p^{n-1}) \Gamma = p \sum (R(p))(\Gamma')$ $\Gamma' \subset \Gamma$ $(\Gamma:\Gamma')=p^{n-1}$

$= p(\sum \Gamma'')$ Γ'' with $(p\Gamma:\Gamma'')=p^{n-1}$

PS R(p)作用不一样

Γ'' of index p^{n+1} in Γ Fix such a lattice

Let a be the # of times it ~~times~~ occures in first sum

b be the number of times occures in the last sum

need to prove $a = 1 + pb$

two cases

NO.
DATE

T'' 固定

1. $\boxed{T'' \not\subset pT} \Rightarrow b=0$ a is the number of lattices of T' contains T'' and of index p in T, T' contains pT. given T'', T' is ~~defined~~ unique. (Since the subgps of T of index p. 1-1 corresponds to subgps of T/pT (of index p))
$\Rightarrow a=1$ $T' = T'' + pT$

2. $T'' \subset pT$ $b=1$ (count T' s.t. $(T:T')=p$, this is $p+1$
$\Downarrow$
$T'' \subset pT \subset T'$ 自动满足

Cor 1. $T(p^n)$ are polynomials on $T(p)$ and R_p

2. The algebra generated by $T(p)$ and R_p p prime H is commutative and contains $T(n)$ for all n

These are trivial from previous results

3. $T(m) \circ T(n) = \sum_{d \mid \gcd(m,n),\, d>0} d \cdot R_d \circ T(mn/d^2)$

Prove. for prime case then use (1)

~~prove by induction Suppose true for $s+1 \leq r$~~

~~$T(p^{s+1}) T(p^r) = T(p) T(p^s) T(p^r) - p^{k-1} T(p^{s-1}) T(p^r)$~~

~~$= T(p) \sum_{0 \leq n \leq s} T(p^{r+s-2n}) p^{n(k-1)} - p^{k-1} \sum_{0 \leq n \leq s-1} T(p^{r+s-2n-1}) p^{n(k-1)}$~~

$$T(p^r) \circ T(p^s) = \sum_{i \leq \min(r,s)} p^i R(p^i) \circ T(p^{r+s-2i})$$

$$\begin{aligned} T(p^r) \circ T(p^s) &= T(p^r) \circ (T(p^{s-1}) \circ T(p) - p R(p) T(p^{s-2})) \\ &= T(p^r) \circ T(p^{s-1}) \circ T(p) - p R_p T(p^r) \circ T(p^{s-2}) \\ &= \left(\sum_{i \leq \min(r,s-1)} p^i R(p^i) T(p^{r+s-2i-1}) \right) \circ T_p - \sum \end{aligned}$$

$$= \sum_{i \leq \min(r,s-1)} p^i R(p^i) (T(p^{r+s-2i}) + pR_p T(p^{r+s-2i-2})) - \sum_{i \leq \min(r,s-2)} p^{i+1} R(p^{i+1}) T(p^{r+s-2i-2})$$

$r > s$ 时 $= \sum_{i \leq \min(r,s)} p^i R(p^i) \circ T(p^{r+s-2i})$

见 Elliptic and Modular functions from Gauss to ... P4 1/2

So if $n = p_1^{n_1} \cdots p_k^{n_k}$ $\quad m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$

$$T(n)T(m) = T(p_1^{n_1})T(p_1^{m_1}) \cdots T(p_k^{n_k})T(p_k^{m_k})$$

$$= \Big(\sum_{0 \le i \le n_1, m_1} T(p_1^{n_1+m_1-2i})\, p_1^{i} R(p^{i}) \Big) \cdots$$

$$= \sum_{d \mid n, m} d \cdot R(d) \circ T(mn/d^2)$$

Let F be function on $\mathcal{R}$ of weight $2k$

define $R_\lambda F(T) = F(\lambda T) = \lambda^{-2k} F(T)$ $\quad R_\lambda F = \lambda^{-2k} F$

$\therefore R_\lambda (T_n (F)) = T_n (R_\lambda F) = \lambda^{-2k} T(n) F$ $\quad \underline{T F(T) = F(T(T))}$

交换性

(5) $T(m) T(n) F = T(mn) F$

$T(p) T(p^n) F = T(p^{n+1}) F + p^{1-2k} T(p^{n-1}) F$ $\Big\}$ from (3), (4)

matrix lemma (Explain what is Γ' $(\Gamma : \Gamma') = n$)

Lemma

Let S_n be the set of integer matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$, $a \ge 1$

$0 \le b < d$ if $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is contained in S_n Let Γ_σ be the sublattices of Γ having for basis

$w_1' = a w_1 + b w_2$ $\quad w_2' = d w_2$

Γ_σ consists of all Γ' of index n. denote $\Gamma(n)$

$\Gamma_\sigma \subseteq \Gamma(n)$ Since $\det(\sigma) = n$

Conversely $\Gamma' \in \Gamma(n)$

Γ' index n $\quad$ then $\Gamma = \langle w_1, w_2 \rangle$ $\quad$ of order d in $\mathbb{Z} w_2$

$Y_1 = \Gamma / (\Gamma' + \mathbb{Z} w_2)$ $\quad Y_2 = \mathbb{Z} w_2 / (\Gamma' \cap \mathbb{Z} w_2)$ $\quad$ 故而 $w_2' = d w_2$ 在 $\Gamma' \cap \mathbb{Z} w_2$ 中

cyclic group order a $\quad$ cyclic groups order d 因

$0 \to Y_2 \xrightarrow{i} \Gamma/\Gamma' \xrightarrow{\pi} Y_1 \to 0$ exact

$\operatorname{Im} i = \ker \pi$ $\quad$ ~~$\Gamma/\Gamma' = Y_2 \oplus Y_1$~~ ; i injection, π projection

$Y_1 = \Gamma/\Gamma' / \operatorname{im} i$

No.
DATE / /

$w_2' = d w_2$ the $w \in T'$

$w_1' \in T'$ $w_1' \equiv a w_1 \pmod{\mathbb{Z} w_2}$

w_1' w_2' form a basis of T'

$w' = a w_1 + b w_2$ b is unique by modulo d

A: 2×2 matrix determinant n. exist U $U \cdot A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ $ad = n$. a, b, d are unique

Rmk. Let $M(n)$ be the set of 2×2 matrices with coefficients in $\mathbb{Z}$ and determinant n. Then

$$M(n) = \bigcup SL_2(\mathbb{Z}) \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Ex p prime S_p are the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and the p matrices $\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$ $0 \le b \le p$

Action of $T(n)$ on modular function

$$F(\Gamma(w_1, w_2)) = w_2^{-2k} f(w_1/w_2)$$

f be a weakly modular function of weight $2k$ F corresponds to f

define $T(n)f$ on H associated to $n^{2k-1} T(n) F$ and

$$T(n)f(z) = n^{2k-1} T(n) F(\Gamma(z, 1))$$

by lemma $T(n)f(z) = n^{2k-1} \sum_{\substack{a \ge 1,\ ad = n \\ 0 \le b < d}} d^{-2k} f\left(\frac{az+b}{d}\right)$ $T(n) = \sum \Gamma'$, Γ' of form in lemma 2

(5) $T(m) T(n) f = T(mn)$

(6) $T(p) T(p^n) f = T(p^{n+1}) f + p^{2k-1} T(p^{n-1}) f$

pf (5) (6) follows from (3) (4)

Behavior at infinity

f a modular function is meromorphic at infinity

$$f(z) = \sum_{m \in \mathbb{Z}} c(m) q^m \quad q = e^{2\pi i z}$$

Prop: $T(n) f(z) = \sum_{m \in \mathbb{Z}} \gamma(m) q^m$

here $\gamma(m) = \sum_{\substack{a \mid (n,m) \\ a \ge 1}} a^{2k-1} c\left(\frac{mn}{a^2}\right)$

By def we have

$$T_n f(z) := n^{2k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} d^{-2k} \sum_{m \in \mathbb{Z}} c(m)\, e^{2\pi i m (az+b)/d}$$ (代入上面的式子得到 $\Gamma = \Sigma \Gamma'$ and $\Gamma' = (a\ m + b\ \mathrm{mod}\ d)$)

$$\sum_{0 \le b < d} e^{2\pi i b m/d} = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$$ Thus we put $m/d = m'$

$$T_n f(z) := n^{2k-1} \sum_{\substack{ad=n \\ a \ge 1}} \sum_{m' \in \mathbb{Z}} d^{-2k+1} c(m' d)\, q^{a m'}$$

对于 q 的次方为 $a m' = u$

$$T_n f(z) := \sum_{u \in \mathbb{Z}} q^u \underbrace{\sum_{\substack{a \mid (n,u) \\ a \ge 1}} \left(\frac{n}{d}\right)^{2k-1} c\left(\frac{ud}{a}\right)}_{\gamma(u)}$$ Since $ad = n$

f meromphic at infinity $c(m) = 0$ if $m \le -N$, $c\left(\frac{ud}{a}\right)$ zero for $u \le -nN$. $T_n f$ is also meromophic, it's a modular function

Cor1 $\gamma(0) = \sigma_{2k-1}(n)\, c(0)$ $\quad \gamma(1) = c(n)$

Cor2. $n = p$, p prime

$\gamma(m) = c(pm)$ if $m \not\equiv 0 \pmod p$

$\gamma(m) = c(pm) + p^{2k-1} c(m/p)$ if $m \equiv 0\ (p)$

Cor3. If f a modular form so is $T(n) f$

$T(n)$ acts on M_k and M_k^0 ?

Eigenfunctions of $T(n)$

$f(z) = \sum c(n) q^n$ weight $2k$

$T(n) f = \lambda(n) f$ for all $n \ge 1$ f is an eigenfunction

Then $\gamma(1) = c(n) = \lambda(n)$

Consider the coefficients of q

by $T_n f = \lambda_n f$

$\gamma(n) = c(n) = \lambda_n c(1)$

Thm 7. a) The coefficient $c(1)$ of q in f is $\neq 0$

b) If f is normalized by $c(1) = 1$ then $c(n) = \lambda_n$ for all $n \geq 1$

Corollary 1.

modular forms of weight $2k$, $k > 0$, which are eigen functions with the same eigenvalues $\lambda(n)$ and are normalized coincide

$T_n f = \lambda_n f$, $c(n) = \lambda_n$

Cor 2.

$c(m)\, c(n) = c(mn)$ $\quad (m, n) = 1$ $\quad$ by T_n 性质

$c(p)\, c(p^n) = c(p^{n+1}) + p^{2k-1} c(p^{n-1})$

Pf: $T(m) T(n) f = T(mn) f$

$\lambda(m) \lambda(n) f = \lambda(mn) f$

$T(p) T(p^n) f = T(p^{n+1}) f + p^{2k-1} T(p^{n-1}) f$

λ - -

Let ~~$\phi_f(s) = \sum_{n=1}^{\infty}$~~ $\phi_f(s) = \sum_{n=1}^{\infty} c(n)/n^s$ be the D-series and defined by $c(n)$ this series converges absolutely for $\Re(s) > 2k$

Cor: $\phi_f(s) = \prod_{p \in P} \frac{1}{1 - c(p) p^{-s} + p^{2k-1-2s}}$

By V2.2 $\quad \phi_f(s) = \prod_{p \text{ prime}} \sum_{n=0}^{\infty} c(p^n) p^{-ns}$

Let $p^{-s} = T$

Need to prove $\sum_{n=0}^{\infty} c(p^n) T^n = \frac{1}{\phi_{f,p}(T)}$ $\quad \phi_{f,p}(T) = 1 - c(p) T + p^{2k-1} T^2$

form the series $\psi(T) = \left(\sum_{n=0}^{\infty} c(p^n) T^n\right)\left(1 - c(p) T + p^{2k-1} T^2\right)$

The coefficient of T in ψ is $c(p) - c(p) = 0$

for T^{n+1} it's $c(p^{n+1}) - c(p) c(p^n) + p^{2k-1} c(p^{n-1}) = 0$ by previous lemma

$\psi = c(1) = 1$ $\Rightarrow$ proves the Cor

Remark. 1) Conversely the formula implies Cor 2.

2) Hecke proved that Φ_f extends to a meromorphic function on the $\mathbb{C}$ plane

$$X_f(s) = (2\pi)^{-s}\,\Gamma(s)\Phi_f(s)$$

satisfies the functional equation

$$X_f(s) = (-1)^k X_f(2k-s)$$

The pf use Mellin's formula $\quad F(s) = \int_0^\infty x^{s-1} f(x)\,dx$

$$X_f(s) = \int_0^\infty (f(iy) - f(\infty))\, y^s \frac{dy}{y} \quad \text{combined with } f(-\tfrac{1}{z}) = z^{2k} f(z)$$

Also proved a converse

every Dirichlet series ϕ which satisfies a functional equation of this type. and some regularity and growth hypothesis, comes from a modular form of weight $2k$

f is a normalized eigenfunction of $T(n)$ iff ϕ is an Eulerian product of type $\boxed{\ }$ in Cor 3

Examples of eigenfunction

Eisenstein series G_k with eigenvalue $\sigma_{2k-1}(n)$

$$(-1)^k \frac{B_k}{4k} E_k = (-1)^k \frac{B_k}{4k} + \sum_{k=1}^{\infty} \sigma_{2k-1}(n)\, q^n$$

$G_k = 2\zeta(2k)\, E_k$

$E_k = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n$

$\gamma_k = (-1)^k \frac{4k}{B_k}$

The corresponding Dirichlet series is $\zeta(s)\,\zeta(s-2k+1)$

suffices to do this for $T(p)$, p prime

G_k is an eigenfunction

$$G_k(\Gamma) = \sum_{\gamma\in\Gamma}' \frac{1}{\gamma^{2k}}$$

$$T(p)\, G_k(\Gamma) = \sum_{(\Gamma:\Gamma')=p} \sum_{\gamma\in\Gamma'}' \frac{1}{\gamma^{2k}}$$

$\gamma \in p\Gamma \Rightarrow \gamma$ belongs to $p+1$ sublattices of Γ of index p

$\gamma \in \Gamma - p\Gamma$ then γ belongs to only one sublattice of index p

No

DATE / /

Thus $T(p)G_k(T) = G_k(T) + p\sum_{\gamma(pT)} \frac{1}{\gamma^{2k}} = G_k(T) + pG_k(pT)$

$= (1+p^{1-2k})G_k(T)$

$\Rightarrow$ G_k with eigen value $(1+p^{1-2k})$

viewed as a modular form G_k is of eigenvalue $p^{2k-1}(1+p^{1-2k})$?

$= \sigma_{2k-1}(p)$

the eigenvalues of $T_{(n)}$ are $\sigma_{2k-1}(n)$

$\sum \sigma_{2k-1}(n)/n^s = \sum_{a,d\geq 1} a^{2k-1}/d^s a^s$

$= \left(\sum_{d\geq 1}\frac{1}{d^s}\right)\left(\sum_{a\geq 1}\frac{1}{a^{s+1-2k}}\right)$

$= \zeta(s)\zeta(s-2k+1)$

The Δ function

Δ function is an eigenfunction of $T_{(n)}$

with normalized eigen function is

$(2\pi)^{-12}\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n$

Since the space of cusp form of weight 12 is of dimension 1

$\Rightarrow$ it must be stable

Cor $\tau(n)\tau(m) = \tau(nm)$

$\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$

this is immediately from previous results

$\dim(M_k^0) = 1$ happens for $k = 6, 8, 10, 11, 13$. with basis

Δ & ΔG_2, ΔG_3, ΔG_4, ΔG_5, ΔG_7

Complements

1. Peterson inner product

Let f and g be the cusp forms of weight $2k > 0$.

differential $f\overline{g(z)} \cdot y^{2k} dx dy$ is invariant under the action of $SL_2(\mathbb{R})$

Here $z = x + iy$

Pf $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$f(\gamma z) = (cz+d)^{2k} f(z)$ $\quad \overline{g(\gamma z)} = \overline{(cz+d)^{2k}}\, \overline{g(z)}$

$\cancel{y^{2k-2} \xrightarrow{\gamma} |cz+d|^{4-4k}\, y^{2k-2}}$

$\cancel{\gamma^*(dx\,dy) = |cz+d|^4}$

$\cancel{\gamma z = \frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}}$

$\cancel{\Rightarrow y^{2k-2}}$

$\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{\operatorname{Im}(adz+bc\bar{z})}{|cz+d|^2}$

$\operatorname{Im}(adz+bc\bar{z}) = (ad-bc)\operatorname{Im}(z) = \operatorname{Im}(z)$

$\Rightarrow y \xrightarrow{\gamma} \frac{y}{|cz+d|^2}$

$z \to \gamma z$ multiplies area by $\left|\frac{d\gamma}{dz}\right|^2$

$\gamma^*(dx\,dy) = \frac{dx\,dy}{|cz+d|^4}$ $\quad \frac{d\gamma}{dz} = \frac{1}{(cz+d)^2}$ $\quad \gamma(z) \to \frac{az+b}{cz+d}$

$\Rightarrow f(\gamma z)\cdot\overline{g(\gamma z)}\,(\text{Im})^{2k-2} dx\,dy) = |cz+d|^{4k} \cdot \frac{1}{|cz+d|^{4k-4}} f(z)\overline{g(z)}\,dx\,dy \cdot \frac{1}{|cz+d|^4}$

Then $\underbrace{\iint_D f(z)\overline{g(z)}\, y^{2k-2}\, dx\,dy}_{\langle f, g\rangle}$ is invariant, when f, g cusp forms it converges

petersson inner product

$f(z) = a_1 q^1 + \dots + a_n q^n$ $\quad f(x+iy) = O(e^{-2\pi y})$ as $y \to +\infty$

Thus $\langle f, g\rangle = \int_0^1 \int_{\sqrt{1-x^2}}^{\infty} f(x+iy)\,\overline{g(x+iy)}\, y^{k-2}\,dx\,dy = \int_0^1\int_{\sqrt{1-x^2}}^{\infty} O(e^{-2\pi y})\, O(1)\, y^{k-2}\,dx\,dy < \infty$

Similarly the other cusps

$\cancel{\langle f, g}$

$\langle f, g\rangle = \overline{\langle g, f\rangle}$ $\quad \langle f, f\rangle > 0$ for all $f \neq 0$

$\langle T_n f, g\rangle = \langle f, T_n g\rangle$ $\quad T_n$ is a Hermitian operator

Suffices to prove for $T(p)$

not proved today for M.F.

通过 $f|_k \alpha = (\det\alpha)^k (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ $\quad \alpha \in GL_2(\mathbb{R})^+$

$\Rightarrow \langle f|_k\alpha, g|_k\alpha\rangle = \langle f, g\rangle$

NO

DATE / /

orthogonal basis of $M_k^?$ made of eigenvectors of $T(n)$ and the eigenvectors of $T(n)$ are real numbers

$M_k(\mathbb{Z})$ be the set of modular forms

$f = \sum_{n=0}^{\infty} c(n) q^n$ of weight $2k$ whose coefficients $c(n)$ are integers

there is a $\mathbb{Z}$-basis of $M_k(\mathbb{Z})$ which is a $\mathbb{C}$-basis of M_k

E_2 has degree 4 E_3 has degree 6

$f = \sum a_n q^n$ $f = a_0 E_2^a E_3^b + \Delta \cdot g$ $4a + 6b = 2k$ $g \in M_{k-12}$

the coefficients of the characteristic polynomials of $T(n)$ acting on M_k are integers in particular eigenvalues of T_n are algebraic integers

5.6.2 The Ramanujan-Petersson conjecture

$f = \sum c_n q^n$ $c(1) = 1$ a cusp form of weight $2k$, normalized of T_n

$\Phi_{f,p}(T) = 1 - c(p)T + p^{2k-1}T^2$

$\Phi_{f,p}(T) = (1 - \alpha_p T)(1 - \alpha_p' T)$

$\alpha + \alpha_p' = c(p)$, $\alpha \cdot \alpha_p' = p^{2k-1}$, α, α_p' are complex conjugate

equivalently $|\alpha_p| = |\alpha_p'| = p^{\frac{k-1}{2}}$

$|c(p)| \le 2p^{k-1/2}$ or $|c(n)| \le n^{k-1/2} \sigma_0(n)$

$k = 6$ we have $|\tau(p)| \le 2p^{11/2}$