Zeta function of the hypersurface:

$$Z_f(u) = \exp\left(\sum_{s=1}^{\infty} \frac{N_s u^s}{s}\right)$$ (Hasse-Weil Zeta Function)

首先定义 $F = \mathbb{Z}/p\mathbb{Z}$, F_s = field containing F and has p^s elements.

$f(y) \in F[y_0, \cdots, y_n]$ homogeneous poly.

N_s is the number of points on the projective hyperface.

$\Updownarrow$

N_s is the number of zeros of f in $P^n(F_s)$.

As for $P^n(F)$, we first consider $A^{n+1}(F) \Rightarrow$ vector space $=(n+1, F)$.

And $P^n(F)$ has a bijection between lines in $A^{n+1}(F)$ that passes through origin.

$$Z_f(u) = \exp\left(\sum_{s=1}^{\infty} \frac{N_s \cdot u^s}{s}\right) \qquad \{u \in \mathbb{C} \mid |u| < q^{-n}\}.$$

① As for f, $\{[y_0, \cdots, y_n]\}$ only has $(p^s)^{n+1}$ elements.

We consider if zeta function is rational function.

$$Z_f(u) = \frac{P(u)}{Q(u)} = \exp\left(\sum_{s=1}^{\infty} \frac{N_s u^s}{s}\right)$$

We may assume $P(0) = Q(0) = 1$ $\quad Z_f(0) = 1 \Rightarrow P(0) = Q(0) \Rightarrow P(0) = Q(0) = 1$.

then zeta function can be factored as follows:

$$Z_f(u) = \frac{\prod_i (1 - \alpha_i u)}{\prod_j (1 - \beta_j u)}$$

Prop 1. The zeta function is rational iff there exist complex numbers α_i and β_j s.t. $N_s = \sum_j \beta_j^s - \sum_j \alpha_i^s$.

$$Z(u) = \frac{\prod_i (1 - \alpha_i u)}{\prod_j (1 - \beta_j u)}$$

$$\frac{Z'(u)}{Z(u)} = \frac{1}{\left(\prod_j (1-\beta_j u)\right)^2}\Big[\prod_i (1-\alpha_i u)\cdot\Big(\sum_i \frac{-\alpha_i}{1-\alpha_i u}\Big)\cdot \prod_j (1-\beta_j u)$$
$$- \prod_i (1-\alpha_i u)\cdot \prod_j (1-\beta_j u)\cdot\Big(\sum_j \frac{-\beta_j}{1-\beta_j u}\Big)\Big]$$
$$\times \frac{\prod_j (1-\beta_j u)}{\prod_i (1-\alpha_i u)}$$

$$= \sum_i \frac{-\alpha_i}{1-\alpha_i u} - \sum_j \frac{-\beta_j}{1-\beta_j u} \qquad ①$$

$$Z(u) = \exp\Big(\sum_{s=1}^{\infty} \frac{N_s\cdot u^s}{s}\Big) \qquad \frac{Z'(u)}{Z(u)} = \Big(\sum_{s=1}^{\infty}\frac{N_s\cdot u^s}{s}\Big)' = \sum_{s=1}^{\infty}\frac{N_s\cdot s\cdot u^{s-1}}{s} = \sum_{s=1}^{\infty} N_s\cdot u^{s-1} \qquad ②$$

对 ①, $\frac{Z'(u)}{Z(u)} = \sum_i \frac{-\alpha_i}{1-\alpha_i u} - \sum_j \frac{-\beta_j}{1-\beta_j u}$

$$u\frac{Z'(u)}{Z(u)} = \sum_i \Big(-\alpha_i u \sum_{k=0}^{\infty} (\alpha_i u)^k\Big) - \sum_j \Big(-\beta_j u \sum_{k=0}^{\infty} (\beta_j u)^k\Big)$$

$$= \sum_j \Big(\sum_{k=1}^{\infty} (\beta_j u)^k\Big) - \sum_i \Big(\sum_{k=1}^{\infty} (\alpha_i u)^k\Big)$$

$$= \sum_{k=1}^{\infty} \Big(\sum_j \beta_j^k - \sum_i \alpha_i^k\Big) u^k$$

Then $\Rightarrow$ $N_k = \sum_j \beta_j^k - \sum_i \alpha_i^k$

"$\Rightarrow$" We need a converse proof in detail. □.

RMK: (重中之重):

If the series $f(z) = \sum a_n e^{-\lambda_n z}$ converges for $z = z_0$, it converges uniformly in every domain of the form $R(z-z_0)\geq 0$, $\mathrm{Arg}(z-z_0)\leq \alpha$.

Dirichlet L-function

§1. The Zeta function.

The Riemann zeta function $\zeta(s)$ is defined by $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$. It converges for $s>1$ and converges uniformly for $s\geq 1+\delta>1$, for each $\delta>0$.

Prop 1. For $s>1$,

$$\zeta(s)=\prod_{p}(1-p^{-s})^{-1}$$

Pf. $\frac{1}{1-p^{-s}}=\sum_{m=0}^{\infty}(p^{-s})^{m}=\sum_{m=0}^{\infty}p^{-ms}$ 证明董哥已经讲过

$$\prod_{p\leq N}(1-p^{-s})^{-1}=\sum_{n\leq N}n^{-s}+R_N(s) \leq \sum_{n=1}^{\infty} n^{-s}.$$

For converges, $R_N(s)\to 0,\ s\to\infty$.

Prop. Assume $s>1$, then

$$\lim_{s\to 1}(s-1)\zeta(s)=1.$$

pf. $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$.

$\forall s>1$, $\zeta(s)$ converges. $(n+1)^{-s}\leq\int_n^{n+1}t^{-s}\,dt\leq n^{-s}$

Then $\zeta(s)-1\leq\int_1^{\infty}t^{-s}\,ds\leq\zeta(s)$

Then $\zeta(s)-1\leq\frac{1}{s-1}\leq\zeta(s)$.

$$\Rightarrow\begin{cases}(s-1)\zeta(s)-(s-1)\leq 1\\ 1\leq(s-1)\zeta(s)\end{cases}\Rightarrow 1\leq(s-1)\zeta(s)\leq s.$$

when $s\to 1$, $\lim_{s\to 1}(s-1)\zeta(s)=1$.

Corollary. As $s\to 1$, we have $\frac{\ln\zeta(s)}{\ln(s-1)^{-1}}\to 1$

$$|(s-1)\zeta(s)-1|<\varepsilon \quad \text{when} \quad s-1<\delta.$$

$$1-\varepsilon<(s-1)\zeta(s)<\varepsilon+1$$

① $(s-1)\zeta(s) < \varepsilon+1$

$$\frac{\ln(s-1)+\ln(\zeta(s))}{-\ln(s-1)} < \frac{\ln(\varepsilon+1)}{-\ln(s-1)}$$

$$\frac{\ln(\zeta(s))}{\ln[(s-1)^{-1}]} < 1 + \frac{\ln(\varepsilon+1)}{\ln[(s-1)^{-1}]}$$

② $1-\varepsilon < (s-1)\zeta(s)$

$$\frac{\ln(1-\varepsilon)}{-\ln(s-1)} < \frac{\ln(s-1)}{-\ln(s-1)} + \frac{\ln\zeta(s)}{-\ln(s-1)}$$

$$\frac{\ln(1-\varepsilon)}{\ln[(s-1)^{-1}]} + 1 < \frac{\ln(\zeta(s))}{\ln[(s-1)^{-1}]}.$$

③ $\lim \frac{\ln(\zeta(s))}{\ln[(s-1)^{-1}]} = 1.$ $\square$.

讨论: Dirichlet Density 与 Natural Density 的关系:

Def. A set of positive primes $\mathcal{P}$ is said to have Dirichlet density if

$$\lim_{s\to 1} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\ln(s-1)^{-1}}$$

Prop. Let $\mathcal{P}$ be a set of positive prime numbers. Then

(a) If $\mathcal{P}$ is finite, then $d(\mathcal{P})=0$

(b) If $\mathcal{P}$ consists of all but finitely many primes, then $d(\mathcal{P})=1$.

(c) If $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint and $d(\mathcal{P}_1)$ and

$d(e_2)$ both

Natural Density:

$$p = \frac{\left|\left\{ \begin{matrix} a \in A \\ a \le x \end{matrix} \;\middle|\; A \text{ is a subset of } \mathbb{N} \right\}\right|}{x}.$$

那么，在考虑质数分布下，我们

考虑 $p = \lim\limits_{x\to\infty}. \dfrac{|\{a \in A, a \le x \mid A \text{ is a subset of primes}\}|}{|\{p \le x \mid p \text{ is prime}\}|}$

$s>1,$

Prop. $\ln \zeta(s) = \sum\limits_p p^{-s} + R(s)$ where $R(s)$ remains bounded as $s \to 1$. need a proof.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}$$

$$\ln(\zeta(s)) = \sum_p -\ln(1-p^{-s}) = \sum_p \left(\sum_{k=1}^{\infty} \frac{1}{k} p^{-ks}\right) = \sum_p p^{-s} + \left(\sum_p \sum_{k=2} \frac{1}{k} p^{-ks}\right)$$

$$\sum_p \sum_{k=2} \frac{1}{k} p^{-ks} \le \sum_p \sum p^{-ks} = \sum_p p^{-2s} \cdot \sum_{k=0}^{\infty} p^{-ks} = \sum_p p^{-2s}(1-p^{-s})^{-1}$$

$$\sum_p p^{-2s} \cdot (1-p^{-s})^{-1} \le (1-2^{-s})^{-1} \cdot \left(\sum_p p^{-2s}\right) \le 2 \cdot \sum_n \frac{1}{n^{2s}} \le 2\zeta(2).$$ 有界?

那么，我们就可以直接考虑

$$\lim_{s\to 1^+} \frac{\sum\limits_{p\in A} p^{-s}}{\ln(\frac{1}{s-1})} = \lim_{s\to 1^+} \frac{\sum\limits_{p\in A} p^{-s}}{\ln(\zeta(s))} = \lim_{s\to 1^+} \frac{\sum\limits_{p\in A} p^{-s}}{\sum\limits_p p^{-s} + R(s)} = \lim_{s\to 1^+} \frac{\sum\limits_{p\in A} p^{-s}}{\sum\limits_p p^{-s}}.$$

它们之间的关系.

我们现在证明定理:

设: $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ 为一个 Dirichlet series;

$$S(x) = \sum_{n \le x} a_n .$$

① 当 $\lim_{x\to\infty} \frac{S(x)}{x} = A$ 时, $\lim_{s\to 1^+} (s-1) f(s) = A$.

Pf.

$$\sum_{n=m}^{N} \frac{a_n}{n^s} = \sum_{n=m}^{N} S(n)\cdot\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + S(N)\cdot\frac{1}{(N+1)^s} - \frac{S(m-1)}{m^3}$$

显然, 对于 $s>1$,

$$\sum_{n=m}^{\infty} \frac{a_n}{n^s} = \sum_{n=m}^{\infty} S(n)\cdot\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{S(m-1)}{m^3}$$

又, 对 $\forall \delta > 0$, 总 $\exists N$, $n \ge N$时, $|S(n) - An| < \delta n$.
$N(\delta)$.

此时, $\sum_{n=N}^{\infty} \frac{a_n}{n^s} - \sum_{n=N}^{\infty} A\cdot\frac{1}{n^s}$

$$= \sum_{n=N}^{\infty} \frac{a_n - A}{n^s}$$

$$= \sum_{n=N}^{\infty} (S(n) - An)\cdot\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{S(N-1) - A(N-1)}{(N-1)^s}.$$

Remark:

$$\sum_{n=m}^{N} a_n b_n = \sum_{n=m}^{N-1} A_n (b_n - b_{n+1}) - A_{m-1}\cdot b_m + A_N\cdot b_{N+1}$$

$$= \sum_{n=m}^{N-1} B_n (a_n - a_{n+1}) - B_{m-1}\cdot a_m + B_N\cdot a_{N+1}$$

$= OR$.

$$|OR| \le \delta\cdot\sum_{n=N}^{\infty} n\cdot\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \frac{\delta(N-1)}{(N-1)^s}$$

$$\sum_{n=N}^{\infty} n\cdot\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = \frac{N}{N^s} + \sum_{n=N}^{\infty} \frac{1}{(n+1)^s} + \frac{1}{N^s} - \frac{1}{N^s}$$

$\frac{N}{N^s}$ - N + N+1 - 1 N-1

$$N^s \qquad (N+1)^s \qquad (N+1)^s \qquad (N+2)^s \qquad N^s$$

$$|OR| \le \delta\left(\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{N-1}{N^s}\right)+\frac{\delta(N-1)}{(N-1)^s}$$
$$\le \delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{2\delta(N-1)}{(N-1)^s}$$

即，$$\left|\sum_{n=N}^{\infty}\frac{a_n}{n^s}-A\sum_{n=N}^{\infty}\frac{1}{n^s}\right| \le \delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{2\delta(N-1)}{(N-1)^s}.$$

$$A\sum_{n=N}^{\infty}\frac{1}{n^s}-\delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}-\frac{2\delta(N-1)}{(N-1)^s} \le \sum_{n=N}^{\infty}\frac{a_n}{n^s} \le A\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}+\underbrace{\delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{2\delta(N-1)}{(N-1)^s}}_{\delta\cdot\zeta(s)+2\delta}.$$

$\Rightarrow$ 两边同乘 $(s-1)$ 再加上 $\sum_{n=1}^{N-1}\frac{a_n}{n^s}$.

$\delta\zeta(s)+2\delta$

就有，
$$\begin{cases}(s-1)\left(\sum_{n=1}^{N}\frac{a_n}{n^s}+A\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}-A\cdot\sum_{n=1}^{N-1}\frac{1}{n^s}\underbrace{-\delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}-\frac{2\delta(N-1)}{(N-1)^s}}\right)\le(s-1)f(s)\\ (s-1)f(s)\le\left(\sum_{n=1}^{N}\frac{a_n}{n^s}+A\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}-A\cdot\sum_{n=1}^{N}\frac{1}{n^s}+\underbrace{\delta\cdot\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{2\delta(N-1)}{(N-1)^s}}_{\delta\cdot\zeta(s)+2\delta}\right)(s-1).\end{cases}$$

$$(s-1)f(s)\ge-(s-1)\cdot M+A\cdot(s-1)\cdot\zeta(s)-(s-1)\cdot\delta\cdot\zeta(s)-2\delta\cdot(s-1)$$
$$(s-1)f(s)\le(s-1)\cdot M+A\cdot(s-1)\cdot\zeta(s)+(s-1)\cdot\delta\,\zeta(s)+2\delta\cdot(s-1).$$

$$\Rightarrow \lim_{s\to1^+}(s-1)f(s)=A.$$

再证：
$$\begin{cases}\limsup_{s\to1^+} f(s)\le\limsup_{x\to\infty}\frac{S(x)}{x}\\ \liminf_{s\to1^+} f(s)\ge\liminf_{x\to\infty}\frac{S(x)}{x}.\end{cases}$$

pf. $$\limsup_{x\to\infty}\frac{S(x)}{x}=A,$$

1. $A=-\infty$. $\exists\,\omega>0$, $n\ge N$时, $S(n)<-\omega n$.

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\sum_{n=1}^{\infty}S_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right).$$

$$\leq \sum_{n=1}^{N}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)-w\sum_{n=N}^{\infty}n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$$

$$=\sum_{n=1}^{N}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)-w\sum_{n=N}^{\infty}\frac{1}{n^s}(n-(n-1))-\frac{w(N-1)}{N^s}$$

$$\Rightarrow (s-1)f(s)<(s-1)\left|\sum_{n=1}^{N}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right|-w(s-1)\sum_{n=N}^{\infty}\frac{1}{n^s}.$$

$\Rightarrow \forall\ 1<s<1+\varepsilon(w)$,

$$(s-1)f(s)\leq -\frac{1}{2}\cdot w.$$

$$\Rightarrow \lim (s-1)f(s)=-\infty.$$

2. $A>-\infty$,

$\exists N$, $n\geqslant N$时, $S(n)<(A+\varepsilon)n$.

$$(s-1)f(s)<\sum_{n=1}^{N-1}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+(A+\varepsilon)\sum_{n=N}^{\infty}n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$$

$$=\sum_{n=1}^{N-1}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+(A+\varepsilon)\sum_{n=N}^{\infty}\frac{1}{n^s}+\frac{(A+\delta)(N-1)}{N^s}.$$

$$\Rightarrow \limsup_{s\to 1}(s-1)f(s)\leq A+\varepsilon.$$

$\square$.

设 $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$

$S(x)=\sum_{n\le x} a_n$.

那么,

$$\begin{cases} \limsup_{s\to 1^+} \frac{1}{\log(\frac{1}{s-1})}\cdot f(s) \le \limsup_{x\to\infty} \frac{S(x)}{\frac{x}{\log(x)}}. & ① \\ \liminf_{s\to 1^+} \frac{1}{\log(\frac{1}{s-1})} f(s) \ge \limsup_{x\to\infty} \frac{S(x)}{\frac{x}{\log x}}. & ② \end{cases}$$

我们只考虑①即可, 设 $\limsup_{x\to\infty}\frac{S(x)}{\frac{x}{\log(x)}}=A$. 对于 $\forall B>A$,

$S(n) < B\frac{n}{\log n}$ $\exists\ n\ge N=N(B)$, 使左式成立.

那么, $f(s)\le \sum_{n=1}^{N-1} S(n)\cdot\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+\sum_{n=N}^{\infty} B\cdot\frac{n}{\log n}\cdot\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$.

$= B\cdot\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+g_1(s)$.

$g_1(s)=\sum_{n=1}^{N-1} S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)-\sum_{n=2}^{N-1}\frac{Bn}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$.

$=\sum_{n=1}^{N-1}\left(S(n)-\frac{Bn}{\log n}\right)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$

$g_1(s)$ 由于均为有限求和, $\lim_{s\to 1^+} g_1(s)$ 一定存在.

且

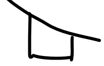
$\lim_{s\to 1^+}\frac{1}{\log(\frac{1}{s-1})}\cdot g_1(s)=0$ 必成立.

第二部分: $\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}-\frac{s}{n^{s+1}}\right|=\left|s\int_n^{n+1}\frac{1}{x^{s+1}}dx-\frac{s}{n^{s+1}}\right|$ $\quad < -\frac{s}{(n+1)^{s+1}}+\frac{s}{n^{s+1}}$

$<\left|s\cdot\left[\frac{1}{(n+1)^{s+1}}+\frac{1}{n^{s+1}}\right]\cdot\frac{1}{2}-\frac{s}{n^{s+1}}\right|$

$<\left|\frac{s}{2}\cdot\frac{1}{n^{s+1}}-\frac{s}{2}\cdot\frac{1}{(n+1)^{s+1}}\right|$

$<\left|s\cdot\frac{1}{n^{s+1}}-s\cdot\frac{1}{(n+1)^{s+1}}\right|$



$$= s(s+1)\cdot\int_{n}^{n+1}\frac{1}{x^{s+2}}\,dx$$

$$< \frac{s(s+1)}{n^{s+2}} \qquad (s>1).$$

$$\le \frac{s(s+1)}{n^{3}}.$$

那么, $\left|\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}-\frac{s}{n^{s+1}}\right)\right| < s(s+1)\cdot\sum_{n=2}^{\infty}\frac{1}{n^2\log n}$

设 $g_2(s)$,

使 $\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right) = s\sum_{n=2}^{\infty}\frac{1}{n^s\log n} + g_2(s).$

$$\left|\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}-\frac{s}{n^{s+1}}\right)\right| \le s(s+1)\sum_{n=2}^{\infty}\frac{1}{n^2\log n}.$$

$$|g_2(s)| = \left|\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right) - s\sum_{n=2}^{\infty}\frac{1}{n^s\log n}\right|$$

$$= \left|\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}-\frac{s}{n^{s+1}}\right) + \sum_{n=2}^{\infty}\frac{s}{\log n}\left(\frac{1}{n^s}\right) - s\sum_{n=2}^{\infty}\frac{1}{n^s\log n}\right|$$

$$= \left|\sum_{n=2}^{\infty}\frac{n}{\log n}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right|$$

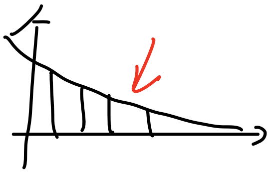
$$\le s(s+1)\cdot\sum_{n=2}^{\infty}\frac{1}{n^2\log n}$$

$$\le M\cdot s(s+1).$$

那么 $\left|\frac{g_2(s)}{\log\left(\frac{1}{s-1}\right)}\right| \le \frac{M\cdot s\cdot(s+1)}{\log\left(\frac{1}{s-1}\right)} \to 0.$

即, $g_2(s)$ bounded.

对于 $s>1$, $\sum_{n=2}^{\infty}\frac{1}{n^s\log n} > \int_2^{\infty}\frac{dx}{x^s\log x}$



$$\frac{1}{(n+1)^s\log(n+1)} \le \int_n^{n+1}\frac{dx}{x^s\log x} \le \frac{1}{n^s\log n}$$

$$\sum_{n=2}^{\infty}\frac{1}{(n+1)^s\cdot\log(n+1)} \le \int_2^{\infty}\frac{dx}{x^s\log x} \le \sum_{n=2}^{\infty}\frac{1}{n^s\log n}$$

$$\Rightarrow \begin{cases} \sum_{n=2}^{\infty}\frac{1}{n^s\log n} \le \frac{1}{2^s\log 2} + \int_2^{\infty}\frac{dx}{x^s\log x} \\ \sum_{n=2}^{\infty}\frac{1}{n^s\log n} \ge \int_2^{\infty}\frac{dx}{x^s\log x}. \end{cases}$$

bounded. 通过

那么，设 $g_3(s)$，我们有，

$$\sum_{n=2}^{\infty}\frac{1}{n^s\log n} = \int_2^{\infty}\frac{dx}{x^s\log x} + g_3(s)$$

$\Downarrow$

令 $e^t = x$

$t = \log x$

$$\sum_{n=2}^{\infty}\frac{1}{n^s\log n} = \int_{\log 2}^{\infty}\frac{e^t}{e^{ts}\cdot t}dt + g_3(s)$$

$$= \int_{\log 2}^{\infty} e^{-t(s-1)}\cdot\frac{1}{t}dt + g_3(s).$$

$\Downarrow$

令 $h = (s-1)t$

$$= \int_{(s-1)\log 2}^{\infty} e^{-h}\cdot\frac{1}{h}\cdot dh + g_3(s)$$

$$= M + \int_{(s-1)\log 2}^{1} e^{-h}\frac{1}{h}dh + g_3(s)$$

此时，考虑 $\int_{(s-1)}^{1}\frac{e^{-h}-1}{h}dh$，由于 $\lim_{h\to0}\frac{e^{-h}-1}{h} = \lim_{h\to0}\frac{-e^{-h}}{1} = -1$，有界，

我们知上式广义积分存在；

此时，

$$\sum_{n=2}^{\infty} \frac{1}{n^s \log n} = \int_{(s-1)}^{1} \frac{dw}{w} + g_4(s) \quad \text{bounded}$$

$$= -\log((s-1)\log 2) + g_4(s)$$

$$= \log\left(\frac{1}{s-1}\right) + g_5(s). \quad \text{bounded}$$

此时，我们有，

$$f(s) \leq B \cdot \sum_{n=2}^{\infty} \frac{n}{\log n}\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + g_1(s),$$

$$\sum_{n=2}^{\infty} \frac{n}{\log n}\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = s\sum_{n=2}^{\infty} \frac{1}{n^s \log n} + g_2(s).$$

$$\sum_{n=2}^{\infty} \frac{1}{n^s \log n} = \log\left(\frac{1}{s-1}\right) + g_5(s).$$

$$\Rightarrow \frac{f(s)}{\log\left(\frac{1}{s-1}\right)} \leq Bs + \frac{g_6(s)}{\log\left(\frac{1}{s-1}\right)}.$$

Theorem. Suppose $a, m \in \mathbb{Z}$, with $(a,m)=1$. Let $\mathcal{P}(a:m)$ be the set of positive primes p such that $p \equiv a\ (m)$. Then $d(\mathcal{P}(a:m)) = \frac{1}{\phi(m)}$

A special case.

We will first prove the above theorem when $m=4$.

First, we

define a function X from $\mathbb{Z}$ to $\{0, \pm1\}$ as follows: $X(n)=0$ if

Define function X from $\mathbb{Z}$ to $\{0, 1, -1\}$ as follows: $X(n) = 0$ if n is even, $X(n) = 1$ if $n \equiv 1\ (4)$ and $X(n) = -1$ if $n \equiv 3\ (4)$. It is easily seen that $X(mn) = X(m)X(n)$ for all $m, n \in \mathbb{Z}$.

RMK:

$$X(m)X(n) = X(mn)$$

n \ m	$\equiv 1\ (4)$	$\equiv 3\ (4)$	even
$\equiv 1(4)$	$(1, 1)$	$(-1, -1)$	$(0, 0)$
$\equiv 3(4)$	$(-1, -1)$	$3 \times 3 \equiv 1\ (4)$ $(1, 1)$	$(0, 0)$
even	$(0, 0)$	$(0, 0)$	$(0, 0)$

Define $L(s, X) = \sum_{n=1}^{\infty} X(n) n^{-s}$

$= 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots$ for all n, we have $|X(n) n^{-s}| \leq n^{-s}$

Thus, it converges.

Then we want to show

$$L(s, X) = \prod_p (1 - X(p) p^{-s})^{-1}$$ (这个董哥讲过)

Then we have $L(s, X) = \prod_p (1 - X(p) p^{-s})^{-1}$

Then $\begin{cases} \zeta(s) = \sum_{n \geq 1}^{\infty} \frac{1}{n^s} \\ \zeta^*(s) = \sum_{n \text{ odd}} \frac{1}{n^s} \end{cases} \Rightarrow \zeta(s) = \sum_{n \text{ odd}} \frac{1}{n^s} + \sum_{n \text{ even}} \frac{1}{n^s}$

$$= \sum_{n \text{ odd}} \frac{1}{n^s} + 2^{-s} \cdot \zeta(s)$$

$\zeta^*(s) = (1 - 2^{-s}) \cdot \zeta(s)$.

$$= (1 - 2^{-s}) \cdot \prod_p (1 - p^{-s})^{-1} = \prod_{p \text{ odd}} (1 - p^{-s})^{-1}$$

$\begin{cases} \zeta^*(s) = \sum_{n \text{ odd}} \frac{1}{n^s} = \prod_{p \text{ odd}} (1 - p^{-s})^{-1} \\ L(s, X) = \sum^{\infty} \frac{X(n)}{n^s} = \prod (1 - X(p) p^{-s})^{-1} \end{cases}$

$$\ln \zeta^*(s) = \sum_{p\ odd} -\ln(1-p^{-s}) = \sum_{p\ odd}\left(\sum_{k=1}^{\infty}\frac{1}{k}\cdot p^{-ks}\right) = \sum_{p\ odd} p^{-s} + \overset{(*)}{\sum_{p\ odd}\sum_{k=2}^{\infty}\frac{1}{k}p^{-ks}}.$$

$$(*) \le \sum_{p\ odd} p^{-2s}\sum_{k=0}^{\infty}p^{-ks} = \sum_{p\ odd} p^{-2s}(1-p^{-s})^{-1} \le (1-2^{-s})\left(\sum_{p\ odd}p^{-2s}\right).$$

And,

$$\ln\left(L(s,\chi)\right) = \sum_{p\ odd} -\ln\left(1-\chi(p)p^{-s}\right) = \sum_{p\ odd}\sum_{k=1}^{\infty}\frac{\chi(p^k)p^{-ks}}{k}$$

$$= \sum_{p\ odd}\chi(p)\cdot p^{-s} + \overset{(※)}{\sum_{p}\sum_{k=2}^{\infty}\frac{\chi(p^k)\cdot p^{-ks}}{k}} \quad 有界!$$

$$(※) \le \sum_{p}\chi(p^2)\cdot p^{-2s}\cdot\sum_{k=0}^{\infty}\frac{\chi(p^k)\cdot p^{-ks}}{k+2}$$

bounded obviously.

Then, ① $\ln\zeta^*(s) + \ln\left(L(s,\chi)\right) = \sum_{p\ odd}\left(1+\chi(p)\right)p^{-s} + R_1$

② $\ln\zeta^*(s) - \ln\left(L(s,\chi)\right) = \sum_{p\ odd}\left(1-\chi(p)\right)p^{-s} + R_2.$

$$1+\chi(p) = \begin{cases}2, & p\equiv 1\ (4)\\ 0, & p\equiv 3\ (4)\end{cases} \qquad 1-\chi(p) = \begin{cases}0, & p\equiv 1\ (4)\\ 2, & p\equiv 3\ (4)\end{cases}$$

Then, ① $:= \ln\zeta^*(s) + \ln(L(s,\chi)) = 2\sum_{p\equiv 1(4)} p^{-s} + R_1$

③ $:= \ln\zeta^*(s) - \ln(L(s,\chi)) = 2\sum_{p\equiv 3(4)} p^{-s} + R_2.$

Now, we need the final striking !!!

we need a proof of the boundness of $\ln(L(s,\chi))$. $(s>1)$

pf. $L(s,\chi) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots$

$L(s,\chi) = \left(1-\frac{1}{3^s}\right) + \left(\frac{1}{5^s}-\frac{1}{7^s}\right) + \cdots\cdots \quad > \frac{2}{3}$

$L(s,\chi) = 1 - \left(\frac{1}{3^s}-\frac{1}{5^s}\right) - \left(\frac{1}{7^s}-\frac{1}{9^s}\right) - \cdots \quad < 1$

$\ln(L(s,\chi)) \in [\ln(\frac{?}{3}), 0]$.

$\ln \zeta^*(s) = \ln(1-2^{-s})^{-1} + \ln(\zeta(s))$.

$\Rightarrow$ As for ①, ②

①: $\underbrace{\ln(\zeta(s)) - \ln(1-2^{-s}) + \ln(L(s,\chi))}_{-\ln(s-1)} = \underbrace{2\sum_{p\equiv 1(4)} p^{-s} + R_1}_{-\ln(s-1)}$

②: $\underbrace{\ln(\zeta(s)) - \ln(1-2^{-s}) - \ln(L(s,\chi))}_{-\ln(s-1)} = \underbrace{2\sum_{p\equiv 3(4)} p^{-s} + R_2}_{-\ln(s-1)}$

$\Rightarrow \quad d(\mathcal{P}(1:4)) = d(\mathcal{P}(3:4)) = \frac{1}{2}$.

Recall some properties of Dirichlet character.

Let $\mathcal{X}: (\mathbb{Z}/m\mathbb{Z}) \to \mathbb{C}^*$ be a homomorphism.

We use $\mathcal{X}$ to define χ.

$\chi(n): \mathbb{Z} \to \mathbb{C}^*$

If $\begin{cases} (n,m)>1, & \chi(n)=0 \\ (n,m)=1, & \chi(n) = \mathcal{X}(n+m\mathbb{Z}). \end{cases}$

prop. $\begin{cases} \chi(n+m) = \chi(n) \\ \chi(kn) = \chi(k)\chi(n) \\ \chi(n) \neq 0 \iff (n,m)=1. \end{cases}$

注：如此，我们在这里就与董哥那部分再一次接轨.

即通过上面的定义，我们得知 χ 与 $\hat{\chi}$ 一一对应.
不妨用 χ 代表它与其 $\hat{\chi}$ 属性：
那么，我们有 G 与 $\hat{G}$ 同构，由于 G 为一个模 m 的一个剩余类.

此时考虑 $\chi, \psi \in \hat{G}$，$\delta(x,y) = \begin{cases} 0, & x \neq y \\ 1, & x = y, \end{cases}$

prop.

1. $\sum_{a \in G} \chi(a)\bar{\psi}(a) = n\,\delta(\chi,\psi)$ $\quad n = |G|$

$$\begin{cases} \bar{\psi} = \chi^{-1} \Rightarrow \chi_0 \\ \bar{\psi} \neq \chi^{-1} \end{cases}$$

2. $\sum_{\chi \in \hat{G}} \chi(a)\bar{\chi}(b) = n\,\delta(a,b)$

$a = b$, $\sum_{\chi \in \hat{G}} \chi \cdot \bar{\chi}(a) = n$

$a \neq b$, $\sum_{\chi \in \hat{G}} \chi(a)\bar{\chi}(b) = \sum_{\chi \in \hat{G}} \chi(a \cdot b^{-1}) \cdot \chi(b) \cdot \bar{\chi}(b)$

$= \sum_{\chi \in \hat{G}} \chi(a \cdot b^{-1}) = \sum_{\chi \in \hat{G}} \chi(c)$

$\exists\, \psi$, $\psi(c) \neq 1$, $\Rightarrow \sum_{\chi \in \hat{G}} \psi \cdot \chi(c) = \sum_{\chi \in \hat{G}} \psi(c) \cdot \chi(c)$

$= \psi(c) \sum_{\chi \in \hat{G}} \chi(c)$

$= \sum_{\chi \in \hat{G}} \chi(c)$

$\Rightarrow \sum_{\chi \in \hat{G}} \chi(c) = 0.$

Then for a Dirichlet character,

$$\begin{cases} \sum_{a=0}^{m-1} \chi(a)\,\bar{\psi}(a) = \phi(m)\,\delta(\chi,\psi) \\ \sum_{\chi} \chi(a) \cdot \bar{\chi}(b) = \phi(m)\,\delta(a,b). \end{cases}$$

Dirichlet character ??

Dirichlet L-function

全形式

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\cdot n^{-s} = \prod_{p} (1-\chi(p)p^{-s})^{-1}$$

Consider $\mathbb{Z}/m\mathbb{Z}$, $\chi(p)=0$ for $p \mid m$.

Then, $L(s,\chi) = \prod_{p\nmid m} (1-\chi(p)p^{-s})^{-1} = \prod_{p\mid m}(1-p^{-s})\cdot \zeta(s)$

我们想讨论 $\ln L(s,\chi)$，但遇到问题，它定义在复平面上，所以我们考虑 $e^{G}=L$.

$$e^{G(s,\chi)} = e^{\sum_p \sum_{k=1} \frac{1}{k}\chi(p^k)\cdot p^{-ks}} = \prod_p e^{\sum \frac{1}{k}\chi(p^k)p^{-ks}}$$

Define $G(s,\chi) = \sum_p \sum_{k\geq 1} \frac{1}{k}\chi(p^k)p^{-ks}$ $\quad \left|\frac{1}{k}\chi(p^k)p^{-ks}\right| \leq p^{-ks}$

$$e^{G(s,\chi)} = e^{\sum_p \sum_{k=1} \frac{1}{k}\chi(p^k)\cdot p^{-ks}} = \prod_p e^{\sum \frac{1}{k}\chi(p^k)p^{-ks}} =$$

Then we have $G(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k}\chi(p^k)p^{-ks}$

$$G(s,\chi) = \sum_p \sum_{k=1} \frac{1}{k}\chi(p^k)p^{-ks} = \sum_{p\nmid m}\chi(p)p^{-s} + \underbrace{\sum_p\sum_{k=2}\frac{1}{k}\chi(p^k)p^{-ks}}_{}$$

R_1

$$\leq \sum_p p^{-2s}\cdot \underline{\sum_{k=0}^{\infty} p^{-ks}}$$

$$\sum_p p^{-2s}(1-p^{-s})^{-1}$$

$$\leq (1-2^{-s})^{-1}\cdot \sum_p p^{-2s}$$

Then,

$\bar{\chi}(a)G(s,\chi)$

$= \sum_{p\nmid m} \bar{\chi}(a)\chi(p)p^{-s} + \bar{\chi}(a)\cdot R_1$

$$\Rightarrow \sum_{\chi}\bar{\chi}(a)G(s,\chi) = \sum_{p\nmid m}\left(\sum_{\chi}\bar{\chi}(a)\chi(p)p^{-s}\right) + \sum_{\chi}\bar{\chi}(a)R_1$$

$$= \sum_{p \nmid m} \left(\phi(m)\delta(a,p)\, p^{-s}\right) + \sum_{\chi} \bar{\chi}(a) R_1$$

$$= \phi(m) \sum_{p \equiv a(m)} p^{-s} + \sum_{\chi} \bar{\chi}(a) R_1$$

Consider $\sum_{\chi} \bar{\chi}(a)\, G(s,\chi)$,

① χ_0 is trivial,

$$e^{G(s,\chi_0)} = L(s,\chi_0) = \prod_{p \mid m} (1-p^{-s}) \cdot \zeta(s)$$

$$\frac{G(s,\chi_0)}{\ln(\frac{1}{s-1})} = \frac{\overset{0}{\sum_{p \mid m} \ln(1-p^{-s})}}{\ln(\frac{1}{s-1})} + \frac{\overset{1}{\ln(\zeta(s))}}{\ln(\frac{1}{s-1})}$$

② χ is nontrivial, quite difficult.

注：这里我们要讨论的是 $\frac{G(s,\chi)}{\ln(\frac{1}{s-1})}$ 行为在 $s \to 1$ 时，

那么，我们

Recall, $A_N = \sum_{n=1}^{N} a_n$, $B_N = \sum_{n=1}^{N} b_n$

$$S_N = \sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N.$$

Then, if $A_N b_N \to 0$ as $N \to \infty$,

we have $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} A_n (b_n - b_{n+1})$.

Lemma1. $\zeta(s) - (s-1)^{-1}$ can be continued to an analytic function on the region $\{s \in \mathbb{C} \mid \sigma > 0\}$.

As for $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} 1 \times \frac{1}{n^s} = \sum_{n=1}^{\infty} n \times \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).$$

$$= \sum_{n=1}^{\infty} n \times s \int_n^{n+1} x^{-s-1}\, dx$$

$$= s \cdot \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1}\, dx$$

$$= s \cdot \sum_{n=1}^{\infty} \int_n^{n+1} [x] \cdot x^{-s-1}\, dx$$

$$= s \cdot \int_1^{\infty} [x] \cdot x^{-s-1}\, dx \qquad \langle x \rangle = x - [x]$$

$$= s \cdot \int_1^{\infty} x^{-s}\, dx - s \cdot \int_1^{\infty} \langle x \rangle\, x^{-s-1}\, dx$$

$$= s \cdot \frac{1}{s-1} - s \cdot \int_1^{\infty} \langle x \rangle\, x^{-s-1}\, dx$$

$$\zeta(s) - \frac{1}{s-1} = 1 - s \cdot \int_1^{\infty} \langle x \rangle \cdot x^{-s-1}\, dx \qquad \square.$$

Lemma2. Let χ be a nontrivial character modulo m. For all $N > 0$, we have $\left|\sum_{n=0}^{N} \chi(n)\right| \leq \phi(m)$.

pf. $N = qm + r$.

$$\sum_{n=0}^{N} \chi(n) = \left|\sum_{n=qm+1}^{qm+r} \chi(n)\right| \leq \sum_{n=0}^{m} |\chi(n)| = \phi(m)$$

Prop. Let χ be a nontrivial Dirichlet character modulo m. Then, $L(s, \chi)$ can be continued to an analytic function in the region $\{s \in \mathbb{C} \mid \sigma > 0\}$.

pf. Define $S(N) = \sum_{n \leq N} \chi(n)$.

$\sigma>1,$

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\cdot\frac{1}{n^s}$$

$$=\sum_{n=1}^{\infty}S(n)\cdot\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$$

$$=s\cdot\sum_{n=1}^{\infty}S(n)\cdot\int_n^{n+1}x^{-s-1}\,dx$$

$$=s\cdot\int_1^{\infty}S(x)\,x^{-s-1}\,dx.$$

Then we can extend $L(s,\chi)$ to $\sigma>0$.

Now, we consider $L(1,\chi)$, we want $L(1,\chi)\neq 0$ when χ is nontrivial.

I. χ is a complex character.

Prop: $F(s)=\prod_{\chi}L(s,\chi)$ where the product is over all Dirichlet characters modulo m. for s real and $s>1$, we have $F(s)\geq 1$.

Pf. $e^{G}=L \Rightarrow e^{\sum_{\chi}G(s,\chi)}=\prod_{\chi}L(s,\chi)$.

Consider $\sum_{\chi}G(s,\chi)$,

$$=\sum_{\chi}\sum_{p}\sum_{k=1}^{\infty}\frac{1}{k}\chi(p^k)\,p^{-ks}$$

$$=\sum_{p}\sum_{k=1}^{\infty}\frac{1}{k}\underline{\left(\sum_{\chi}\chi(p^k)\right)}\,p^{-ks}$$

$p^k=1$ 时 $\sum_{\chi}\chi(p^k)=\phi(m)$

否则为0.

$$=\phi(m)\sum_{p^k\equiv 1\,(m)}\frac{1}{k}\cdot p^{-ks}\Rightarrow\ \geq 0.$$

$$\Rightarrow\ F=e^{\sum_{\chi}G}\geq 1.$$

Then if χ is a complex character,

$\overline{L(s,\chi)} = L(s,\bar{\chi})$

$F = \prod_{\chi} L(s,\chi)$. If $L(1,\chi)=0$,

① χ_0 is trivial, $L(s,\chi_0)$ is a simple pole.

$$L(s,\chi_0) = \prod_{p|m}(1-p^{-s})\cdot \zeta(s)$$

② But $L(s,\chi)$, $L(s,\bar{\chi})$ distinct and they get two zeros,

Then $F(1)=0$, but $F\geq 1$ for all $s>1$. Contradiction. $\square$

II. χ is a real Dirichlet character modulo m.

We want to prove $L(1,\chi)\neq 0$.

Lemma:

Suppose f is a nonnegative, multiplicative function on $\mathbb{Z}^+$, i.e. for all $m,n>0$ with $(m,n)=1$, $f(mn)=f(m)f(n)$.

Assume there is a constant c s.t. $f(p^k)<c$ for all prime powers p^k. Then $\sum_{n=1}^{\infty} f(n)\cdot n^{-s}$ converges for all $s>1$.

Moreover,
$$\sum_{n=1}^{\infty} f(n)\cdot n^{-s} = \prod_p \left(1+\sum_{k=1}^{\infty} f(p^k)\, p^{-ks}\right)$$

Pf. Fix $s>1$. Let $a(p) = \sum_{k=1}^{\infty} f(p^k)\, p^{-ks}$

Then $a(p) < c\cdot p^{-s}\cdot \sum_{k=0}^{\infty} p^{-ks} = c\cdot p^{-s}\cdot (1-p^{-s})^{-1} < 2c\cdot p^{-s}$;

Then, $\prod_{p \le N} (1+a(p)) \le \prod_{p \le N} e^{a(p)} = \exp\left(\sum_{p \le N} a(p)\right)$.

And $\sum_{p \le N} a(p) \le 2C \cdot \sum_{p \le N} p^{-s} = M(N)$.

Moreover,

bounded.

$$\sum_{n=1}^{N} f(n) \cdot n^{-s} = \prod_{p \le N} \left(1 + \sum_{k=1}^{\infty} f(p^k) \cdot p^{-ks}\right) + R$$

$$\sum_{n=1}^{N} f(n) \cdot n^{-s} = \prod_{p \le N} \left(1 + \sum_{k=1}^{\infty} f(p^k) \cdot p^{-ks}\right) + R.$$

Then $\sum_{n=1}^{\infty} f(n) \cdot n^{-s}$ is bounded for every N.

And $\sum_{n=1}^{N} f(n) \cdot n^{-s} \nearrow$, $\sum_{n=1}^{\infty} f(n) \cdot n^{-s}$ is convergent. !

pf. Consider the function

$$\psi(s) = \frac{L(s,\chi)\, L(s,\chi_0)}{L(2s,\chi_0)}$$

Assume $L(1,\chi) = 0$.

$\begin{cases} L(s,\chi) \text{ has zero at } s=1. \\ L(s,\chi_0) \text{ has simple pole at } s=1. \end{cases}$

$\Rightarrow$ $\psi(1)$ analytic on $\sigma > \frac{1}{2}$.

And $\psi(s)$ has a zero at $s = \frac{1}{2}$.

And $\psi(s) = \prod_p (1-\chi(p)p^{-s})^{-1}(1-\chi_0(p)p^{-s})^{-1}(1-\chi_0(p)p^{-2s})$

$$= \prod_{p\nmid m} \frac{(1-p^{-2s})}{(1-p^{-s})(1-\chi(p)p^{-s})}$$

$\chi(p)=-1$, $\quad \dfrac{1-p^{-2s}}{(1-p^{-s})(1+p^{-s})} = 1.$

$$\Rightarrow \quad \psi(s) = \prod_{\chi(p)=1} \frac{1-p^{-2s}}{(1-p^{-s})(1-p^{-s})} = \prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}}.$$

For convergence, we need a proof.

As for $\dfrac{1+p^{-s}}{1-p^{-s}}$, $\dfrac{1+p^{-s}}{1-p^{-s}} = (1+p^{-s})(\sum_{k=0}^{\infty} p^{-ks}) = 1 + 2p^{-s} + 2p^{-2s} + \cdots$

Then $\psi(s) = \sum_{n=1}^{\infty} a_n \cdot n^{-s}$ where $a_n \geq 0$ and the series for $s>1$.

$a_1 = 1.$

Then we consider $\psi(s)$ as a complex variable function and we expand it in a power series about $s=2$. $\psi(s) = \sum_{m=0}^{\infty} b_m \cdot (s-2)^m.$

Since $\psi(s)$ is analytic for $\sigma > \frac{1}{2}$, Need a proof. !!!

To compute the b_m we use Taylor's theorem. i.e. $b_m = \dfrac{\psi^{(m)}(2)}{m!}$,

Since $\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $\quad \psi^{(m)}(2) = \sum_{n=1}^{\infty} (-\ln n)^m \cdot a_n \cdot n^{-2}$

$$= (-1)^m \cdot \sum_{n=1}^{\infty} a_n \cdot n^{-2} (\ln n)^m$$

$c_m \geq 0. \qquad = (-1)^m \cdot c_m$

$\psi(s) = \sum_{n=0}^{\infty} c(m) \cdot (2-s)^m$, $\quad c_0 = \psi(2) \geq a_1 = 1.$

Then $\psi(s) \geq 1$ when $s \to \frac{1}{2}$, this is a contradiction with $\psi(s) \to 0$ when $s \to \frac{1}{2}$.

Evaluating $L(s,\chi)$ at Negative Integers.

Recall: $\Gamma(s) = \int_0^\infty e^{-t} \cdot t^{s-1}\, dt$.

$\Gamma(s) = (s-1)\Gamma(s-1)$.

$\Gamma(n+1) = n!$ since $\Gamma(1) = 1$.

If $\sigma > -1$, we define $\Gamma_1(s) = \Gamma_1(s) = \frac{1}{s}\Gamma(s+1) = \int_0^\infty e^{-t} \cdot t^s dt \cdot \frac{1}{s}$

For $\sigma > 0$, $\Gamma_1(s) = \Gamma(s)$.

Similarly, if k is a positive integer, we define

$$\Gamma_k(s) = \frac{1}{s(s+1)\cdots(s+k-1)} \cdot \Gamma(s+k).$$

which is analytic on $\{s \in \mathbb{C} \mid \sigma > -k\}$

except for some simple poles at $s = 0, -1, \cdots, 1-k$.

Then we expand $\Gamma(s)$.

For $\sigma > 1$, $n^{-s}\Gamma(s) = \int_0^\infty e^{-t} \cdot t^{s-1} dt$

令 $nx = t$

$$原式 = \int_0^\infty e^{-nx} \cdot n^{s-1} \cdot x^{s-1} \cdot n\,dx \cdot \frac{1}{n^s}$$

$$= \int_0^\infty e^{-nx} \cdot x^{s-1} dx.$$

$$\Rightarrow \left(\sum_{n=1}^\infty n^{-s}\right) \cdot \Gamma(s) = \int_0^\infty x^{s-1} \cdot \left(\sum_{n=1}^\infty e^{-nx}\, dx\right)$$

$$= \int_0^\infty x^{s-1} \cdot e^{-x} \cdot \left(\sum_{n=0}^\infty e^{-nx}\right) dx$$

$$= \int_0^\infty x^{s-1} \cdot e^{-x} \cdot \frac{1}{1-e^{-x}}\, dx$$

$$= \int_0^\infty x^{s-1} \cdot \frac{e^{-x}}{1-e^{-x}}\, dx.$$

$$\Gamma(s)\cdot\zeta(s) = \int_0^\infty x^{s-1}\cdot\frac{e^{-x}}{1-e^{-x}}dx.$$

算不了.

则 $x=2t$.

$$\Gamma(s)\cdot\zeta(s) = \int_0^\infty 2^{s-1}\cdot t^{s-1}\cdot\frac{e^{-2t}}{1-e^{-2t}}\cdot 2\cdot dt.$$

$$= 2^s\cdot\int_0^\infty t^{s-1}\cdot\frac{e^{-2t}}{1-e^{-2t}}dt$$

$$\Rightarrow 2^{1-s}\cdot\Gamma(s)\cdot\zeta(s) = 2\cdot\int_0^\infty t^{s-1}\cdot\frac{e^{-2t}}{1-e^{-2t}}dt$$

$$\zeta^*(s) = (1-2^{1-s})\zeta(s)$$

$$\Gamma(s)\cdot\zeta(s) - 2^{1-s}\cdot\Gamma(s)\zeta(s) = \zeta^*(s)\cdot\Gamma(s)$$

$$= \int_0^\infty t^{s-1}\cdot\frac{e^{-t}}{1-e^{-t}}dt - 2\int_0^\infty t^{s-1}\cdot\frac{e^{-2t}}{1-e^{-2t}}dt$$

$$= \int_0^\infty t^{s-1}\cdot\frac{e^{-t}(1-e^{-t})}{1-e^{-2t}}dt$$

$$= \int_0^\infty t^{s-1}\cdot\frac{e^{-t}}{1+e^{-t}}dt$$

Let $R(x) = \frac{x}{1+x}$

$$\Rightarrow R(e^{-t}) = \frac{e^{-t}}{1+e^{-t}}$$

$$\Rightarrow \Gamma(s)\cdot\zeta^*(s) = \int_0^\infty t^{s-1}\cdot R(e^{-t})dt.$$

Let $R_0(t) = R(e^{-t})$

$$\Rightarrow \forall m\in\mathbb{N}^+,\quad R_m(t) = \frac{d^m R}{dt^m}(R(e^{-t}))$$

$$\frac{d}{dt}\frac{e^{-t}}{1+e^{-t}} = \frac{-e^{-t}(1+e^{-t})-e^{-t}(-e^{-t})}{(1+e^{-t})^2} = \frac{-e^{-t}}{(1+e^{-t})^2}$$

$$\frac{-e^{-t}-(e^{-t})^2+(e^{-t})^2}{(1+e^{-t})^2}$$

$$\left(\frac{d}{dt}\right)^2\left(\frac{e^{-t}}{1+e^{-t}}\right) = \frac{\qquad}{(1+e^{-t})^4}$$

Then we will figure.

$$\begin{cases} ① \ R_m(t) = e^{-t} \cdot P_m(e^{-t})(1+e^{-t})^{-2m}, \quad P_m \text{ is a poly.} \\ ② \ R_m(0) \text{ is finite.} \\ ③ \ \frac{R_m(t)}{e^{-t}} \text{ is bounded as } t \to \infty. \end{cases}$$

Take $u = R(e^{-t})$, $v = \frac{t^s}{s}$

$$\Gamma(s) \cdot \zeta^*(s) = \int_0^\infty t^{s-1} \cdot R(e^{-t})\, dt.$$

$$= \frac{1}{s} \cdot t^s \cdot R_0(t) \Big|_0^\infty - \frac{1}{s} \int_0^\infty t^s \cdot R_1(t)\, dt$$

$$= -\frac{1}{s} \int_0^\infty t^s \cdot R_1(t)\, dt$$

$$\Gamma(s+1) \cdot \zeta^*(s) = -\int_0^\infty R_1(t) \cdot t^s\, dt$$

$$\Downarrow$$

$$\Gamma(s+k) \cdot \zeta^*(s) = (-1)^k \cdot \int_0^\infty R_k(t) \cdot t^{s+k-1}\, dt.$$

Prop. Let k be a positive integer. Then $\zeta(0) = -\frac{1}{2}$, and for $k > 1$, $\zeta(1-k) = -\frac{B_k}{k}$ where B_k is the kth Beronulli number.

Pf. From above,

$$\Gamma(s+k) \cdot \zeta^*(s) = (-1)^k \cdot \int_0^\infty R_k(t) \cdot t^{s+k-1}\, dt.$$

$$\Gamma(1) \cdot \zeta^*(1-k) = (-1)^k \cdot \int_0^\infty R_k(t)\, dt$$

we deduce.

$$\Rightarrow \quad (1-2^k)\cdot\Gamma(1)\cdot\zeta(1-k) = (-1)^k\cdot\int_0^\infty R_k(t)\,dt.$$

$$\Downarrow$$

$$(-1)^k\int_0^\infty R_k(t)\,dt = R_{k-1}(t)\Big|_0^\infty\cdot(-1)^k = (-1)^{k-1}R_{k-1}(t)$$

$$R_0(t) = \frac{e^{-t}}{1+e^{-t}} = \frac{e^{-t}}{1-e^{-t}} - \frac{2\cdot e^{-2t}}{1-e^{-2t}}$$

$$= \frac{1}{t}\left(\frac{t}{e^t-1} - \frac{2t}{e^{2t}-1}\right)$$

$$\Downarrow$$

$$= \frac{1}{t}\left(\sum_{k=0}^\infty\left(\frac{B_k}{k!}\right)\cdot t^k - \sum_{k=0}^\infty\left(\frac{B_k}{k!}\right)\cdot(2t)^k\right)$$

$$= \left[\sum_{k=0}^\infty\left(\frac{B_k}{k!}\right)\cdot t^k(1-2^k)\right]\cdot\frac{1}{t}.$$

$$= \sum_{k=1}^\infty\left(\frac{B_k}{k!}\right)\cdot t^{k-1}\cdot(1-2^k).$$

$$\Rightarrow \quad (1-2^k)\cdot\Gamma(1)\cdot\zeta(1-k) = (-1)^{k-1}\cdot\frac{B_k}{k!}\cdot(1-2^k)$$

$$\zeta(1-k) = (-1)^{k-1}\cdot\frac{B_k}{k!}$$

RMK: $\begin{cases} B_1 = -\frac{1}{2} \\ B_{odd} = 0 \end{cases} \Rightarrow \begin{cases} \zeta(0) = -\frac{1}{2} \\ \zeta(1-k) = -\frac{B_k}{k!}. \end{cases}$ $\square$.

A

$$\begin{cases} \Gamma(s)\cdot L(s,x) = \int_0^\infty F_x(e^{-t})\,t^{s-1}\,dt. \\ F_x(e^{-t}) = \sum_{n=1}^\infty x(n)e^{-t} \end{cases}$$

$$= \sum_{a=1}^m x(a)\sum_{k=0}^\infty e^{-(a+km)t}$$

$$= \sum_{a=1}^m x(a)\cdot\frac{e^{-at}}{1-e^{-mt}}.$$

suppose $L^*(s,x) = (1-2^{1-s})L(s,x).$

$$\Gamma(s) L^*(s,x) = \int_0^\infty R_x(e^{-t}) t^{s-1}\, dt.$$

$$R_x(x) = \frac{x f(x)}{1+x+\cdots+x^{2m-1}}, \quad f(x) \text{ is a polynomial.}$$

We also have :

$$\Gamma(s+k) L^*(s,x) = (-1)^k \int_0^\infty R_{x,k}(t)\, t^{s+k-1}\, dt.$$

Def. Let χ be a nontrivial Dirichlet character modulo m. The generalized Bernoulli number $B_{n,\chi}$ is defined by:

$$\sum_{a=1}^{m} \chi(a) \frac{te^{at}}{e^{mt}-1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} \cdot t^n.$$

Lemma: $t F_x(e^{-t}) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{B_{n,x}}{n!}\right) t^n.$

Prop: Let k be a positive integer. Then $L(1-k, \chi) = -\frac{B_{k,\chi}}{k}$.

Pf. For $\Gamma(s+k) L^*(s,\chi) = (-1)^k \int_0^\infty R_{x,k}(t)\, t^{s+k-1}\, dt.$

$\Rightarrow s = 1-t,$

$\Rightarrow (1-2^k) L(1-k, \chi) = (-1)^k \int_0^\infty R_{x,k}(t)\, dt.$

And $R_x(e^{-t}) = F_x(e^{-t}) - 2F_x(e^{-t}) = \frac{1}{t} \sum_{k=1}^{\infty} (-1)^k \cdot (1-2^k) \cdot \frac{B_{k,x}}{k!} \cdot t^k$

$\Rightarrow (-1)^{k-1} R_{x,k-1}(0) = -(1-2^k) \cdot \left(\frac{B_{k,x}}{k}\right).$

Thus, $L(1-k, x) = \frac{-B_{k,x}}{k}.$

RMK: $\sum_{n=0}^{\infty} \frac{B_{n,\chi_0}}{n!} \cdot t^n = \frac{te^t}{e^t-1} = 1 + \frac{1}{2} t + \sum_{n=2}^{\infty} \frac{B_n}{n!} \cdot t^n.$

$\Rightarrow$ They are the same. $\square$.

Equations on Finite Field.

If F is a finite field with q elements, then clearly $A^n(F)$ has q^n elements. $P^n(F)$ has $\frac{q^{n+1}-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + q + 1$

Lemma 1. Let $\bar{H}$ be the set of $[x] \in P^n(F)$ (it is the representative element of equivalent class $\{ x = (a_0, \cdots, a_n) \mid (a_0, \cdots, a_n) \sim (b_0, \cdots, b_n)$ if $\exists \gamma \in F^*$, s.t. $(a_0, \cdots, a_n) = \gamma (b_0, \cdots, b_n) \}$) such that $x_0 = 0$. Then ϕ maps $P^n(F) - \bar{H}$ to $A^n(F)$ and this map is one to one and onto.

$\phi([x]) = \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)$

pf. Consider $x \in P^n(F) - \bar{H}$, $x = (x_0, \cdots, x_n)$, $x_0 \neq 0$.

If $\phi([x]) = \phi([y])$, $\frac{x_1}{x_0} = \frac{y_1}{y_0}, \cdots$

$\Downarrow$

$x_1 = \gamma y_1, \cdots, x_n = \gamma y_n \Rightarrow [x] = [y]$.

$\forall x = (x_1, \cdots, x_n) \Rightarrow$ we set $[x]$ representation is $(1, x_1, \cdots, x_n)$ in $P^n(F) - \bar{H}$

□

The set $\bar{H}$ is called the hyperplane at infinity. It is to see that $\bar{H}$ has the structure of $P^{n-1}(F)$.

Denoting : $P^n(F) = \underbrace{P^{n-1}(F)}_{\text{points at infinity}} + \underbrace{A^n(F)}_{\text{finite points}}$

$f \in F[x_1, \cdots, x_n]$, $f = \sum_{(i_1, \cdots, i_n)} a_{i_1 i_2 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$.

Suppose that K is a field containing F. If $f(x) \in F[x_1, \cdots, x_n]$, and $a \in A^n(K)$, we can substitute a_i for x_i and compute $f(a)$.

$f(a) = 0 \iff a$ is a zero of $f(x)$.

$H_f(K) = \{ a \in A^n(K) \mid f(a) = 0 \}$. hypersurface defined by f in $A^n(K)$.

Define a projective hypersurface, let $h(x) \in F[x_0, x_1, \cdots, x_n]$ be a nonzero homogeneous polynomial of degree d.

k is a field containing F, for $\gamma \in K^*$, we have $h(\gamma x) = \gamma^d h(x)$

If $a \in A^{n+1}(k)$. $h(a)=0$. $\Rightarrow h(\gamma a)=0 \Rightarrow [a]$: the representative of $h(\gamma a)=0$.

$\Rightarrow$ All of zeros representative, then we may define

$\bar{H}_n(k) = \{[a] \in P^n(k) \mid h(a) = 0\}$ This set is called the hypersurface defined by h in $P^n(k)$.

RMK: Chevalley's Theorem.

F is a finite field with q elements.

We want the theorem:

Let $f(x) \in F[x_1, \cdots, x_n]$ and suppose that $\begin{cases} a.\ f(0,\cdots,0)=0 \\ b.\ n > d = \deg f. \end{cases}$

Then f has at least two zeros in $A^n(F)$.

Lemma 1. Let $f(x_1, \cdots, x_n)$ be a polynomial that is of degree less than q in each of its variables. Then if f vanishes on all of $A^n(F)$, it is the zero polynomial.

pf. 1. $n=1$ $f(x)$ degree less than q. $f(x)=0$ on all of $A^1(F)$,
$x=0$ obviously.

2. $n-1$ is true.

3. As for n. $f(x_1, \cdots, x_n) = \sum_{i=0}^{q-1} g_i(x_1, \cdots, x_{n-1}) x_n^i$

任意性 Select $a_1, \cdots, a_{n-1}$,

$\sum_{i=0}^{q-1} g_i(a_1, \cdots, a_{n-1}) x_n^i = 0$ q roots.

$\Rightarrow g_i(a_1, \cdots, a_{n-1}) = 0, i = 0, \cdots, q-1$.

$\forall g_i$, $g_i = 0$.

① If a polynomial is of degree less than q in each variable, it is said to be reduced.

② If $f(a)=g(a)$ for all $a \in A^n(F)$ we write $f \sim g$.

Lemma 2. Each polynomial $f(x) \in F[x_1, \cdots, x_n]$ is equivalent to a reduced polynomial.

pf. we will obviously have $x^q \sim x$ $(q = |F|)$

Then, we can use induction. ▯.

Recall:

We want the theorem:

Let $f(x) \in F[x_1, \cdots, x_n]$ and suppose that $\begin{cases} a. & f(0, \cdots 0) = 0 \\ b. & n > d = \deg f. \end{cases}$

Then f has at least two zeros in $A^n(F)$.

pf. Need to complete. !

Gauss and Jacobi sums over Finite Fields

Define F has $q=p^n$ elements, For $\alpha \in F$, define $tr(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$ which is called the trace of α.

$F_p : \mathbb{Z}/p\mathbb{Z}$.

Prop: 1. $tr(\alpha) \in F_p$

2. $tr(\alpha+\beta) = tr(\alpha) + tr(\beta)$

3. $tr(a\alpha) = a\,tr(\alpha)$

4. tr maps F onto F_p.

Define ψ: $\psi(x) = \xi_p^{tr(x)}$

$(e^{\frac{2\pi i}{p}} = \xi_p)$

① Prop. ① $\psi(\alpha+\beta) = \psi(\alpha)\cdot\psi(\beta)$

② there is an $\alpha \in F$ s.t. $\psi(\alpha) \neq 1$.

③ $\sum_{\alpha \in F} \psi(\alpha) = 0$

② $\frac{1}{q}\sum_{x\in F} \psi(\alpha(x-y)) = \delta(x,y)$.

Def. Let χ be a character of F and $\alpha \in F^*$. Let $g_\alpha(\chi) = \sum_{t\in F} \chi(t)\cdot\psi(\alpha t)$

$g_\alpha(\chi)$ is called a Gauss sum on F belonging to the character χ.

Theorem. Suppose that F is a field with q elements and $q \equiv 1 \pmod m$. The homogeneous equation $a_0 y_0^m + \cdots + a_n y_n^m = 0$, $a_1, \cdots, a_n \in F^*$, defines a hypersurface in $P^n(F)$. The number of points on this hypersurface is given by

$$q^{n-1} + \cdots + q + 1 + \frac{1}{q-1} \sum_{\chi_0 \cdots \chi_n} \chi_0(a_0^{-1}) \cdots \chi_n(a_n^{-1})\, J_0(\chi_0, \cdots, \chi_n).$$