1 The modular group

1.1 *Definitions*

Let $\mathbf{SL_2}(\mathbb{R})$ be the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with real coefficients, such that ad-bc = 1. We make $\mathbf{SL_2}(\mathbb{R})$ act on $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in the following way:

if g = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL_2}(\mathbb{R})$, and if $z \in \tilde{\mathbb{C}}$, we put $gz = \frac{az+b}{cz+d}$. Check that is a group action indeed.

One checks $Im(gz) = \frac{Im(z)}{|cz+d|^2}$ (1) This shows that H is stable under the action of $\mathbf{SL_2}(\mathbb{R})$. Note that the element $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathbf{SL_2}(\mathbb{R})$ acts trivially on H. We can consider that it is the group $\mathbf{PSL_2}(\mathbb{R}) = \mathbf{SL_2}(\mathbb{R})/\{\pm 1\}$ which operates, and this group acts faithfully.

Recall that $\mathbf{SL_2}(\mathbb{Z})$ is the subgroup of $\mathbf{SL_2}(\mathbb{R})$ and it is a discrete subgroup of $\mathbf{SL_2}(\mathbb{Z})$.

Definition 1. The group $G = \mathbf{SL_2}(\mathbb{Z})/\{\pm 1\}$ is called the modular group; it is the image of $\mathbf{SL_2}(\mathbb{Z})$ in $\mathbf{PSL_2}(\mathbb{R})$.

1.2 *Fundamental domain in the modular group G*

Recall that $\mathbf{Aut}(\mathbb{H}) \cong \mathbf{SL_2}(\mathbb{Z})$, where an automorphism of an open set is a holomorphic bijection from the open set to itself. The goal of this section is to determine the structure of G and how G acts on H.

Let S and T be the elements of G defined respectively by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which are generators of G. One has :

$$Sz = -1/z \qquad Tz = z+1 \qquad S^2 = 1 \qquad (ST)^3 = 1$$

On the other hand, let D be the subset of H formed of all points z such that $|z| \geq 1$ and $|Re(z)| \leq 1/2$. We will show that D is a fundamental domain for the action of G on $\mathbb{H}$, where fundamental domain means D contains precisely one point of every orbit of $\mathbb{H}$ under the action of G. More precisely:

Theorem 1. (1) For every $z \in \mathbb{H}$, there exists $g \in G$ such that $gz \in D$.

(2) Suppose that two distinct points z, z' of D are congruent modulo G. Then, $Re(z) = \pm\frac{1}{2}$ and $z = z' \pm 1$, or $|z| = 1$ and $z' = -\frac{1}{z}$.

(3) Let $z \in D$ and let $G_z = \{g | g \in G, gz = z\}$ the stabilizer of z in G. One has $G_z = 1$ except following three cases:

$z = i$, in which case G_z is the group of order 2 generated by S;

$z = \rho = e^(2\pi i/3)$, in which case G_z is the group of order 3 generated by ST;

$z = -\overline{\rho} = e^(\pi i/3)$, in which case G_z is the group of order 3 generated by TS;

Corollary. The canonical map $D \to H/G$ is surjective and its restriction to the interior of D is injective.

Theorem 2. The group G is generated by S and T.

proof of theorems 1 and 2. (1) Let G' be the subgroup of G generated by S and T. Consider $Im(gz) = \frac{Im(z)}{|cz+d|^2}$. Since c and d are integers, the number of pairs (c, d) such that $|cz+d| < r$ for some r is finite, so there exists $g \in G'$ such that $Im(gz)$ is maximum. Let $T^n gz$ has real part between $\pm 1/2$, then it belongs to D. It suffices to see that $|z'| \geq 1$, if not -1/z' would have an imaginary part strictly larger than $Im(z')$, Contrdiction.

(2) Also consider $Im(gz) = \frac{Im(z)}{|cz+d|^2}$. We may assume $Im(gz) \geq Im(z)$, the possible value of c belongs to $\{-1, 0, 1\}$.

It remains to prove $G' = G$. Let $z = gz_0$ for some $z_0 \in D^\circ, g \in G$, there exists $g' \in G'$ such that $gz \in D$, so $g'g = 1, G = G'$.□

Actually, $< S, T; S^2, (ST)^3 >$ is a presentation of G.

2 Modular functions

2.1 *Definitions*

Definition 2. Let k be an integer. We say a function f is weakly modular fo weight $2k$ if f is meromorphic on the half plane $\mathbb{H}$ and verifies the relation $f(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d})$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL_2}(\mathbb{Z})$ (2).

Let g be the image in G of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $\frac{d(gz)}{dz} = (cz+d)^{-2}$. The relation can be written:

$$\frac{f(gz)}{f(z)} = (\frac{d(gz)}{dz})^{-k} \text{ or } f(gz)d(gz)^k = f(z)dz^k. \qquad (3)$$

It means that the "differential form of weight k" $f(z)dz^k$ is invariant under G.

Proposition 1. Let f be meromorphic on $\mathbb{H}$. The function f is a weakly modular function of weight 2k if and only if it satisfies following two relations:

$$f(z+1) = f(z) \qquad (4)$$

$$f(-1/z) = z^{2k} f(z) \qquad (5)$$

proof. If f is weakly modular, f satisfies the two relations is obvious.Conversely, only need to prove (3) holds for $g = S$ or T, since G is generated by S and T.□

Suppose the relation (4) is verified. We can let $q = e^{2\pi i z}$, so $f(z) = \tilde{f}(q)$. We need to check if $\tilde{f}$ is well-defined, *i.e.* all possible value of $e^{2\pi i z} \to z$ is $z+n$, where $n \in \mathbb{Z}$. $\tilde{f}$ is meromorphic in the dsik $|q| < 1$ with the origin removed, since $Im(z) > 0$.

The equation $\lim_{z\to 0} \tilde{f}(z) = \lim_{Im(z)\to+\infty} f(z) = \lim_{Im(z+)\to+\infty} f(z+1)$ allows followings. If $\tilde{f}$ extends to meromorphic at the origin, we say that f is meromorphic at infinity. This means $\tilde{f}$ admits a Laurent expansion in a neighbourhood of the origin $\tilde{f}(q) = \sum_{-\infty}^{+\infty} a_n q^n$, where the a_n are zero for n small enough.

Definition 3. A weakly modular function is called modular if it is meromorphic at infinity. When f is holomorphic at infinity, we set $f(\infty) = \tilde{f}(0)$. This is the value of f at infinity.

Definition 4. A modular function is holomorphic everywhere (including infinity) is called a modular form; if such a function is zero at infinity, it is called a cusp form.

A modular form of weight 2k is thus given by a series

$$f(z) = \sum_0^\infty a_n q^n = \sum_0^\infty a_n e^{2\pi i n z} \qquad (6)$$

which converges for $|q| < 1$ (*i.e.* for $Im(z) > 0$), and which verifies (5). It is a cusp form if $a_0 = 0$

Examples

1) If f and f' are modular forms of weight 2k and 2k', the product ff' is modular form of weight $2k + 2k'$.

2) A cusp form of weight 12

$$q \prod_{n=1}^\infty (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + ...$$

2.2 *Lattice functions and modular functions*

Recall what is a lattice in a real vector space V of finite dimension. It is a subgroup Γ of V verifying one of the following equivalent conditions:

1) Γ is discrete and V/Γ is compact;

2) Γ is discrete and generates the $\mathbb{R}$-vector space V;

3) There exists an $\mathbb{R}$-basis $(e_1, ..., e_n)$ of V which is a $\mathbb{Z}$-basis of Γ (*i.e.* $\Gamma = \oplus_{i=1}^n \mathbb{Z} e_i$)

Let $\mathscr{R}$ be the set of lattices of $\mathbb{C}$ considered as an $\mathbb{R}$-vector space. Let M be the set of pairs(ω_1, ω_2) of elements of $\mathbb{C}^*$ such that $Im(\omega_1/\omega_2) > 0$; to such a pair we associate the lattice $\Gamma(\omega 1, \omega 2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with basis ω_1, ω_2. We thus obtain a map $M \to \mathscr{R}$ which is clearly *surjective.*

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL_2}(\mathbb{Z})$ and let $(\omega 1, \omega 2) \in$ **M**. We put $\omega_1' = a\omega_1 + b\omega_2, \omega_2' = c\omega_1 + d\omega_2$, which can be checked as a group action. It is clear that ω_1', ω_2' is the basis of $\Gamma(\omega_1, \omega_2)$, since $g \in \mathbf{SL_2}(\mathbb{Z})$. Moreover, if we set $z = \omega_1/\omega_2$ and $z' = \omega_1'/\omega_2'$, we have $z' = \frac{az+b}{ca+d} = gz$. This shows that $Im(z') > 0$, hence (ω_1', ω_2') belongs to M.

Proposition 2. For two elements of M to define the same lattice it is necessary and sufficient that they are congruent modulo $\mathrm{SL}_2(\mathbb{Z})$

proof. The condition is sufficient has been shown above. Next we prove it is necessary. Suppose that $(\omega_1', \omega_2') = g(\omega_1, \omega_2)$, we have $g \in \mathbf{M_2}(\mathbb{Z})$, and furthermore, $det(g) = \pm 1$, since the lattice are same. And if $det(g) < 0$, the sign of $Im(\omega_1'/\omega_2')$ would be opposite of $Im(\omega_1/\omega_2)$.□

Hence, $\mathscr{R} \longleftrightarrow M/\mathbf{SL_2}(\mathbb{Z})$.

Consider C^* acts on $\mathscr{R}$ and M by:

$$\Gamma \to \lambda\Gamma \qquad (\omega_1, \omega_2) \to (\lambda\omega_1, \lambda\omega_2), \qquad \lambda \in \mathbb{C}^*$$

which means the angle of (ω_1, ω_2) and the ratio of lengths are invariant. $M/\mathbb{C}^* \longleftarrow \mathbb{H}$ by $(\omega_1, \omega_2) \mapsto z = \omega_1/\omega_2$. So $\mathbf{SL_2}(\mathbb{Z})$ acts on M is transformed to G acts on H.

Proposition 3. The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ is a bijection of $\mathscr{R}/\mathbb{C}^*$ onto $\mathbb{H}/G$. (Thus, an element of $\mathbb{H}/G$ can be identified with alattice of $\mathbb{C}$ defined up to a homothety) *proof.* Choose a element of $\mathscr{R}/\mathbb{C}^*$ whose image in M has the form (ω_1, ω_2) and check. Conversely similar to above.□

Remark. Consider $(\omega_1, \omega_2), g(\omega_1, \omega_2), \lambda g(\omega_1, \omega_2)$ generates the same lattice precisely. And Consider their images under the map in Proposition3., which are in the same orbit by the group action G on $\mathbb{H}$.

Let us pass now to modular functions. Let F be a function on $\mathscr{R}$, with complex numbers, and let $k \in \mathbb{Z}$. We say that F is of weight 2k if $F(\lambda\Gamma) = \lambda^{-2k}F(\Gamma)$ (7) for all lattices Γ and all $\lambda \in \mathbb{C}^*$. If we use (ω_1, ω_2) denote $\Gamma(\omega_1, \omega_2)$, $F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k}F(\omega_1, \omega_2)$ (8). Moreover, $F(\omega_1, \omega_2)$ is invariant by the action of $\mathbf{SL_2}(\mathbb{Z})$ on M. There exists a function f on $\mathbb{H}$ such that $F(\omega_1, \omega_2) = {\omega_2}^{-2k} f(\omega_1/\omega_2)$ (9). We see that f satisfies the identity (2). Conversely, if f verifies (2), formula (9) associates to it a function F on $\mathscr{R}$ which is of weight 2k.

2.3 *Examples of modular functions; Eisenstein series*

Lemma 1. Let Γ be a lattice in $\mathbb{C}$. The series $\sum'_{\gamma\in\Gamma} 1/|\gamma|^\sigma$ is convergent for $\sigma > 2$.

Let k be an integer > 1. Put $G_k(\Gamma) = \sum'_{\gamma\in\Gamma} 1/\gamma^{2k}$. It is converges absolutely, and of weight 2k. It is called *Eisenstein series* of index k. $G_k(\omega_1, \omega_2) = \sum'_{m,n} 1/(m\omega_1 + n\omega_2)^{2k}$. By formula (9), we have $G_k(z) = \sum'_{m,n} 1/(mz+n)^{2k}$.

Proposition 4. Let k be an integer > 1. The Eisenstein SERIES $G_k(z)$ is a modular form of weight 2k. We have $G_k(\infty) = 2\zeta(2k)$. *proof.* $G_k(z)$ is a convergent series, so it is holomorphic on $\mathbb{H}$. We only need to prove it holomorphic at ∞ and find its value. Since it is converges absolutely, so we can exchange integral and limit. $\lim_{Im(z)\to+\infty} G_k(z) = \sum' 1/n^{2k} = 2\zeta(2k)$.□

Examples. The Eisenstein series of lowest weights are G_2 and G_3, which are of weight 4 and 6. Let $g_2 = 60G_2, g_3 = 140G_3$. We have $g_2(\infty) = 120\zeta(4)$ and $g_3(\infty) = 280\zeta(6)$. One find $g_2(\infty) = \frac{4}{3}\pi^4$ and $g_3(\infty) = \frac{8}{27}\pi^6$. If we put $\Delta = {g_2}^3 - 27g_3^2$, we have $\Delta(\infty) = 0$; that is to say, Δ is a cusp form of weight 12.