Intersection Forms

Assumption: M is a closed oriented 4-mfd.

Several statements:

Denote the intersection form of M by $Q_M: H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$

Q_M can be represented by an integer symmetrical matrix A.

Indeed, let $e_1, \dots, e_k$ be a basis of $H_2(M;\mathbb{Z})$

$$A = \begin{bmatrix} Q_M(e_1,e_1) & \cdots & Q_M(e_1,e_k) \\ \vdots & & \vdots \\ Q_M(e_k,e_1) & \cdots & Q_M(e_k,e_k) \end{bmatrix}_{\cancel{k\times k}}$$

Remark: The matrix representing Q_M is not unique.

let $\alpha_1, \dots, \alpha_k$ be a basis of $H^2(M;\mathbb{Z})$, $\beta_1, \dots, \beta_k$ be another basis of $H^2(M;\mathbb{Z})$, Let $Q_M(\alpha,\alpha) = \alpha^T A \alpha$, $Q_M(\beta,\beta) = \beta^T B \beta$, where $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_k)$, $\beta = \alpha \cdot C$

then we have $B = C^{-1} A C$.

Ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

1. Def (Unimodularity) Q_M is unimodular if $\det Q_M = \pm 1$

$\Leftrightarrow Q_M$ is invertible over $\mathbb{Z}$.

Remark: Q_M is unimodular iff $\forall$ $\mathbb{Z}$-linear function $f: H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$, $\forall y \in H_2(M;\mathbb{Z})$, $\exists !\, x \in H_2(M;\mathbb{Z})$ s.t. $f(x) = x \cdot y = Q(x,y)$

2. Invariants of intersection forms:

(rank, signature, definiteness, parity)

Let M be a closed oriented 4-mfd, let Q_M be the intersection form.

Def:

(1) the rank of Q_M: $\operatorname{rank} Q_M = \operatorname{rank}_{\mathbb{Z}} H_2(M;\mathbb{Z})$

(2) the signature of Q_M: $\operatorname{sign} Q_M = b_2^+ - b_2^+$

(diagonalize Q_M into a rational canonical form, and then into the normal form $\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{bmatrix}$

b_2^+ (b_2^-) is the number of $+1$'s (-1's) on the diagonal.

(3) the definiteness of Q_M: • Q_M is called positive definite if $\operatorname{rank} Q_M = \operatorname{sign} Q_M$

(or if $Q_M(\alpha_i, \alpha_i) > 0$ for every $\alpha_i \in H_2(M;\mathbb{Z})$)

• Q_M is called negative definite if $\operatorname{rank} Q_M = -\operatorname{sign} Q_M$

(or if $Q_M(\alpha_i, \alpha_i) < 0$, for every $\alpha_i \in H_2(M;\mathbb{Z})$)

• Q_M is called indefinite otherwise.

(4) the parity of Q_M: Q_M is called even if $Q_M(\alpha_i, \alpha_i)$ is even for all classes $\alpha_i \in H_2(M;\mathbb{Z})$

(or if every metrix representing Q_M has even diagonal).

Remark: ① $\operatorname{Sign}(Q_1 \oplus Q_2) = \operatorname{sign} Q_1 + \operatorname{sign} Q_2$

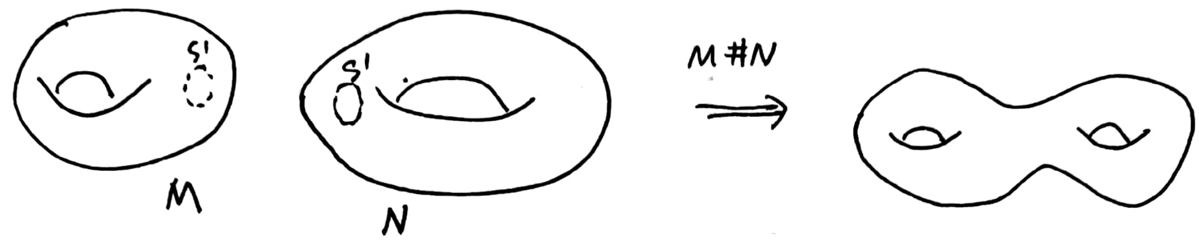
$\operatorname{Sign}(M \# N) = \operatorname{sign} M + \operatorname{sign} N$

$\operatorname{sign} \bar{M} = -\operatorname{sign} M$, where $\bar{M}$ is the manifold M with opposite orientation.

② $Q_{\bar{M}} = -Q_M$ (by definition)

3. (Connected Sum) The connected sum of two oriented mfds M^m and N^n, denoted by $M \# N$, is a mfd formed by deleting a ball inside each mafold and gluing together the resulting boundary spheres:

Ex:



$$M\#N = (M \setminus i_1(0)) \sqcup (N \setminus i_2(0)) \Big/ i_1(tu) \sim i_2((1-t)u) \quad u \in S^{n-1},\ 0 \le t \le 1$$

where $i_1 : D_1 \to M_1$, $i_2 : D_2 \to M_1$ are imbeddings.

Th 1. Let M, N be a connected closed oriented 4-mfds with intersection forms Q_M and Q_N respectively, then $\boxed{Q_{M\#N} = Q_M \oplus Q_N}$

Pf: M° : M − a 4-ball $\cong$ M − a 4-handle

N° : N − a 4 ball $\cong$ N − a 4-handle

since 2-homology of M, N is determined by 1, 2, 3 handles.

$$H_2(C_*) = \ker \partial_2 / \operatorname{Im} \partial_3.$$

$$\partial_k : C_k \to C_{k-1}$$

$$C_k = \mathbb{Z}\{k\text{-handles } h_\alpha^k\}$$

then 2-homlogy of $M\#N$ is the gathering of 2-homologies of M and N.

then $Q_{M\#N} = Q_M \oplus Q_N$

Remark: the results also holds for gluing space $M \cup_\partial N$.

Th2: If indefinite unimodular forms Q_1, Q_2 have the same rank, signature, parity, then $Q_1 \cong Q_2$ (i.e. Q_1 is equivalent to Q_2)

Ex: $H \oplus [-1] \cong 2[-1] \oplus [1]$

They both have rank 3, signature -1 and they are odd.

4. The signature vanishes for boundaries.

Th3. If M^4 is the boundary of some oriented 5-mfd W^5, then $\text{sign}\, Q_M = 0$

Corallary: If two mfds M, N are cobordant, then $\text{sign}\, Q_M = \text{sign}\, Q_N$.

Pf: If $\partial W = \bar{M} \sqcup N$, then $0 = \text{sign}(\bar{M} \sqcup N) = \text{sign}(-Q_M) + \text{sign}(Q_N)$
$= \text{sign}\, Q_N - \text{sign}\, Q_M$

Th4. (V. Rokhlin) If a smooth oriented 4-mfd M has $\text{sign}\, Q_M = 0$ then there is a smooth oriented 5-mfd W s.t. $\partial W = M$.

Corallary: (1) M^4 is the boundary of some oriented 5-manifold W^5 iff $\text{sign}\, Q_M = 0$

(2) Two 4-mfds have the same signature iff they are cobordant.

5. Examples of unimodular intersection forms.

(1) S^4

$H_2(S^4;\mathbb{Z})=0$, thus Q_{S^4} is trivial

(2) CP^2

$H_2(CP^2;\mathbb{Z})=\mathbb{Z}[\alpha]$, where $[\alpha]$ is a class of a complex projective plain.

any two projective line intersects at 1 point.

thus $Q_{CP^2}=[1]$, $\begin{cases} \text{rank} = 1 \\ \text{sign} = 1 \\ \text{positive definite} \\ \text{odd} \end{cases}$

$Q_{\overline{CP^2}} = -Q_{CP^2} = [-1]$, $\begin{cases} \text{rank} = 1 \\ \text{sign} = 1 \\ \text{negative definite} \\ \text{odd} \end{cases}$

(3) $S^2\times S^2$

$H_2(S^2\times S^2;\mathbb{Z}) = \mathbb{Z}[\alpha]\oplus\mathbb{Z}[\beta]$, where $\alpha=[S^2\times\{pt\}]$, $\beta=[\{pt\}\times S^2]$

form the basis of $H_2(S^2\times S^2;\mathbb{Z})$

thus $Q_{S^2\times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, denote it by H

$\begin{cases} \text{rank} = 2 \\ \text{Sign} = 0 \\ \text{indefinite} \\ \text{parity of } Q_{S^2\times S^2} \ ? \end{cases}$

$\boxed{\text{Question}}$: I can not tell the parity of $Q_{S^2\times S^2}$,

since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, let $C=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = C^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}C$

but $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has even diagonal, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has odd diagonal.

14) Connected sum $M = 3CP^2 \# \overline{CP^2} \# 2S^2 \times S^2$

$$Q_M = Q(3CP^2 \# \overline{CP^2} \# 2S^2 \times S^2)$$

$$= 3Q_{CP^2} \oplus Q_{\overline{CP^2}} \oplus 2H$$

$$= 3[+1] \oplus [-1] \oplus H$$

$$\begin{cases} \text{rank} = 6 \\ \text{sign} = 2 \\ \text{indefinite} \\ \text{odd} \end{cases}$$

15) the E_8-manifold $M_{E_8} = P_{E_8} \cup_{\Sigma_p} \Delta$, where Δ is a fake 4-ball, Σ_p is a homology 3-sphere, P_{E_8} is a space gathering 8 sphere with self intersection 2 and intersecting the other sphere 0 or 1 as the form $Q_{E_8} =: \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}_{8\times 8}$

$$Q_{M_{E_8}} = Q(P_{E_8} \cup_{\Sigma_p} \Delta) = Q_{P_{E_8}} \oplus Q_\Delta = Q_{P_{E_8}}$$

$$\begin{cases} \text{rank} = 8, \\ \text{sign} = 8 \\ \text{positive-definite} \\ \text{even} \end{cases}$$

Remark: $Q_{M_{E_8}}$ can also be represented by $\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & 0 & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}_{8\times 8}$

Th5. (Whitehead) Two simply connected, closed, oriented 4-mfds X_1, X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$.

Th6. (Freedman) For every unimodular symmetric bilinear form Q, there exists a simply closed topological 4-mfd X s.t. $Q_X = X$. If Q is even, then X is unique up to homeomorphism.

Corallary: A simply connected closed 4-mfd X is homeomorphic to S^4 iff X is homology equivalent to S^4.