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Examples: $g_2 = 60G_2$, $g_3 = 140G_3$

$g_2(\infty) = 120\zeta(4) = \frac{4}{3}\pi^4$, $g_3(\infty) = 280\zeta(6) = \frac{8}{27}\pi^6$

define $\Delta = g_2^3 - 27g_3^2$ then $\Delta(\infty) = 0$

Δ is a cusp form of weight 12

elliptic curves

elliptic functions: nonconstant, doubly periodic, meromorphic functions

$f(z+w_1) = f(z+w_2) = f(z)$ for some $w_1, w_2 \in \mathbb{C}$, $w_1/w_2 \notin \mathbb{R}$

f can be defined in $\mathbb{C}/\langle w_1, w_2\rangle$

construction of elliptic functions:

$\sum_{w\in\Lambda} \frac{1}{(z+w)^2}$ does not converge absolutely

define Weierstrass function $p(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*}\left(\frac{1}{(z+w)^2} - \frac{1}{w^2}\right)$

$\left|\frac{1}{(z+w)^2} - \frac{1}{w^2}\right| = \frac{|z(z+2w)|}{|z+w|^2|w|^2} \le C\frac{1}{|w|^3}$ when $|w|$ is large enough

so $\sum_{w\in\Lambda^*}\left(\frac{1}{(z+w)^2} - \frac{1}{w^2}\right)$ converges absolutely $(z \notin \Lambda^*)$

p is doubly periodic: $p'(z) = -\frac{2}{z^3} + \sum_{w\in\Lambda^*}\frac{-2}{(z+w)^3} = -2\sum_{w\in\Lambda}\frac{1}{(z+w)^3}$

$p'(z) = p'(z+w_1) = p'(z+w_2) \Rightarrow p(z) = p(z+w_1) + c_1$, $p(z) = p(z+w_2) + c_2$

p' is odd $\Rightarrow p$ is even $\Rightarrow p(-\frac{w_1}{2}) = p(\frac{w_1}{2})$, $p(-\frac{w_2}{2}) = p(\frac{w_2}{2})$

$\Rightarrow c_1 = c_2 = 0$

expansion for p near 0: $p(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{k+1}(\Lambda)z^{2k}$

pf: $\frac{1}{(z-w)^2} = \frac{1}{w^2}\frac{1}{(\frac{z}{w}-1)^2} = \frac{1}{w^2}\sum_{l=0}^{\infty}(l+1)\left(\frac{z}{w}\right)^l = \frac{1}{w^2} + \frac{1}{w^2}\sum_{l=1}^{\infty}(l+1)\left(\frac{z}{w}\right)^l$

$p(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*}\frac{1}{(z-w)^2} - \frac{1}{w^2}$

$= \frac{1}{z^2} + \sum_{w\in\Lambda^*}\frac{1}{w^2}\sum_{l=1}^{\infty}(l+1)\frac{1}{w^l}z^l$

$= \frac{1}{z^2} + \sum_{l=1}^{\infty}\left(\sum_{w\in\Lambda^*}\frac{l+1}{w^{l+2}}\right)z^l$ $\left(\sum_{w\in\Lambda^*}\frac{1}{w^{l+2}} = 0 \text{ if } l \text{ is odd}\right)$

$= \frac{1}{z^2} + \sum_{k=1}^{\infty}\left(\sum_{w\in\Lambda^*}\frac{2k+1}{w^{2k+2}}\right)z^{2k}$

$= \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{k+1}z^{2k}$

$p'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty}2k(2k+1)G_{k+1}z^{2k-1}$

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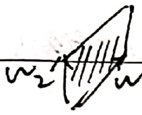
$$p(z) = \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots$$

$$p'(z) = -\frac{2}{z^3} + 6G_2 z + 20 G_3 z^3 + \cdots$$

$$(p'(z))^2 = \frac{4}{z^6} - \frac{24 G_2}{z^2} - 80 G_3 + \cdots$$

$$(p(z))^3 = \frac{1}{z^6} + \frac{9G_2}{z^2} + 15 G_3 + \cdots$$

then $h = (p')^2 - 4p^3 + 60G_2 p + 140 G_3$ is holomorphic, $h(0) = 0$



h is bounded in ▨, doubly periodic, thus is bounded in $\mathbb{C}$

so $h \equiv 0$, $(p')^2 = 4p^3 - 60G_2 p - 140 G_3$

$(p(z), p'(z))$ is on the curve $y^2 = 4x^3 - g_2 x - g_3$

the discriminant of $4x^3 - g_2 x - g_3 = 4(z-e_1)(z-e_2)(z-e_3)$

is $((e_1-e_2)(e_2-e_3)(e_3-e_1))^2 = \frac{1}{16}(g_2^3 - 27g_3^2) = \frac{\Delta}{16}$

fact: e_1, e_2, e_3 are different from each other, $\Delta \neq 0$

$$\Phi : \mathbb{C}/\Lambda \to E = \{(x,y) \mid y^2 = 4x^3 - g_2 x - g_3\} \cup \{\infty\}$$

$$[z] \mapsto \begin{cases} (p(z), p'(z)), & \text{if } z \text{ is not lattice point} \\ \infty, & \text{if } z \text{ is lattice point} \end{cases}$$

Φ is a holomorphic isomorphism between Riemann surfaces

§3. Spaces of modular forms

3.1 zeros and poles of a modular function

def: Let f be a mero. function on H, not identically zero,

$p \in H$, define the order of f at p to be the integer n s.t.

$\frac{f(z)}{(z-p)^n}$ is holomorphic, non-zero at p, denoted by $V_p(f)$

($V_p(f)$ = $\pm$ multiplicity of zero/pole at p)

If f is a modular function, then $V_p(f) = V_{g(p)}(f)$ for $g \in G$

$V_p(f)$ only depends on the image of p in H/G

$V_\infty(f)$ is the order for $q=0$ in $\tilde{f}(q)$

Thm 3: f is a modular function of weight $2k$, not identically 0

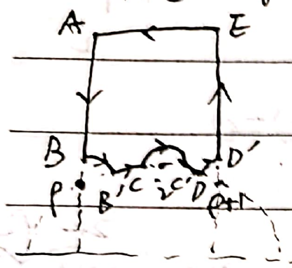
then $$V_\infty(f) + \frac{1}{2}V_i(f) + \frac{1}{3}V_\rho(f) + \sum_{\substack{p \in H/G \\ p \neq [i], [\rho]}} V_p(f) = \frac{k}{6}$$

pf: (the sum makes sense since f only has finite number of zeros, poles in H/G. $\tilde{f}$ has no poles for $0<|q|<r$

$\Rightarrow f$ has no poles for $\mathrm{Im}\, z > r'$

f has finite poles/zeros in $D \cap \{\mathrm{Im}\, z \le r'\}$)

Apply argument principle : $\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{df}{f} = \sum_{p \in \mathrm{Int}(\Gamma_\varepsilon)} v_p(f)$



first assume no poles/zeros on ∂D

$$\frac{1}{2\pi i}\int_{\leftarrow} \frac{df}{f} = \frac{1}{2\pi i}\int_{\circlearrowleft} \frac{d\tilde{f}}{\tilde{f}} = -v_\infty(f)$$

$$\frac{1}{2\pi i}\left(\int_{\frown} + \int_{\frown}\right)\frac{df}{f} = \frac{1}{\pi i}\int_{\frown}\frac{df}{f} \to \frac{-1}{3} v_\rho(f)$$

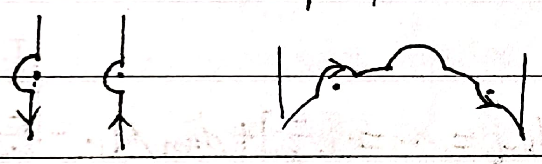
$$\frac{1}{2\pi i}\int_{\frown} \frac{df}{f} \to \frac{-1}{2} v_i(f)$$

$$\int_{\downarrow} \frac{df}{f} + \int_{\uparrow} \frac{df}{f} = \int_{\downarrow} \frac{df(z)}{f(z)} - \int_{\downarrow} \frac{df(Tz)}{f(Tz)} = 0$$

$$\frac{1}{2\pi i}\left(\int_{\frown} \frac{df}{f} + \int_{\frown} \frac{df}{f}\right) = \frac{1}{2\pi i}\int_{\frown} \frac{df(z)}{f(z)} - \int_{\frown} \frac{df(Sz)}{f(Sz)} = \frac{1}{2\pi i}\int_{\frown} -2k\frac{dz}{z} \to \frac{k}{6}$$

$\varepsilon \to 0$, $\quad -v_\infty(f) - \frac{1}{3}v_\rho(f) - \frac{1}{2}v_i(f) + \frac{k}{6} = \sum^*_{p \in H/G} v_p(f)$

if there are poles/zeros on ∂D, change Γ_ε to avoid them



3.2 Vector space of modular forms

M_k: $\mathbb{C}$ vector space of modular forms of weight $2k$.

M_k^0: - - - cusp forms - - -

$\varphi: M_k \to \mathbb{R}$, $f \mapsto f(\infty)$ is a homomorphism.

$\mathrm{Ker}\,\varphi = M_k^0$, $\quad M_k/M_k^0 \simeq \mathrm{Im}\,\varphi < \mathbb{R} \Rightarrow \dim(M_k/M_k^0) \le 1$

when $k \geq 2$, $G_k \in M_k \setminus M_k^0$, so $\dim M_k/M_k^0 = 1$

$$M_k = M_k^0 \oplus \mathbb{C}G_k$$

Thm 4 (1) $M_k = 0$ for $k<0$ and $k=1$

(2) for $k=0,2,3,4,5$, M_k is 1-dim vector space with basis $1, G_2, G_3, G_4, G_5$, and $M_k^0 = 0$

(3) $\psi: M_{k-6} \to M_k^0$, $f \mapsto \Delta f$ is isomorphism

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pf: (1) $V_\infty(f)+\frac{1}{2}V_i(f)+\frac{1}{3}V_\rho(f)+\sum^* V_p(f)=\frac{k}{6}$

LHS ≥ 0, so no solution if $k<0$.

$n+\frac{1}{2}n'+\frac{1}{3}n''=\frac{1}{6}$ also has no non-negative solution

(2) $V_\infty(f)$ must be 0 for $k=0,2,3,4,5$ since $\frac{k}{6}<1$

(3) apply thm 3 for G_2 shows G_2 only has a single zero at ρ

G_3 i

so $\Delta=g_2^3-27g_3^2$ is nonvanishing at i,ρ, $\Delta(\infty)=0$

so $V_\infty(\Delta)=1$, $V_p(\Delta)=0$ for $p\neq\infty$

for $\forall g\in M_k^0$, let $f=\frac{g}{\Delta}$, then f is a modular function of weight $2k-12$

$V_p(f)=V_p(g)-V_p(\Delta)=\begin{cases}V_p(g), & p\neq\infty\\ V_p(g)-1, & p=\infty\end{cases}$

so $V_p(f)\geq 0$ $\Rightarrow$ g is a modular form

Cor: $\dim M_k=\begin{cases}[\frac{k}{6}], & k\equiv 1 \pmod 6, & k\geq 0\\ [\frac{k}{6}]+1, & k\not\equiv 1\pmod 6, & k\geq 0\end{cases}$

pf: let $k=6[\frac{k}{6}]+r$, $0\leq r\leq 5$

then $\dim M_k=1+\dim M_k^0=1+\dim M_{k-6}=\cdots=[\frac{k}{6}]+\dim M_r=\begin{cases}[\frac{k}{6}]+1, & r\neq 1\\ [\frac{k}{6}], & r=1\end{cases}$

Cor: $\{G_2^\alpha G_3^\beta \mid 2\alpha+3\beta=k,\ \alpha,\beta\geq 0\}$ is a basis for M_k

pf: $k\leq 3$ is easy to check. when $k\geq 4$, do induction

1) for $\forall f\in M_k$, take $\alpha,\beta\geq 0$ s.t. $2\alpha+3\beta=k$

$\exists\lambda\in\mathbb{C}$ s.t. $f-\lambda G_2^\alpha G_3^\beta\in M_k^0$

so $f=\lambda G_2^\alpha G_3^\beta+\Delta h$, $h\in M_{k-6}$ is a linear combination for $\{G_2^{\alpha'}G_3^{\beta'} \mid 2\alpha'+3\beta'=k-6,\ \alpha,\beta\geq 0\}$

2) If $\{G_2^\alpha G_3^\beta \mid 2\alpha+3\beta=k,\alpha,\beta\geq 0\}$ are linearly dependent

then $\exists c_i\in\mathbb{C}$, $\sum_{i=1}^n c_i G_2^\alpha G_3^\beta=0 \Rightarrow (\sum_{i=1}^n c_i G_2^\alpha G_3^\beta)^6=0$ $(*)$

$G_2^{\alpha_1}G_3^{\beta_1}\cdots G_2^{\alpha_6}G_3^{\beta_6}=G_2^{\alpha_1+\cdots+\alpha_6}G_3^{\beta_1+\cdots+\beta_6}=(G_2^3)^{\alpha'}(G_3^2)^{\beta'}$, $\alpha'+\beta'=k$, $\alpha',\beta'\in\mathbb{Z}$

since $2(\alpha_1+\cdots+\alpha_6)+3(\beta_1+\cdots+\beta_6)=6k \Rightarrow 3\mid\alpha_1+\cdots+\alpha_6,\ 2\mid\beta_1+\cdots+\beta_6$

so $(*)\Rightarrow \sum_{j=1}^m c_j'\left(\frac{G_2^3}{G_3^2}\right)^j=0 \Rightarrow \frac{G_2^3}{G_3^2}$ is constant

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3.3 modular invariant

$j = 1728\, g_2^3/\Delta$ $\qquad 1728 = 12^3$

Prop 5 (1) j is a modular function of weight 0

(2) j is holomorphic in H, has a simple pole at ∞

(3) $j: \widehat{H/G} \to \hat{\mathbb{C}}$ is bijection

pf: (1) obvious

(2) Δ has no zero in H, has a single zero at ∞

(3) $\forall \lambda \in \mathbb{C}$, $h = g_2^3 - \lambda\Delta \in M_6$

$$n + \frac{n'}{2} + \frac{n''}{3} = \frac{6}{6} \Rightarrow n = 1$$

$\Rightarrow V_\infty(h) = 1$ or $V_p(h) = 1$ for some $p \in H/G$, $p \neq i, \rho$

$\Rightarrow h$ has only one zero in H/G $\Rightarrow j: H/G \to \mathbb{C}$ is bijection

$j(\infty) = \infty$ $\quad \Rightarrow j: \widehat{H/G} \to \hat{\mathbb{C}}$ bijection

Prop 6 Let f be a meromorphic function on H, the following are equivalent:

(1) f is a modular function of weight 0

(2) f is a quotient of two modular forms of same weight

(3) f is a rational function of j

pf: (3)$\Rightarrow$(2)$\Rightarrow$(1) is obvious

(1)$\Rightarrow$(3): we can multiply polynomials of j to make f holomorphic

(if f has a pole of order k at z_0, then $(j(z) - j(z_0))^k f$ is holo. at z_0)

$\exists n \geq 0$ s.t. $\Delta^n f$ is holo at ∞, $\Delta^n f \in M_{6n}$

so $\Delta^n f$ is a linear combination of $G_2^\alpha G_3^\beta$, $2\alpha + 3\beta = 6n$

$\Rightarrow 3|\alpha, 2|\beta$ $\quad$ we only need to show $\frac{G_2^3}{\Delta}, \frac{G_3^2}{\Delta}$ is rational

function of j. $\quad j = \square \frac{g_2^3}{\Delta} = \square \frac{G_2^3}{\Delta}$, $\frac{G_3^2}{\Delta} = \square \frac{G_2^3 - \square\Delta}{\Delta} = \square j - \square$

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Why is j a modular invariant ?

claim: $\mathbb{C}/\Lambda \simeq \mathbb{C}/\tilde{\Lambda} \Leftrightarrow j(\Lambda)=j(\tilde{\Lambda})$

$\Leftarrow$: $j(\Lambda)=j(\tilde{\Lambda}) \Rightarrow j(E)=j(\tilde{E})$, where $E=\Phi(\mathbb{C}/\Lambda)$, $\tilde{E}=\Phi(\mathbb{C}/\tilde{\Lambda})$

easy to prove $\Rightarrow E\simeq\tilde{E} \Rightarrow \mathbb{C}/\Lambda \simeq \mathbb{C}/\tilde{\Lambda}$

$\Rightarrow$: $\mathbb{C}/\Lambda \simeq \mathbb{C}/\tilde{\Lambda} \Rightarrow \mathbb{C}/\langle 1,\tau\rangle \simeq \mathbb{C}/\langle 1,\tilde{\tau}\rangle$, $\tau=\frac{\omega_2}{\omega_1}$, $\tilde{\tau}=\frac{\tilde{\omega}_2}{\tilde{\omega}_1}$

$\exists \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$, $\tilde{\tau}=\frac{a\tau+b}{c\tau+d}$

$$\tilde{G}_2 = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tilde{\tau}+n)^4} = \sum{}' \frac{(c\tau+d)^4}{(m(a\tau+b)+n(c\tau+d))^4}$$

$$= (c\tau+d)^4 \sum{}' \cancel{\frac{1}{((ma+nc)\tau+(bm+nd))^4}}$$

$$\begin{pmatrix} ma+nc \\ bm+nd\end{pmatrix} = \begin{pmatrix} a & c \\ b & d\end{pmatrix}\begin{pmatrix} m \\ n\end{pmatrix}$$

$$= (c\tau+d)^4\, G_2$$

$$\tilde{G}_3 = (c\tau+d)^6 G_3 \Rightarrow j(\tau)=j(\tilde{\tau})$$

$$\Rightarrow j(\Lambda)=j(\tilde{\Lambda})$$

for each $E=\{(x,y)\mid y^2=4x^3-g_2x-g_3\}\cup\{\infty\}$

we can find $\tau\in H$ s.t. $j(\tau)=j(E)$ (since $j: H\to\mathbb{C}$ surjective)

then $\mathbb{C}/\langle 1,\tau\rangle \simeq E$

§4 Expansion at infinity

4.1 Bernoulli number B_k

$$\frac{x}{e^x-1} = 1-\frac{x}{2}+\sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{x^{2k}}{(2k)!}$$

Prop 7 $\zeta(2k)=\frac{2^{2k-1}}{(2k)!}B_k\pi^{2k}$

pf: $x=2iz \Rightarrow z\cot z = 1-\sum_{k=1}^{\infty}B_k\frac{2^{2k}z^{2k}}{(2k)!}$

on the other hand, $z\cot z = 1+2\sum_{n=1}^{\infty}\frac{z^2}{z^2-n^2\pi^2}$

$$=1-2\sum_{n=1}^{\infty}\frac{z^2}{n^2\pi^2}\frac{1}{1-\left(\frac{z}{n\pi}\right)^2}$$

$$=1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{z}{n\pi}\right)^{2k} = 1-2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(n\pi)^{2k}}\right)z^{2k}$$

$$\frac{1}{\pi^{2k}}\zeta(2k) = B_k\frac{2^{2k}}{(2k)!}$$

$\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{2\cdot3^2\cdot5}$, $\zeta(6)=\frac{\pi^6}{3^3\cdot5\cdot7}$

4.2 expansions of G_k

$$\pi\cot\pi z = \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z+m}+\frac{1}{z-m}\right)$$

$$= \pi\frac{\cos\pi z}{\sin\pi z} = i\pi\frac{q+1}{q-1} = i\pi - \frac{2\pi i}{q} = i\pi - 2i\pi\sum_{n=0}^{\infty}q^n$$

take differentiation : $\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^2} = (-2\pi i)^2\sum_{n=1}^{\infty}nq^{n-1}q$

$$\sum_{m\in\mathbb{Z}}\frac{-2}{(z+m)^3} = -(2\pi i)^3\sum_{n=1}^{\infty}n^2q^n$$

$$\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^3} = \frac{1}{2}(-2\pi i)^3\sum_{n=1}^{\infty}n^2q^n$$

...

$$\sum_{m\in\mathbb{Z}}\frac{1}{(m+z)^k} = \frac{1}{(k-1)!}(-2\pi i)^k\sum_{n=1}^{\infty}n^{k-1}q^n$$

Prop 8 $\quad G_k(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$

where $\sigma_k(n) = \sum_{d|n}d^k$

$$G_k(z) = \sum{}'\frac{1}{(nz+m)^{2k}}$$

$$= \sum_{m=1}^{\infty}\frac{1}{m^{2k}} + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(nz+m)^{2k}}$$

$$= \zeta(2k) + 2\sum_{n=1}^{\infty}\frac{1}{(2k-1)!}(-2\pi i)^{2k}\sum_{m=1}^{\infty}m^{2k-1}q^{mn}$$

$$= \zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\left(\sum_{d|n}d^{2k-1}\right)q^n$$

$$= \zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$$

Cor: $G_k = 2\zeta(2k)E_k, \quad E_k = 1+(-1)^k\frac{4k}{B_k}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$

$\left(2\frac{(2\pi i)^{2k}}{(2k-1)!} = 2\zeta(2k)(-1)^k\frac{4k}{B_k}\right)$

$E_2 = 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \quad g_2 = (2\pi)^4\frac{1}{2^2\cdot 3}E_2$

$E_3 = 1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \quad g_3 = (2\pi)^6\frac{1}{2^3\cdot 3^3}E_3$

$E_4 = 1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n$

remark: $\dim M_4 = \dim M_5 = 1 \Rightarrow E_2^2 = E_4, \; E_2E_3 = E_5$

$$\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)^2 = 1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n$$

$$\Rightarrow 240\sigma_3(n)\cdot 2 + \sum_{k=1}^{n-1}240^2\sigma_3(k)\sigma_3(n-k) = 480\sigma_7(n)$$

$$\Rightarrow 120\sum_{k=1}^{n-1}\sigma_3(k)\sigma_3(n-k) = \sigma_7(n)-\sigma_3(n)$$

similarly, $11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{m=1}^{n-1}\sigma_3(m)\sigma_5(n-m)$

$\Delta = g_2^3 - 27g_3^2 = (2\pi)^{12} \cdot 2^{-6}3^{-3}(E_2^3 - E_3^2)$

$= (2\pi)^{12} \frac{1}{12^3}[(1+240X)^3 - (1-504Y)^2]$

$= (2\pi)^{12} \frac{1}{12^3}[720X + 1008Y + 12^3(\cdots)]$

$= (2\pi)^{12} [\frac{5X}{12} + \frac{7}{12}Y + (\cdots)]$

need to show $\frac{5}{12}\sigma_3(n) + \frac{7}{12}\sigma_5(n) \in \mathbb{Z}$

$d^5 - d^3 = d^3(d-1)(d+1) \equiv 0 \pmod{12}$

4.3 estimate of coefficients of modular functions

$f(z) = \sum_{n=0}^{\infty} a_n q^n$ how about growth of $|a_n|$?

Prop 9: If $f = G_{2k}$, $\exists\, A, B > 0$ s.t. $An^{2k-1} \le |a_n| \le Bn^{2k-1}$

pf: $|a_n| = \left|2\frac{(2\pi i)^{2k}}{(2k-1)!}\sigma_{2k-1}(n)\right| = C\,\sigma_{2k-1}(n) \geq C \cdot n^{2k-1}$

$\sigma_{2k-1}(n)/n^{2k-1} = \sum_{d|n} \frac{d^{2k-1}}{n^{2k-1}} = \sum_{d'|n} \frac{1}{(d')^{2k-1}} < \zeta(2k-1) < +\infty$

Thm 5 (Hecke) If f is a cusp form of weight $2k$

then $|a_n| = O(n^k)$

pf: $a_0 = 0$ $\quad f = q(a_1 + \sum_{k=2}^{\infty} a_k q^{k-1})$

so $|f| = O(|q|) = O(e^{-2\pi \operatorname{Im} z})$ as $q \to 0$

Let $y = \operatorname{Im} z$, $g(z) = |f(z)|y^k$

then $|g(z)| \le Ce^{-2\pi y}y^k \to 0$ as $y \to \infty$

so g is bounded $|f|y^k \le M$ for some M

$|f(z)| \le My^{-k}$

fix y, $a_n = \frac{1}{2\pi i}\int_{C_y} f(z) q^{-n-1} dq = \frac{1}{2\pi i}\int_0^1 f(x+yi)\, q^{-n-1} \cdot 2\pi i q\, dz$

$|a_n| \le \sup|f(x+yi)| \cdot |q|^{-n} \le My^{-k}e^{2\pi n y}$

take $y = \frac{1}{n}$, then $|a_n| \le Me^{2\pi} n^k$

Cor: If f is not a cusp form, then $|a_n| \neq O(n^{2k-1})$

4.4 Expansion of Δ

Thm 6 (Jacobi) $\Delta = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$

pf: $\dim M_6^0 = \dim M_0 = 1$, coefficience of q is equal

so only need to prove $F(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is a modular form of weight 12

it suffices to show $F(-\frac{1}{z}) = z^{12}F(z)$

$G_1(z) = \sum_n \sum_m' \frac{1}{(m+nz)^2}$ $(m,n)\neq(0,0)$

then $G_1(-\frac{1}{z}) = z^2 G_1(z) - 2\pi i z$ $(*)$ (we'll prove this in the end)

$G_1(z) = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)q^n$

$\frac{dF}{F} = \frac{dq}{q}\left(1 - 24\sum_{n=1}^{\infty}\frac{n}{1-q^n}\right) = 2\pi i\left(1-24\sum_{n=1}^{\infty} n\sum_{m=1}^{\infty} q^{nm}\right)dz$

$= 2\pi i\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)dz = \frac{6i}{\pi}G_1(z)\,dz$

$\frac{dF(-\frac{1}{z})}{F(-\frac{1}{z})} = \frac{6i}{\pi}G_1(-\frac{1}{z})\,d(-\frac{1}{z}) = \frac{dF(z)}{F(z)} + 12\frac{dz}{z}$

$= \frac{d\, z^{12}F(z)}{z^{12}F(z)}$

$\Rightarrow F(-\frac{1}{z}) = z^{12}F(z) + c$

take $z=i$, then $F(i) = F(i) + c \Rightarrow c = 0$

so $\Delta = (2\pi)^{12}F(z)$

4.5 Ramanujan function

$\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$

properties: $\tau(n) = O(n^6)$

$\tau(nm) = \tau(n)\tau(m)$ if $(n,m)=1$

$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$

$\tau(n) \equiv n^2\sigma_7(n) \pmod{27}$, $\tau(n) \equiv n\sigma_3(n) \pmod 7$

$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, ...

open question: Is $\tau(n)\neq 0$ for all $n\geq 1$?

Date / /

$$G_1(z)=\sum_n \sum_m{}' \frac{1}{(m+nz)^2}, \quad G_1(-\tfrac{1}{z})=z^2G_1(z)-2\pi i z$$

pf: $G_1(-\frac{1}{z})=\sum_n\sum_m{}' \frac{1}{(m-\frac{n}{z})^2}=z^2\sum_n\sum_m{}'\frac{1}{(mz-n)^2}=z^2\sum_m\sum_n{}'\frac{1}{(m+nz)^2}$

define $a_{m,n}(z)=\frac{1}{m-1+nz}-\frac{1}{m+nz}=\frac{1}{(m-1+nz)(m+nz)}$

fix $n\neq 0$, $\sum_{m\in\mathbb{Z}} a_{m,n}(z)=\lim_{m\to\infty}\left(\frac{1}{nz-m}-\frac{1}{nz+m}\right)=0$

$$G_1(z)=2\zeta(2)+\sum_{n\neq 0}\sum_{m\in\mathbb{Z}}\frac{1}{(m+nz)^2}-a_{m,n}(z)$$

$$=2\zeta(2)+\sum_{n\neq 0}\sum_m \frac{-1}{(m+nz)^2(m-1+nz)}$$

$\frac{1}{(m+nz)^2(m-1+nz)}=O\left(\frac{1}{(|n|+|m|)^3}\right)$ so converges absolutely

so $G_1(z)=2\zeta(2)+\sum_{m\in\mathbb{Z}}\sum_{n\neq 0}\frac{1}{(m+nz)^2}-a_{m,n}(z)$

$$=z^{-2}G_1(-\tfrac{1}{z})-\sum_{m\in\mathbb{Z}}\sum_{n\neq 0}a_{m,n}(z)$$

$$A_N(z)=\sum_{m=-N+1}^{N}\sum_{n\neq 0}a_{m,n}(z)=\sum_{n\neq 0}\left(\frac{1}{-N+nz}-\frac{1}{N+nz}\right)$$

recall $\pi\cot\pi z-\frac{1}{z}=\sum_{k=1}^{\infty}\left(\frac{1}{z+k}+\frac{1}{z-k}\right)$

$$A_N(z)=2\sum_{n=1}^{\infty}\left(\frac{1}{-N+nz}-\frac{1}{N+nz}\right)=\frac{2}{z}\sum_{n=1}^{\infty}\left(\frac{1}{-\frac{N}{z}+n}+\frac{1}{-\frac{N}{z}-n}\right)$$

$$=\frac{2}{z}\left(\pi\cot(-\tfrac{N\pi}{z})+\frac{z}{N}\right)$$

$$\to\frac{2}{z}(-i+0)=\frac{-2i}{z}$$

so $G_1(z)=z^{-2}G_1(-\frac{1}{z})+\frac{2i}{z}$