Lecture 2 Handle Decomposition

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Theorem (h-cobordism theorem)**.** *Let M^m and N^m be compact simply-connected oriented m-manifolds that are h-cobordant through the simply-connected (m+1)-manifold W^{m+1}. If $m \geq 5$, then there is a diffeomorphism*

$$W \cong M \times [0,1],$$

which can be chosen to be the identity from $M \subset W$ to $M \times 0 \subset M \times [0,1]$. In particular, M and N are diffeomorphic.

h-cobordism theorem could be proved by handle decomposition and Whitney thick.

1 CW complex

Definition 1.1 (CW complex)**.** *(1) Start with a discrete set X^0, whose points are regarded as 0-cells.*

(2) Inductively, form the ***n-skeleton*** *X^n from X^{n-1} by attaching n-cells e^n_α via maps $\psi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sin \psi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.*

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

A space X constructed in this way is called a ***cell complex*** *or* ***CW complex****.*

Morse theory provides a way to see the CW complex structure of manifolds.

Let M be a compact manifold of dim n. A critical point of a smooth map $f : M \to \mathbb{R}$ is a point $p \in M$ where the differential $d_p f \in T^*_p M$ is zero. We say the a critical $p \in M$ of a smooth map f is non-degenerate if df and the zero section are transverse at p

Definition 1.2 (Morse function)**.** *We say f is a Morse function if df is transverse to the zero section, i.e., if all critical points are non-degenerate.*

The critical points of a Morse function are isolated. The Morse Lemma tells that around any critical point p (where $df|_p = 0$), there exists a chart (x, U) around p s.t. the function f can be written locally as

$$fx^{-1}(t) = -t_1^2 - \cdots - t_k^2 + t_{k+1}^2 + \cdots + t_n^2$$

where $t = (t_1, \ldots, t_n) \in U$, n $= \dim M$. The critical point is then called a critical point is then called a **critical point of index k**. Its Hessian matrix is

$$D(D(fx^{-1}))(x(p)) = \begin{pmatrix} -2I_{k\times k} & 0 \\ 0 & 2I_{(n-k)\times(n-k)} \end{pmatrix}$$

Let $M^a = f^{-1}((-\infty, a])$

Proposition 1.1. *Let M be a compact manifold, $f : M \to \mathbb{R}$ a smooth function and $a < b \in \mathbb{R}$. Suppose f has no critical values in [a,b]. Then M^a is diffeomorphic to M^b.*

Theorem 1.1. *Let M be a compact and let f be a Morse function. Assume $f^{-1}[a, b]$ contains a single critical point p with $c = f(p) \in (a, b)$ and let λ be the index of f at p. Then M^b is obtained from M^a by attaching a λ-cell. Then we get a cell decomposition of M.*

Example 1.1.

- *T^2: height function.* $e^0 \cup e^1_1 \cup e^1_2 \cup e^2$
- *$\mathbb{RP}^2$: let $\lambda_0 < \lambda_1 < \lambda_2$ be three distinct positive real numbers. The Morse function of $\mathbb{RP}^2$ is*

  $$f : \mathbb{RP}^2 \to \mathbb{R} : [z_0 : z_1 : z_2] \mapsto \frac{\lambda_0 z_0^2 + \lambda_1 z_1^2 + \lambda_2 z_2^2}{z_0^2 + z_1^2 + z_2^2}$$

  Critical points: [1:0:0] index 0, [0:1:0] index 1, [0:0:1] index 2. $e^0 \cup e^1 \cup e^2$.

The cell decomposition is very sensitive to deformations of the Morse function.

2 Handle decomposition

2.1 Handles

The handle decomposition is similar to the cell decomposition. Here, we require each part is of the same dimension. A naive way is to thick k-cell by times a $n - k$ disk:

$$k - cell \mapsto k - cell \times \mathbb{D}^{n-k}$$

Things like $k - cell \times \mathbb{D}^{n-k}$ is a n-dimensional **k-handle**.

Definition 2.1. *For $0 \leq k \leq n$, an n-dimensional k-handle h is a copy of $\mathbb{D}^k \times \mathbb{D}^{n-k}$, attached to the boundary of an n-manifold M along $\partial\mathbb{D}^k \times \mathbb{D}^{n-k}$ by an embedding $\varphi : \partial\mathbb{D}^k \times \mathbb{D}^{n-k} \to \partial M$.*

Definition 2.2 (Handle decompositions)**.** *Let M be a compact n-manifold with boundary ∂M decomposed as a disjoint union $\partial_+ M \coprod \partial_- M$ of two compact submanifolds (either of which may be empty). If X is oriented, orient $\partial_\pm M$ so that $\partial M = \partial_+ M \coprod \bar{\partial_-} M$ in the boundary orientation. A **handle decomposition** of M (relative to $\partial_- M$) is an identification of M with a manifold obtained from $I \times \partial_- X$ by attaching handles, such that $\partial_- M$ corresponds to $\{0\} \times \partial_- M$ in the obvious way. A manifold M with a given handle decomposition is called a relative handlebody built on $\partial_- M$, or if $\partial_- M = \emptyset$ it is called a handlebody.*

Theorem. *Every smooth, compact manifold pair $(M, \partial_- M)$ as above admits a handle decomposition.*

Proof. (idea)An important fact is that any smooth function $f : M \to [0,1]$ with $f^{-1}(0) = \partial_- M$ and $f^{-1}(1) = \partial_+ M$ can be perturbed into a Morse function (with no critical points on ∂M). Then each smooth, compact manifold pair $(M, \partial_- M)$ admits a handle decomposition. The k-handle corresponds to index k of critical point. □

In similar, a noncompact manifold with compact boundary will admit a proper Morse function $f : M \to [0, \infty)$ with $f^{-1}(0) = \partial M$, providing a theory of handle decompositions of noncompact manifolds. The difference is that the noncompact case may need infinite many handles.

We say the handle decomposition is topological (smooth) if the attaching maps are homeomorphic (diffeomorphic) embeddings.

Theorem 2.1 (Moise (n=3); Kirby, Siebenmann ($n \geq 6$); Freedman, Quinn (n=5))**.** *A topological manifold pair $(M, \partial_- M)$ with $\dim M = n \neq 4$ always admits a topological handle decomposition.*

Theorem 2.2. *If $n = 4$, then $(M, \partial M)$ admits a topological handle decomposition if and only if M is smoothable.*

Proof. Idea: Any homeomorphic embedding of smooth 3-manifolds is uniquely smoothable [Mo], so a handle decomposition of a topological 4-manifold determines a smooth structure. □

Anatomy of a handle: core; attaching sphere; belt sphere

Example 2.1. *Freedman's closed 4-manifold with intersection form E_8 admits no handle decomposition.*

$$E_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

Definition 2.3 (intersection form)**.** *Let M be a compact, oriented, topological 4-manifold M; let $[M]$ be its fundamental class $[M] \in H_4(M, \partial M; \mathbb{Z})$. The symmetric bilinear form*

$$Q_M : H^2(M, \partial M; \mathbb{Z}) \times H^2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_M(a, b) = \langle a \cup b, [M] \rangle = a \cdot b \in \mathbb{Z}$ is called the intersection form of M. Since by Poincare duality $H_2(M; \mathbb{Z}) \cong H^2(M, \partial M; \mathbb{Z})$, Q_M is defined on $H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z})$ as well.

Since $Q_M(a, b) = 0$ if a or b is a torsion element. It reduces to a bilinear form on a free $\mathbb{Z}$ module.

Example 2.2. *torus;* $\mathbb{RP}^2$

2.2 Handle sliding

The dimension argument about attaching sphere of h^i and belt sphere of h^j tells that: if $i \leq j$ then the two sphere generically do not meet. Geometrically, h^i (lower-order) can be slid off h^j (higher-order) if h^i is glued on h^j.

2.3 Homology from handles

Since a k-handle is merely a thickened k-cell, it should be no surprise that the homology $H_*(M, \partial_- M; \mathbb{Z})$ can be retrieved directly from the handle decomposition of M.

A chain complex with groups

$$C_k = \mathbb{Z}\{\text{k-handles } h^k_\alpha\}$$

and boundary maps $\partial_k : C_k \to C_{k-1}$, given by

$$\partial_k(h^k_\alpha) = \sum \langle h^k_\alpha | h^{k-1}_\beta \rangle \cdot h^{k-1}_\beta,$$

where $\langle h^k_\alpha | h^{k-1}_\beta \rangle$ is the incidence number of h^k_α with h^{k-1}_β. This coefficient is defined as the intersection number of the attach sphere of h^k_α with the belt sphere of h^{k-1}_β.

The attaching sphere of h^k_α is a (k-1)-sphere, while the belt sphere of h^{k-1}_β is an $(n-k)$-sphere; both are living in the $(n-1)$-dimensional upper boundary M_ρ of the ascending cobordism $\partial_+ M_\rho$. If assumed transverse, their intersection is in isolated points; there points can then be counted with signs to yield the coefficients $\langle h^k_\alpha | h^{k-1}_\beta \rangle$.

Thus we can retrieve the relative homology group $H_*(M, \partial_- M; \mathbb{Z})$.

Example 2.3. *torus;* $\mathbb{RP}^2$

3 Handle moves

3.1 Handle cancellation; Handle creation

If the hole created by adding a (k-1)-handle h^{k-1}_β is filled by the later addition of some k-handle h^k_α, then this **pair** of handles can be eliminated. This is a geometric language: intersection points counted without sign is exactly 1. A necessary condition in algebraic language is that:

$$\partial h^k_\alpha = \pm h^{k-1}_\beta$$

3.2 Handle sliding

Algebraic effect of sliding is that it changes the boundary operator $\partial_k : C_k \to C_{k-1}$. Specifically, sliding h^k_α over h^k_β modifies ∂_k the same way as would changing the basis of C_k by replacing h^k_α by $h^k_\alpha + h^k_\beta$ or $h^k_\alpha - h^k_\beta$.

4 Proof of h-cobordism theorem

Let M^m and N^m be compact simply-connected oriented manifolds, and let W^{m+1} be a simply-connected cobordism between them; assume that $H_*(W, M; \mathbb{Z}) = 0$. We can obtain a handle decomposition of the pair (W, M). An algebraic result shows that, one can change boundary operators looks like:

$$\partial_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad or \qquad \partial_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

by sliding handles and adding pairs of canceling handles. Such handles are "∂-paired". Algebraically, we do the best. The Whitney trick (useful in dimension ≥ 5) could eliminate geometry opposite intersection points. Then we can do can cancel handles in pairs (handle cancellation). Finally, we eliminate all handles on W, so $W \cong$ (diffeo) $M \times [0, 1]$.

References

[1] Robert E. Gompf, Andras I.Stipsicz; 4-Manifolds and Kirby Calculus.

[2] Scorpan; The Wild World of 4-Manifolds.