Topological h-cobordism theorem for 4-manifolds

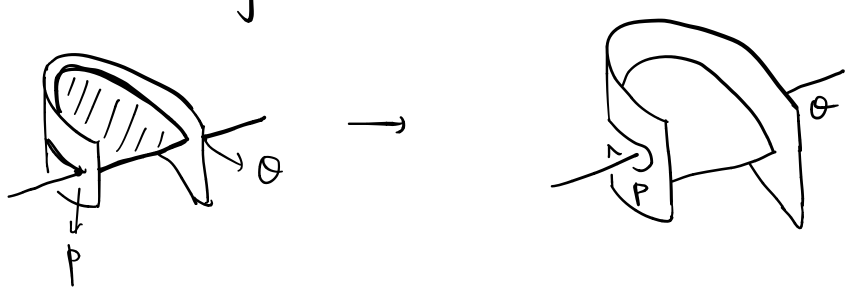
There are many facts we have to admit without giving rigorous proof in this notes. Instead, we will mainly focus on geometric intuition. Please find references yourself if you are interested.

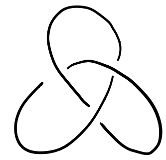
Recall: In high-dimensional case, we have the smooth h-cobordism theorem:

h-cobordism theorem. Higher dimension. Let $m \geq 5$. M^m, N^n be cpt. simply-connected oriented m-mfds and W^{m+1} a simply connected h-cobordism between them. Then W is diffeomorphic to $M \times [0,1]$.

To prove it, we find a handle decomposition on W^{m+1}, do sliding and moving to eliminate the algebraic intersection, do the Whitney trick to equalize the geometric intersection and algebraic one, and finally cancel out the handle to trivialize the cobordism.

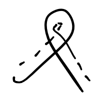
All the processes can go smoothly in dimension 4 except for the Whitney trick.



If $m > 4$, then the embeddings are dense in all maps $D^2 \to M^m$. So a loop in a simply connected space always almost bound an embedded disk. This is not true in dimension 4. For example, a nontrivial knot in $\mathbb{R}^3$ does not bound an embedded disk. 

We will introduce a new trick given by Casson to find an embedded disk.

Set-up

An immersed disk is not embedded because of the transverse double points. They are the enemies. 

Let W^5 be the 5-dimensional cobordism between M^4, N^4. By previous results, we can eliminate every 0-handle and 5-handle, and trade all 1-handles with 4-handles. We are left with 2-handles and 3-handles.

The corresponding chain complex $0 \to C_3 \xrightarrow{\partial} C_2 \to 0$ gives the relative homology $H_*(W;M)$, which are all 0 by the h-cobordism hypothesis. Thus ∂ is an isomorphism, and by sliding & moving, one can make the matrix of ∂ being the identity. Hence $\partial h^3_\alpha = \pm h^2_\beta$ and each 2-handle is paired with a 3-handle.

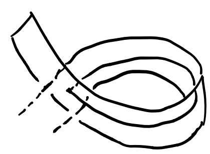
Suppose $\frac{5}{2}$ is a regular value of the Morse function from which the handle decomposition is given. Furthermore, Suppose $W_{\frac{5}{2}}$ consists of all 2-handles but no 3-handle. Let $M_{\frac{5}{2}}$ be its 4-dimensional upper boundary. Then $M_{\frac{5}{2}}$ consists of all the 2-belt spheres and 3-attaching spheres. Let P, Q be two points of their intersections with opposite signs. We can find a loop connecting p, Q, as before. By simply-connectedness of $M_{\frac{1}{2}} - \{p, Q\}$, the loop bounds an immersed disk D, with possibly transverse double points.

We may cut the manifold properly, so that D is immersed in some 4-manifold M, with ∂D sent to ∂M. (For example, we may cut in the interior along the loop.)

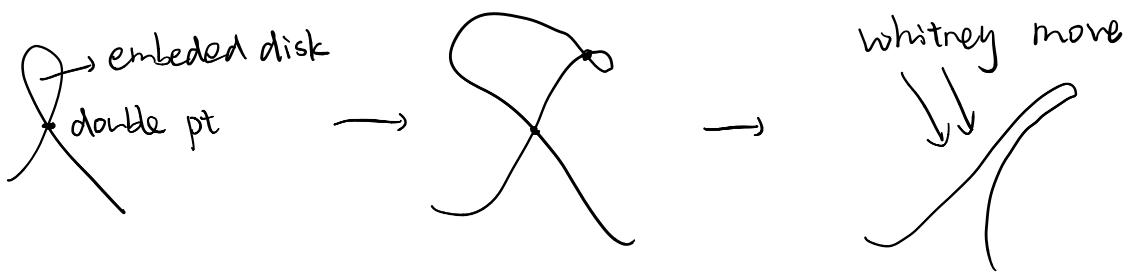
Small Tricks.

Creating self-intersections: Let S be a surface immersed in a 4-manifold. One can create more self-intersections.

Eliminating self-intersections: Under certain conditions, self-intersections can be eliminated.



Choose a loop in D based on the double point. Suppose the loop bound an embedded disk, then we create another self-intersection at the loop and do the Whitney move.

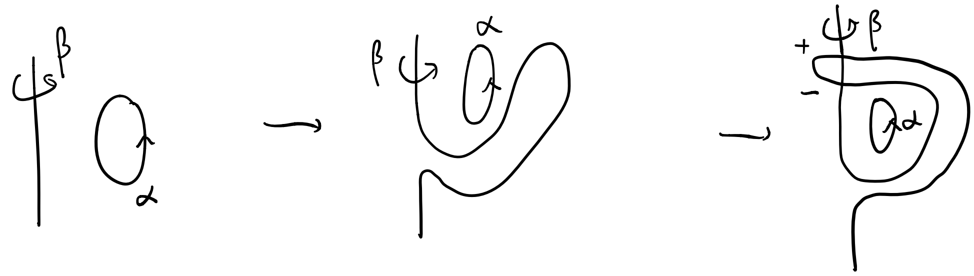


As a result, both self-intersections are eliminated.

Finger moves: For a loop mentioned above to have any chance of bounding an embedded disk, we want the simply connectedness of its complement.

In all the situations we meet, the fundamental group G is perfect, that is, $G=[G,G]$, where $[G,G]$ is the commutator subgroup of G. So it suffices to trivialize the subgroup $[G,G]$.

Suppose $S\subseteq M$ is an immersed surface in a 4-manifold, and $\alpha\subseteq M-S$ is a loop. Let β be a loop around S, we do the finger move:



The result is killing the commutator $[\alpha,\beta]$, but creating a new pair of double points.

To see this, near each double point of an immersed 2-surface, there is a nbhd U homeomorphic to $D^2\times D^2$. Its complementary in M is a torus. β is one of the generator of $\pi_1(T^2)$ and $\alpha\beta\alpha^{-1}$ is another. Since $\pi_1(T^2)$ is commutative, β and $\alpha^{-1}\beta\alpha$ commutes, thus the commutator $[\beta,\alpha^{-1}\beta\alpha]$ vanishes.

Step by step, we reduce the fundamental group to reach a simply connected complement. (no further details)

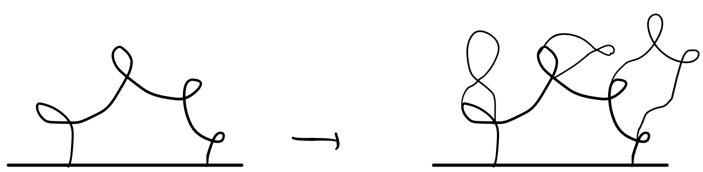
Casson Handle

Now we have an immersed disk $D \subseteq M^4$, with $\partial D \subseteq \partial M$.

By finger move, we have all desired simply-connectedness.

We want to use the "eliminating self-intersection" trick to eliminate the transverse double pts in D.

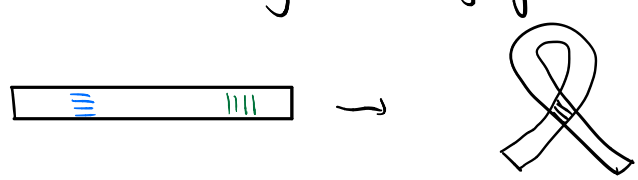
At each point, we find a loop and create another double point. If the loop bound an embedded disk, we are done. Otherwise, it bound an immersed disk with self-intersection double points. Do the same thing for these double points, growing loops and new immersed disks. Continue this process, indefinitely.



The finger moves are also performed at each step. After infinitely many steps, we get "a tower of failures".

Thicken to a Casson handle.

Replace the disks above by thinkened disks $D \times I D^2$ to obtain more flexibility. The Self-intersections is illustrated as follows: pick to small disks D', D'' inside $D \times I D^2$, the identify $D' \times I D^2$ with $D'' \times I D^2$ by identifying $D' \times p \subseteq D' \times I D^2$ with $p \times I D^2 \subseteq D'' \times I D^2$.



The result look much like a 2-handle. It also need to be attached properly, which we will not discuss here. (Before doing thicken, we need only attach a point of the loop to a double point.)

We have $\partial(\mathbb{D}^2\times\mathbb{D}^2) = \mathbb{D}^2\times S^1 \cup S^1\times\mathbb{D}^2$. At each step, $S^1\times\mathbb{D}^2$ is attached to the previous thickened disk, making $\mathbb{D}^2\times S^1$ left. What we want is an embedded disk. So we do not wish to see any boundary inside the interior. Therefore, we simply discard the part $\mathbb{D}^2\times S^1$ of the boundary. Finally, we take the union of all these thickened disks. Call this monster a Casson handle.

<u>Miracle</u>. (Freedman's theorem on Casson handles. 1981)

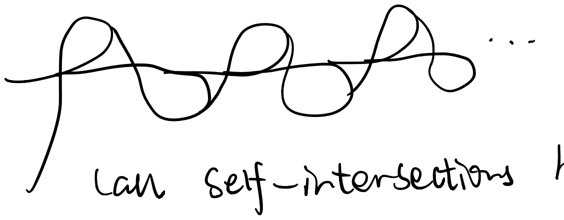
Any Casson handle is homeomorphic a thickened disk $\mathbb{D}^2\times\mathbb{R}^2$, and thus is a genuine 2-handle (open). Its core is a topologically embedded 2-disk.

Thus, by pushing our problems away to infinity, we actually solved the problem!

Cor. (Topological h-cobordism theorem for 4-manifolds)

Let W^5 be an h-cobordism between M^4 and N^4, everybody simply-connected. Then there is a homeomorphism $W \cong M\times[0,1]$. In particular, M and N are homeomorphic.

① The smooth version of 4-dim h-cobordism theorem fails. Since there are exotic (homeo but not diffeo) Casson handles.



(all self-intersections have the same sign)

② If the manifolds do not have a smooth structure, we can still apply the theorem if we have a topological handle decomposition. This indeed works, but the existence of topological handles is nontrivial. Recall we used to get smooth handles from Morse functions. Fact about special 4 case: Any such handle decomposition can be deformed so that is attached via diffeomorphisms. Thus in dim 4. $\exists$ handle decomposition $\Leftrightarrow$ $\exists$ smooth structure. In other dimensions, handle decom. always exist.

Cor. (Topological 4-dim Poincaré Conjecture) If a 4-mfd Σ^4 is homotopic equivalent to S^4, then $\Sigma \cong S^4$.

③ The smth version of this is open. We do not know if there exist exotic 4-spheres.

Possible counter-examples: Let S be a 2-sphere embedded in S^4. We choose a tubular nbhd $S \times D^2$ of it, and delete it from S^4. Then glue it using a boundary diffeo. $S \times S^1$. The resulting thing is $\simeq S^4$, hence $\cong S^4$.

Also, if S is unknotted, it is $\approx S^4$.

If S comes from spinning a knotted circle in 4th dimension, also ditto. If S is knotted more weirdly, may be exotic.