CHARACTERISTIC CLASSES AND SPIN STRUCTURE

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Contents

1 Characteristic classes

"A characteristic class is a way of associating to each principal bundle of X a cohomology class of X. The cohomology class measures the extent the bundle is 'twisted' and whether it possesses sections. Characteristic classes are global invariants that measure the deviation of a local product structure from a global product structure. They are one of the unifying geometric concepts in algebraic topology, differential geometry, and algebraic geometry."
–Wikipedia

Definition 1.0.1. *A characteristic class c of a vector bundle or a principal G-bundle is an assignment to each bundle a class in the cohomology ring of the base space that is natural: if $f : N \to M$ is a map, then $c(f^*E) = f^*(c(E)) \in H^*(N)$, where E is a bundle on M.*

1.1 The Stiefel-Whitney classes

Definition 1.1.1. *The Stiefel-Whitney classes are characteristic classes for a real vector bundle $E \to M$ (may not be orientable). For each $i \geqslant 0$ the i^{th} Stiefel-Whitney class $w_i(E) \in H^i(M, \mathbb{Z}/2)$. The total Chern class $w(E) := w_0(E) + w_1(E) + \cdots$. The Stiefel-Whitney class of M, $w_i(M)$, is defined to be $w_i(TM)$. Stiefel-Whitney classes satisfy the following condition:*

(1) $w_0(E) = 1$.

(2) *The Whitney sum formula: $w(E \oplus F) = w(E)w(F)$. Hence,*

$$w_k(E \oplus F) = \sum_{i+j=k} w_i(E)w_j(E)$$

(3) *Let x be the generator of $H^2(\mathbb{RP}^n, \mathbb{Z}/2) \cong \mathbb{Z}/2$. Then $c(H) = 1 - x$ where H is the tautological bundle of $\mathbb{RP}^2$.*

Before showing that the Stiefel-Whitney classes exist and are unique, let's see some consequences of Stiefel-Whitney classes.

Example 1.1.2. (1) *Let $\epsilon_n \to M$ be the trivial bundle of rank n. Then $w(\epsilon_n) = 1$ since ϵ_n is the pull-back of the trivial bundle over a point.*

(2) *$w(E \oplus \epsilon_n) = w(E)w(\epsilon_n) = w(E)$. Thus if E is stably trivial, then $w(E) = 1$. Hence the total Chern class can tell us a necessary condition for a bundle to be stably trivial.*

(3) *$T\mathbb{RP}^n \oplus \epsilon_1 \cong H^{*\oplus n+1}$ implies $w(\mathbb{RP}^n) = (1 + x)^{n+1}$.*

(4) *If E is a real vector bundle of rank n with a Riemannian metric, which possesses k cross-sections which are nowhere linearly dependent, then*

$$w_{n-k+1}(E) = w_{n-k+2}(E) = \cdots = w_n(E) = 0.$$

For E splits as a whitney sum $\epsilon_k \oplus \epsilon_k^\perp$

There is another way to think about the characteristic class. Given a real bundle $E \to M$ of rank n with a Riemannian metric, we get a principal O_n-bundle, hence a classifying map $f_E : M \to B_{O_n}$. If $a \in H^*(B_{O_n})$, then let $a(E) := f_E^*(a)$. Then $a(E)$ is a characteristic class. On the other hand, all characteristic classes for E arise this way since all principal O_n-bundle are pull-backs of the universal bundle $E_{O_n} \to B_{O_n}$ by Theorem A.0.9. Namely, a characteristic class is a cohomology class of the classifying space.

Now, let's define Stiefel-Whitney classes by classifying space. First, the map

$$O_n \hookrightarrow O_{n+1}, \; A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

induces a map

$$B_{O_n} \to B_{O_{n+1}}$$

This is a direct system so we can take the direct limit

$$B_O := \varinjlim B_{O_n}$$

In fact B_{O_n} has a concrete expression: $Gr_n(\mathbb{R}^\infty)$ (see Example A.0.5 (5)).

By taking the direct limit of Grassmannian, it can be computed that

$$H^*(B_O) \cong \mathbb{Z}[w_1, w_2, \cdots], \; with \; |w_k| = k.$$

Then the k^{th} Stiefel-Whitney class $w_k(E)$ can be defined to be $f_E^*(w_k)$ where f_E is the classifying map of E.

Remark 1.1.3. *This approach allows us to define the Stiefel-Whitney classes for a principale G-bundle, not just vector bundles.*

It is easy to see that the axioms $(1), (3)$ are satisfied. To show the existence, only need to show the Whitney sum formula: $w(E \oplus F) = w(E)w(F)$.

Proof. First note that

$$B_{O_n \times O_m} \simeq B_{O_n} \times B_{O_m} \quad (*).$$

Indeed, by taking the product of uniforsal bundles for O_n and O_m, we get a $O_n \times O_m$-bundle over $B_{O_n} \times B_{O_m}$, with total space $E_{O_n} \times E_{O_m}$:

$$O_n \times O_m \hookrightarrow E_{O_n} \times E_{O_m} \to B_{O_n} \times B_{O_m} \quad (**).$$

Since $\pi_i(E_{O_n} \times E_{O_m}) \cong \pi_i(E_{O_n}) \times \pi_i(E_{O_m}) \cong 0 \; \forall \; i$, it follows that $(**)$ is the universal bundle for $O_n \times O_m$, thus proving $(*)$ by Proposition A.0.11.

Next, the inclusion $O_n \times O_m \hookrightarrow O_{n+m}$ yields a map

$$\phi : B_{O_n \times O_m} \simeq B_{O_n} \times B_{O_m} \to B_{O_{n+m}}.$$

By Künneth formula, one can show that:

$$\phi^* w_k = \sum_{i+j=k} w_i \times w_j.$$

Therefore,

$$\begin{aligned} w_k(E \oplus F) =& w_k(\Delta^*(E \times F)) \\ =& \Delta^* w_k(E \times F) \\ =& \Delta^*(f^*_{E\times F}(w_k)) \\ =& \Delta(f^*_E \times f^*_F)(\phi^* w_k) \\ =& \sum_{i+j=k} \Delta^*(f^*_E(w_i) \times f^*_F(w_j)) \\ =& \sum_{i+j=k} \Delta^*(w_i(E) \times w_j(F)) \\ =& \sum_{i+j=k} w_i(E) w_j(F). \end{aligned}$$

Here, we use the fact that the classifying map for $E \times F$, regarded as an O_{n+m}-bundle is $\phi \circ (f_E \times f_F)$.

□

For the uniqueness of the Stiefel-Whitney class, suppose there are two Stiefel-Whitney classes w and $\tilde{w}$. Then $w(H) = \tilde{w}(H)$. Thus, $w(H \times \cdots \times H) = \tilde{w}(H \times \cdots \times H)$. Now using the existence of a bundle map $H \times \cdots \times H \to H(n)$ (where $H(n)$ is the tautological bundle of $Gr_n(\mathbb{R}^\infty)$) and the fact that $H^*(Gr_n(\mathbb{R}^\infty))$ injects monomorphicaly into $H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty)$, it follows that $w(H(n)) = \tilde{w}(H(n))$.

$\forall$ bundle $E \to B$, choose a classifying map $f_E : B \to Gr_n(\mathbb{R}^\infty)$. It follows that

$$w(E) = f^*_E(w(H(n))) = f^*_E((H(n))) = \tilde{w}(E).$$

Theorem 1.1.4. *If $E \to B$ is a vector bundle of rank n over a CW complex B, then $w_k(E)$ measures the obstruction to finding a field of $n-k+1$ linearly independent vectors over the k-skeleton of B.*

Proof. See [3, page 140-142]. □

Although we don't prove it here, we can see the motivation by considering the low-dimensional case.

For a vector bundle $E \to B$ with B path-connected, orientability is detected by the homomorphism $\pi_1(B) \to \mathbb{Z}/2$ that assigns 0 or 1 to each loop according to whether orientations of fibers are preserved or reversed as one goes around the loop. Since $\mathbb{Z}/2$ is abelian, this homomorphism factors through the abelianization $H_1(B)$ of $\pi_1(B)$, and homomorphisms $H_1(B, \mathbb{Z}/2)$ are identifiable with elements of $H^1(B, \mathbb{Z}/2)$. Thus we have an element of $H^1(B, \mathbb{Z}/2)$ associated to E which is zero exactly when E is orientable. This is exactly $w_1(E)$. If B is a CW complex, then E is orientable iff its restriction to the 1-skeleton K^1 of B is orientable, since all loops in B can be deformed to lie in K^1. Futhermore, a vector bundle over a 1-dimensional CW complex is trivial iff it is orientable, so we can also say that $w_1(E)$ measures whether E is trivial over the 1-skeleton K^1.

What about the higher skeleton? Assuming that E is trivial over K^1, then if the fibers have dimension n we can choose n orthonormal sections over K^1, and so we ask whether these sections extend to orthonormal sections over each 2-cell. If we pull E back to a vector bundle over the disk D^2 via a characteristic map $D^2 \to B$ for a 2-cell, the sections over K^1 pull back to sections of the pullback over ∂D^2. The pullback bundle is trivial since D^2 is contractible, so by choosing a trivializaion the sections over ∂D^2 determine a map $\partial D^2 \to O_n$. If we choose the trivialization to give the same orientation as the trivialization of E over K^1 determined by the sections, we can take this map to have image in SO_n. Thus we have an element of $\pi_1(SO_n)$ for each 2-cell, and it is not hard to see that the sections over K^1 extend to orthonormal sections over K^2 iff this element of $\pi_1(SO_n)$ is zero for each 2-cell. The group $\pi_1(SO_n)$ is 0 for $n = 1$, and it is $\mathbb{Z}$ for $n = 2$. For $n > 2$, $\pi_1(SO_n) = \mathbb{Z}/2$. Thus for any case, the coefficient can be reduced to $\mathbb{Z}/2$.

Definition 1.1.5. *For a compact manifold M, there exists one and only one cohomology class (caled Wu class)*

$$v_k \in H^k(M)$$

which satisfies the identity

$$\langle v_k \cup x, [M] \rangle = \langle Sq^k(x), [M] \rangle$$

for every x. The total Wu class is defined to be

$$v = 1 + v_1 + \cdots + v_n$$

.

Theorem 1.1.6 (Wu's formula)**.** *The total Stiefel-Whitney class w is equal to $Sq(v)$. In other words*

$$w_k = \sum_{i+j=k} Sq^i(v_j).$$

Proof. See [3, page 132-133]. □

If $E^r \to M^n$ is oriented, then there is a class $e(E) \in H^r(M, \mathbb{Z})$, called the Euler class, s.t. $w_r(E) = e(E)\ mod\ 2$. Explicitly, the Euler class can be defined as follows:

Definition 1.1.7. *Suppose $\sigma : M \to E$ is a smooth section that transversly intersects the zero section. Let $Z \subset X$ be the zero locus of σ. Then Z is a codimension r submanifold of X which represents a homology class $[Z] \in H_{n-r}(M, \mathbb{Z})$ and the Euler class $e(E)$ is defined to be the Poincaré dual of $[Z]$.*

By obstruction theory, it is easy to see that $w_n \equiv e \bmod 2$.

Theorem 1.1.8. *If M is a smooth compact oriented manifold, then*

$$\langle e(M), [M] \rangle = \chi(M),$$

where $\chi(M)$ is the Euler character.

Proof. See [3, page 130]. □

1.2 The Chern classes

Definition 1.2.1. *The Chern classes are characteristic classes for a complex vector bundle $E \to M$. For each $i \geqslant 0$ the i^{th} Chern class $c_i(E) \in H^{2i}(M, \mathbb{Z})$. The total Chern class $c(E) := c_0(E) + c_1(E) + \cdots$. The Chern class of M, $c_i(M)$, is defined to be $c_i(TM)$. Chern classes satisfy the following condition:*

(1) *$c_0(E) = 1$.*

(2) *The Whitney sum formula: $c(E \oplus F) = c(E)c(F)$. Hence,*

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E)c_j(E)$$

(3) *Let x be the generator of $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$. Then $c(H) = 1 - x$ where H is the tautological bundle of $\mathbb{CP}^2$.*

Example 1.2.2. (1) *Let $\epsilon_n \to M$ be the trivial bundle of rank n. Then $c(\epsilon_n) = 1$ since ϵ_n is the pull-back of the trivial bundle over a point.*

(2) *$c(E \oplus \epsilon_n) = c(E)c(\epsilon_n) = c(E)$. Thus if E is stably trivial, then $c(E) = 1$. Hence the total Chern class can tell us a necessary condition for a bundle to be stably trivial.*

(3) *$T\mathbb{CP}^n \oplus \epsilon_1 \cong H^{*\oplus n+1}$ implies $c(\mathbb{CP}^n) = (1 + x)^{n+1}$.*

There is another way to think about the characteristic class. Given a complex bundle $E \to M$ of rank n, we get a principal U_n-bundle, hence a classifying map $f_E : M \to B_{U_n}$. If $a \in H^*(B_{U_n})$, then let $a(E) := f_E^*(a)$. Then $a(E)$ is a characteristic class. On the other hand, all characteristic classes for E arise this way since all principal U_n-bundle are pull-backs of the universal bundle $E_{U_n} \to B_{U_n}$ by Theorem A.0.9. Namely, a characteristic class is a cohomology class of the classifying space.

Now, let's define Chern classes by classifying space. First, the map

$$U_n \hookrightarrow U_{n+1},\ A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

induces a map

$$B_{U_n} \to B_{U_{n+1}}$$

This is a direct system so we can take the direct limit

$$B_U := \varinjlim B_{U_n}$$

In fact B_{U_n} has a concrete expression: $Gr_n(\mathbb{C}^\infty)$ (see Example A.0.5 (5)).

By taking the direct limit of Grassmannian, it can be computed that

$$H^*(B_U) \cong \mathbb{Z}[c_1, c_2, \cdots],\ with\ |c_k| = 2k.$$

Then the k^{th} Chern class $c_k(E)$ can be defined to be $f_E^*(c_k)$ where f_E is the classifying map of E.

Remark 1.2.3. *This approach allows us to define the Chern classes for a principale G-bundle, not just vector bundles.*

It is easy to see that the axioms $(1), (3)$ are satisfied. To show the existence, only need to show the Whitney sum formula: $c(E \oplus F) = c(E)c(F)$.

Proof. First note that

$$B_{U_n \times U_m} \simeq B_{U_n} \times B_{U_m} \quad (*).$$

Indeed, by taking the product of uniforsal bundles for U_n and U_m, we get a $U_n \times U_m$-bundle over $B_{U_n} \times B_{U_m}$, with total space $E_{U_n} \times E_{U_m}$:

$$U_n \times U_m \hookrightarrow E_{U_n} \times E_{U_m} \to B_{U_n} \times B_{U_m} \quad (**).$$

Since $\pi_i(E_{U_n} \times E_{U_m}) \cong \pi_i(E_{U_n}) \times \pi_i(E_{U_m}) \cong 0\ \forall\ i$, it follows that $(**)$ is the universal bundle for $U_n \times U_m$, thus proving $(*)$ by Proposition A.0.11.

Next, the inclusion $U_n \times U_m \hookrightarrow U_{n+m}$ yields a map

$$\phi : B_{U_n \times U_m} \simeq B_{U_n} \times B_{U_m} \to B_{U_{n+m}}.$$

By Künneth formula, one can show that:

$$\phi^* w_k = \sum_{i+j=k} w_i \times w_j.$$

Therefore,

$$\begin{aligned}
c_k(E \oplus F) =& c_k(\Delta^*(E \times F)) \\
=& \Delta^* c_k(E \times F) \\
=& \Delta^*(f^*_{E\times F}(c_k)) \\
=& \Delta(f^*_E \times f^*_F)(\phi^* c_k) \\
=& \sum_{i+j=k} \Delta^*(f^*_E(c_i) \times f^*_F(c_j)) \\
=& \sum_{i+j=k} \Delta^*(c_i(E) \times c_j(F)) \\
=& \sum_{i+j=k} c_i(E)c_j(F).
\end{aligned}$$

Here, we use the fact that the classifying map for $E \times F$, regarded as an U_{n+m}-bundle is $\phi \circ (f_E \times f_F)$. □

For the uniqueness of the Chern class, suppose there are two Chern classes c and $\tilde{c}$. Then $c(H) = \tilde{c}(H)$. Thus, $c(H \times \cdots \times H) = \tilde{c}(H \times \cdots \times H)$. Now using the existence of a bundle map $H \times \cdots \times H \to H(n)$ (where $H(n)$ is the tautological bundle of $Gr_n(\mathbb{C}^\infty)$) and the fact that $H^*(Gr_n(\mathbb{C}^\infty))$ injects monomorphicaly into $H^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty)$, it follows that $c(H(n)) = \tilde{c}(H(n))$.

$\forall$ bundle $E \to B$, choose a classifying map $f_E : B \to Gr_n(\mathbb{C}^\infty)$. It follows that

$$c(E) = f^*_E(c(H(n))) = f^*_E((H(n))) = \tilde{c}(E).$$

Definition 1.2.4. *Given a real vector bundle E over M, the k^{th} Pontryjagin class $p_k(E)$ is defined to be*

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M, \mathbb{Z})$$

Proposition 1.2.5. *For a complex bundle E, its Pontrjagin classes are totally determined by the Chern classes by:*

$$1-p_1(E)+p_2(E)-\cdots+(-1)^n p_n(E) = (1+c_1(E)+\cdots+c_n(E))(1-c_1(E)+\cdots+(-1)^n c_n(E)).$$

Proof. Note that

$$E \otimes \mathbb{C} \cong E \oplus \sqrt{-1}E \cong E \oplus \bar{E},$$

and use the Whitney formula. □

Example 1.2.6. (1) $p_1 = c_1^2 - 2c_2$.

(2) $p_2 = 2c_4 - 2c_1c_3 + c_2^2$.

Definition 1.2.7. *Let E be a complex bundle, the Chern polynomial c_t of E is given by:*

$$c_t(E) := 1 + c_1(E)t + \cdots + c_n(E)t^n$$

If we use the axiomatic approach to define the Chern class, then it is easy to see thath the Chern polynomial satisfies the Whitney sum formula:

$$c_t(E \oplus F) = c_t(E)c_t(F)$$

If $E = L_1 \oplus \cdots \oplus L_n$ is a direct sum of complex line bundles, then it follows from the Whitney sum formula that

$$c_t(E) = (1 + \gamma_1(E)t) \cdots (1 + \gamma_n(E)t)$$

where $\gamma_i(E) = c_1(L_i)$. $\gamma_i(E)$ is called the Chern root of E, which determine the coefficients of the Chern polynomial:

$$c_k(E) = e_k(\gamma_1(E), \cdots, \gamma_n(E))$$

where e_k is the k^{th} elementary symmetric polynomial.

Definition 1.2.8. *The Chern character of a complex bundle $E \to X$ is defined to be*

$$\text{ch}(E) := e^{\gamma} + \cdots + e^{\gamma} \in H^*(X, \mathbb{Q})$$

where a_i are Chern roots.

Equivalently, by Chern-Weil theory,

$$\text{ch}(E) = tr\left(exp\left(\frac{i\Omega}{2\pi}\right)\right)$$

where Ω is the curvature matrix.

The Chern character satisfies

$$\text{ch}(E \oplus F) = \text{ch}(E) + \text{ch}(F)$$

$$\text{ch}(E \otimes F) = \text{ch}(E)\text{ch}(F)$$

Definition 1.2.9. *Let (M, J) be a almost complex manifold , $\lambda = [\lambda_1, \cdots, \lambda_k]$ be a partition of n. The Chern number of (M, J) with respect to λ is defined to be*

$$C_\lambda(M) := \int_M c_{\lambda_1} \cdots c_{\lambda_k} \in \mathbb{Z}.$$

where the integral of a characteristic class is the integral of its associated differential form class by de Rham theorem

$$H^n(M, \mathbb{R}) \cong H^n_{dR}(M)$$

2 Spin structure

Recall that for an oriented vector bundle E of rank n, there is an associated principal SO_n-bundle, which we denote by $P_{SO}(E)$. As $\pi_1(SO_n) = \mathbb{Z}/2$ for $n \geqslant 3$, there is a simply connected 2-fold covering of SO_n, known as $Spin(n)$. For $n = 2$, we take $Spin(n)$ to be the circle, double covering itself. As covering of Lie groups are Lie groups, $Spin(n)$ itself is a Lie group. If we can lift the structure group to $Spin(n)$, then we say that our oriented bundle E is spinnable.

Definition 2.0.1. *For a spinnable vector bundle $E \to B$, a spin structure $P_{spin}(E) \to B$ is a principal $Spin(n)$-bundle that nontrivially and equivariantly double covers $P_{SO}(E)$ on each fiber, and the covering map is a bundle morphism.*

Remark 2.0.2. *Two spin structures may be equivalent as bundles, but not equivalent as spin structures. Consider spin structures over S^1. We will see later that spin structures are classified up to equivalence by $H^1(B, \mathbb{Z}/2)$, so there are two distinct spin structures over S^1. However, any principal G-bundle the circle with G connected is bundle isomorphic to the trivial bundle, since such bundles are classified up to isomorphism by homotopy class of maps $[S^1 \to B_G] = \pi_1(B_G) = \pi_0(B) = 0$.*

Theorem 2.0.3. *Let X be a CW complex, $F \to E \to B$ is a fibration. Let $f : X \to B$ be a map and $g : X^n \to E$ a lift of f on the n-skeleton. If F is n-connected, then an obstruction class*

$$[\theta^{n+1}(g)] \in H^{n+1}(X, \pi_n(F))$$

is defined.

If $[\theta^{n+1}(g)]$ vanishes, then g can be redefined over the n-skeleton, then exteded over the $(n+1)$-skeleton X^{n+1}.

Proof. See [4, page 197-202.] □

Proposition 2.0.4. *$E \to B$ is spinnable iff there is a lift of the classifying map $B \to B_{SO_n}$ to a map $B \to B_{Spin(n)}$. Thus, the only obstruction to E being spinnable is the second Stiefel-Whitney class $w_2(E)$.*

The last sentence follows from the fact that the fiber on the right hand sid is $\mathbb{RP}^\infty$ so the only obstruction to extending a section is in $H^2(B_{SO_n}, \pi_1(\mathbb{RP}^\infty)) = H^2(B_{SO_n}, \mathbb{Z}/2)$, which is just $w_2(E)$.

Recall that to trivialize a G-bundle is to find a section of this bundle. Suppose E is spinnable and $\mathcal{P}_{Spin(k)}$ is the associated principal $Spin(k)$-bundle. Then to trivialize E over 2-skeleton, we only need to find a section of $\mathcal{P}_{Spin(k)}$ and then project it to the sections of E. Since $Spin(k)$ is simply-connected, this can be done by the Theorem 2.0.3. Hence, for each spin structure of E, we can find a trivialization of E over 1-skeleton which extends to 2-skeleton. The converse is also true (see [4, page 183-189].

Proposition 2.0.5. *If $E \to B$ is spinnable, then the group $H^1(B, \mathbb{Z}/2)$ acts freely and transitively on the set of equivalence classes of spin structures of E over a fixed principal SO_n-bundle.*

Proof. We can first identify $H^1(B, \mathbb{Z}/2)$ with Čech cohomology, where 1-cocycles α_{ij} : $U_i \cap U_j \to \mathbb{Z}/2$ are defined on a cover of local trivializations of a given spin structure with transition functions $g_{ij} : U_i \cap U_j \to Spin(n)$. Given a 1-cocycle α and a spin structure determined by functions g_{ij}, we can define new transition function h_{ij} on $U_i \cap U_j$ that determine an inequivalent spin structure over the same SO_n-bundle by setting

$$h_{ij} = \begin{cases} g_{ij} \ if \ \alpha_{ij} = 0 \\ -g_{ij} \ if \ \alpha_{ij} = 1 \end{cases}$$

Freeness follows immediately, while transitivity follows from the fact that both spin structures are double covers of the same SO_n-bundle, so that any such two differ by multiplication by -1 in the fiber over appropriate trivializations.

□

Example 2.0.6. (1) *Every parallelizable manifold is clearly spinnable, such as Lie groups, compact orientable* 3*-manifolds, etc.*

(2) *Recall that $w(\mathbb{RP}^n) = (1+x)^{n+1}$. Since we requre both w_1 and w_2 to vanish, this forces n to be odd and $\frac{n(n+1)}{2}$ to be even. Thus $\mathbb{RP}^n$ is spinnable iff $n \equiv 3$ mod* 4.

(3) *Recall that a K_3 surface X has the trivial canonical bundle. Thus, $w_2(X) = c_1(X)$ mod $2 = -c_1(K_X)$ mod $2 = 0$. Hence, every K_3 surface is spinnable.*

3 Applications on 4-dimensional cases

Theorem 3.0.1 (Dold-Whitney theorem)**.** *If two oriented bundles of rank 4 over an oriented 4-manifold have the same second Whitney-Stiefel class, first Pontrjagin class, and Euler class, then they must be isomorphic.*

Recall that given a real vector bundle E over M, the k^{th} Pontrjagin class $p_k(E)$ is defined to be

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M, \mathbb{Z}).$$

Let

$$Q(x) := \frac{x}{tanh(x)} = \sum_{k \geqslant 0} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k} = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \cdots .$$

be a formal power series, where B_i is the i^{th} Bernoulli number.

Theorem 3.0.2 (Hirzebruch's signature theorem)**.** *Let M^{4k} be a closed almost complex manifold. The L-genus is defined to be*

$$L(M) := \prod_i Q(\gamma_i),$$

where γ_i are Chern roots of $TM \otimes \mathbb{C}$. Then

$$\mathrm{sign}(M) = \langle L(M), [M] \rangle = \int_M L(M).$$

Proof. The theorem can be shown by Atiyah-Singer index theorem (but we won't state the index theorem here): Let $D = \Delta = d + d^*$. Define

$$\tau := \sqrt{-1}^{r(r-1)+\frac{n}{2}} * : \mathcal{A}^r(M) \to \mathcal{A}^{n-r}(M),$$

where $*$ is the Hodge star, which satisfies $\tau^2 = Id$ and $\tau D + D\tau = 0$. Then $\mathcal{A}^*(M) = \bigoplus \mathcal{A}^r(M)$ can be decomposed as

$$\mathcal{A}^*(M) = \mathcal{A}^(M) + \oplus \mathcal{A}^-(M)$$

with respect to the eigenvalues $+1$ and -1 respectively. The anti-commutativity $\tau D = -D\tau$ implies that we can define a restriction D^+, and its dual D^-, of D given by

$$D^+ : \mathcal{A}^+(M) \to \mathcal{A}^-(M),$$

$$D^- : \mathcal{A}^-(M) \to \mathcal{A}^+(M).$$

Then the analytical index of D^+ is

$$\mathrm{a} - \mathrm{Ind}(D) = \mathrm{sign}(M)$$

the topological index is

$$\mathrm{t-Ind}(D) = \int_M \mathrm{ch}(D^+)\mathrm{td}(M) = \int_M L(M)$$

Then we get the result. □

Corollary 3.0.3. *For every closed* 4*-manifold* M*, we have*

$$\langle p_1(M), [M]\rangle = 3\mathrm{sign}(M).$$

Proof. By Hirzebruch's signatrure theorem 3.0.2, we have

$$\begin{aligned}
\mathrm{sign}(M) =& \int_M L(M) \\
=& \frac{1}{3}\int_M \sum \gamma_i^2 \\
=& \frac{1}{3}\int_M ((\sum \gamma_i)^2 - 2\sum_{i<j} \gamma_i\gamma_j) \\
=& \frac{1}{3}\int_M (c_1(M)^2 - 2c_2(M)) \quad \textit{by Example } 1.2.6 \\
=& \frac{1}{3}\int_M p_1(M).
\end{aligned}$$

□

Remark 3.0.4. *The signature theorem is also true for non-almost-complex* $4k$*-manifolds, but we need some modifications:*

First, we can define a genus with respect to a formal power series in terms of the Pontrjagin classes by multiplicative series (See [3]).

Next, by this method, the L*-genus corresponds to the formal power series*

$$Q(x) = \frac{\sqrt{x}}{tanh(\sqrt{x})}$$

rather than

$$Q(x) = \frac{x}{tanh(x)}.$$

The statements of the theorem and the corollary will be the same as before.

Proposition 3.0.5. *Let* M *be a* 4*-dimensional manifold. Then* $\forall$ *oriented surfaces* S *embedded in* M*, we have:*

$$w_2 \cdot S \equiv S \cdot S \ mod\ 2.$$

Proof. By Wu's formula 1.1.6,

$$\begin{aligned} w_2 =& Sq^0(V_2) + 2Sq^1(v_1) + Sq^2(v_0) \\ =& v_2 + 2v_1^2 \\ \equiv& v_2 \ mod \ 2 \end{aligned}$$

Let $PD : H^k(M) \to H_{n-k}(M)$ be the Poincaré dual. Then

$$\begin{aligned} w_2 \cdot S =& \langle PD^{-1}([S]), PD^{-1}([S]) \cap [M] \rangle \\ =& \langle PD^{-1}([S]) \cup PD^{-1}([S]), [M] \rangle \\ =& \langle Sq^2([S]), [M] \rangle \\ \equiv& \langle v_2, [S] \rangle \\ =& \langle PD^{-1}([S]) \cup PD^{-1}([S]), [M] \rangle \\ =& S \cdot S \end{aligned}$$

□

Corollary 3.0.6. *The intersection form of a spin* 4*-manifold is even.*

Conversely, if the intersection form of M is even, and $H_1(M)$ has no 2-torsion in addition, then $w_2(M) = 0$.

Corollary 3.0.7. *Any* 4*-manifold without 2-torsion, for example simply-connected, admits spin structures iff its intersection form is even.*

Recall how we define cobordism groups of oriented closed manifolds. Now we want to restrict the cobordism to the equivalence set of closed spin manifolds. The following theorem will help us to calculate some spin-cobordism groups.

Theorem 3.0.8 (V. Rokhlin)**.** *If a closed spin* 4*-manifold M has zero signature, then M bounds a spin* 5*-manifold whose spin structure induces the spin structure of M.*

There is a discussion on how the Hirzebruch's signature theorem implies this theorem on [4, page 167], but I have not understand yet.

Corollary 3.0.9. *If two spin* 4*-manifolds M and N have the same signature, then they can be linked by a cobordism that is a spin* 5*-manifold, and its spin structure induces on M and N their respective spin structures.*

Theorem 3.0.10 (V. Rokhlin)**.** *The signature of a spin* 4*-manifold must be divided by* 16.

This theorem will be discussed by RUAN Xiabing next time.

Corollary 3.0.11. $\Omega_4^{Spin} \cong \mathbb{Z}$*, and a* K_3 *surface is its generator.*

Proof. By the Corollary 3.0.9 and the Theorem 3.0.10, only need to show the signature of a K_3 surface S is ± 16.

Recall that the intersection form of a K_3 surface is given by

$$Q = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H.$$

The dimension of the positive and negative eigenspaces are 3 and 19 respectively. Thus, the signature is -16. $\square$

A Universal bundles and classifying spaces

Every fiber bundle can be pulled back by a continuous maps. After being pulled back, the bundle carries less information than before. So here is a question: Does there exist "the most complicated bundle" s.t. every bundle is a pull-back of this bundle? Under some circumstances, the answer is "yes".

Definition A.0.1 (Principal bundles). *Let G be a topological group. If the continuous map $p : E \to B$ from a G-space E to a topological space B satisfies the following conditions, then (E, B, p) (sometimes denoted as E only) is called a principal G-bundle.*

$\exists$ a countable open covering $\{U_i\}_{i \in I}$ of B and homeomorphisms $\phi_i : U_i \times G \to p^{-1}(U_i)$ satisfying that $\forall\ b \in U_i$ and $g, h \in G$

(1) $p \circ \phi_i(b, g) = b$

(2) $\phi_i(b, gh) = g \cdot \phi_i(b, h)$

Such (U_i, ϕ_i) is called a trivialization and E is called the total space.

Definition A.0.2. *A principal bundle is called universal if the total space is weakly contractible. i.e. every homotopy group of the total space is* 0.

The universal bundle with respect to the topological group G is always denoted as $E_G \to B_G$. B_G is called the classifying space with respect to G.

Remark A.0.3. (1) *This is not the origin definition but it is convenient. Later we will see why it is called "universal".*

(2) *The Whitehead theorem says that if f is a continuous map between CW-complexes X, Y inducing isomorphisms on all homotopy groups, then f is a homotopy equivalence. Thus if a CW-complex is weakly contractible, then it is contractible.*

For an arbitrary topological group G, dose there exist a universal principal G-bundle?

Theorem A.0.4 (Milnor). *Every topological group has a universal bundle.*

Proof. Milnor constructed E_G directly as follows: Define the join of two spaces X,Y as $X \times Y \times [0, 1]/\sim$ where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ (See the figure 2), and it is denoted as $X * Y$. Just let $E_G := G * G * \cdots * G * \cdots$, $B_G := E_G/G$. G has a natural action on E_G and E_G is weakly contractible. Hence $E_G \to B_G$ is a universal bundle with respect to G. □

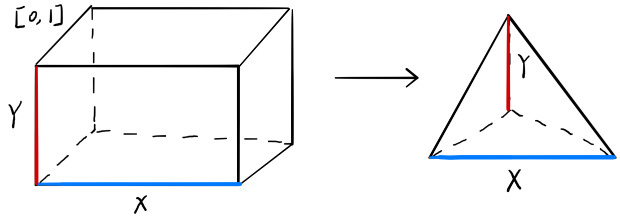

Figure 1: $X \times Y \times [0,1] \to X * Y$

Example A.0.5. (1) *Let $G = \{e\}$, then $pt \to pt$ is the associated universal bundle.*

(2) *Let $G = \mathbb{C}^*$. G acts on $\mathbb{C}^n \backslash \{0\}$ by*

$$c \cdot (x_1, \cdots, x_n) = (cx_1, \cdots, cx_n)$$

Then $\mathbb{C}^n \backslash \{0\} \to \mathbb{CP}^{n-1}$ is a principal G-bundle.

However, this is not a universal bundle, since $\pi_{2n-1}(\mathbb{C}^n \backslash \{0\}) \neq 0$, although the previous homotopy goups are all 0. How to fix this problem? Note $\mathbb{C}^n \backslash \{0\} \hookrightarrow \mathbb{C}^{n+1} \backslash \{0\}$ $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0)$ is G-equivariant. Therefore we get inclusions of G-bundles:

$$\cdots \hookrightarrow \mathbb{C}^n \backslash \{0\} \hookrightarrow \mathbb{C}^{n+1} \backslash \{0\} \hookrightarrow \cdots$$

$$\downarrow \qquad\qquad \downarrow$$

$$\cdots \hookrightarrow \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n \hookrightarrow \cdots$$

This forms a direct system. Take the direct limit. Then we get

$$\mathbb{C}^\infty \backslash \{0\} \to \mathbb{CP}^\infty$$

is a universal principal G-bundle, since $\mathbb{C}^\infty \backslash \{0\}$ is weakly contractible.

(3) *Let $G = S^1$. G acts on S^{2n+1} in the similar way as that in (2). Then $S^{2n+1} \to \mathbb{CP}^n$ is a principal G-bundle. S^{2n+1} is k-connected whenever $k < 2n+1$ but it is not $2n+1$-connected, so this is not a universal bundle. Using the similar method to fix it, we get*

$$S^\infty \to \mathbb{CP}^\infty$$

is the universal bundle with respect to S^1.

(4) *Let $G = GL_n(\mathbb{C})$, $M_{m,n}^{full}$ denote the space of $m \times n$ matrices of full rank and $Gr_n(\mathbb{C}^m)$ denote the Grassmannian. Then G has a natural action on $M_{m,n}^{full}$.*

$$M_{m,n}^{full} \to Gr_n(\mathbb{C}^m), \ [v_1, \cdots, v_n] \to [span\{v_1, \cdots, v_n\}]$$

is a principal G-bundle. It can be shown that $M_{m,n}^{full}$ is k-connected whenever $k \leq m - n$. Hence

$$M_{\infty,n}^{full} \to Gr_n(\mathbb{C}^\infty)$$

is the universal bundle with respect to $GL_n(\mathbb{C})$.

(5) *Let* $G = U_n$. *Then*

$$E_G = \{(e_1, \cdots, e_n) | \langle e_i, e_j \rangle = \delta_{ij},\ e_i \in \mathbb{C}^\infty\}$$
$$B_G = E_G/G = \{V \subset \mathbb{C}^\infty | dimV = n\} = Gr_n(\mathbb{C}^\infty).$$

Similarly, let $G = O_n$. *Then*

$$E_G = \{(e_1, \cdots, e_n) | \langle e_i, e_j \rangle = \delta_{ij},\ e_i \in \mathbb{R}^\infty\}$$
$$B_G = E_G/G = \{V \subset \mathbb{R}^\infty | dimV = n\} = Gr_n(\mathbb{R}^\infty).$$

(6) *Let* G *be a Lie group and* H *is a closed Lie subgroup of* G. *If* $E_G \to B_G$ *is a universal bundle of* G, *then* $E_G \to E_G/H$ *is a universal bundle for* H.

Before explaining the properties of the universal bundles, we need to make some preparations.

Definition A.0.6. *If* X *is a right* G*-space and* Y *is a left* G*-space, the balanced product* $X \times_G Y$ *is the quotient space* $X \times Y/\sim$, *where* $(xg, y) \sim (x, gy)$. *Equivalently, we can regard* $X \times Y$ *as a right* G*-space:* $(x, y)g = (xg, g^{-1}y)$. *Then* $X \times_G Y = (X \times Y)/G$.

Lemma A.0.7. *Every fiber bundle with weakly contractible fibers admits a section.*

Proof. See [2], Lemma 4.0.1. □

Lemma A.0.8. *Given two principal* G*-bundles* $P \to B$ *and* $P' \to B'$. *There is a bijective correspondance between* $Mor_G(P, P')$ *and* $\Gamma(B, P \times_G P')$.

Proof. See [2], Corollary 4.0.1. □

Now, it is ready to explain what makes the universal bundle universal.

Theorem A.0.9. *Suppose* $E_G \to B_G$ *is a universal* G*-bundle. Then* $\forall$ *CW-complex* X, *the map*

$$[X, B_G] \to \mathcal{P}_G X,\ [f] \mapsto [f^* E_G]$$

is bijective, where $[X, B_G]$ *denotes the homotopic classes of continuous map from* X *to* B_G *and* $\mathcal{P}_G X$ *denotes the principal bundles over* X *up to isomorphisms.*

Proof. For surjectivity: Suppse $P \to X$ is a principal G-bundle. It is equivalent to finding a G-equivariant map $\phi : P \to E_G$ and putting $f : X \to B_G$ the induced map.

By the Lemma A.0.7, it suffices to find a section of the bundle $P \times_G E_G \to X$.

By the Lemma A.0.8, it suffices to show $P \times_G E_G$ is weakly contractible.

Since E_G is a universal bundle, it is weakly contractible. P is a principal G-bundle, which implies G acts on P transitively. Hence, $P \times_G E_G$ is weakly contractible.

The injectivity part requires a lot of preparations so it is omitted. One can find the proof in [2], Theorem 4.0.1. □

From this theorem, we know that every pincipal G-bundle is a pull-back of the universal bundle. B_G is called the classifying space for G. If $P \to X$ is a principal G-bundle, then any map $f : X \to B_G$ s.t. $P = f^*(E_G)$ is called a classifying map for P.

Lemma A.0.10. *Let $F \to E \to B$ be a fiber bundle. Then there is a long exact sequence of homotopy groups:*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Proof. See [1, Theorem 4.41, page 376]. □

Proposition A.0.11. *Let $E_G \to B_G$ be a universal bundle. Then*

(1) B_G can be taken to be a CW-complex. Henceforth, B_G's that appear below default to CW-complexes.

(2) E_G is unique up to homotopy equivalence.

Proof. (1) Let $\phi : B'_G \to B_G$ be a CW-approximation. Then we have the following commutative diagram:

$$\begin{array}{ccc} \phi^*(E_G) & \xrightarrow{p_1} & E_G \\ p_2 \downarrow & & \downarrow \\ B'_G & \longrightarrow & B_G \end{array}$$

Only need to show ϕ^* is weakly contractible. We have the following exact sequences and commutative diagram by the Lemma 2.1.10:

$$\begin{array}{ccccccccc} \cdots \to & \pi_n(G) & \to & \pi_n(E_G) & \to & \pi_n(B_G) & \to & \pi_{n-1}(G) & \to \cdots \\ & \uparrow \cong & & \uparrow p_{2*} & & \uparrow \phi_* & & \uparrow \cong & \\ \cdots \to & \pi_n(G) & \to & \pi_n(\phi^*(E_G)) & \to & \pi_n(B'_G) & \to & \pi_{n-1}(G) & \to \cdots \end{array}$$

Since ϕ is a CW-approximation, ϕ_* is an isomorphism. Thus p_{2*} is also an isomorphism by Five Lemma. Hence, $\forall\, n$,

$$\pi_n(\phi^*(E_G)) = \pi_n(E_G) = 0$$

i.e. $\phi^*(E_G)$ is weakly contractible.

(2) Choose two classifying maps

$$f : B'_G \to B_G,\ g : B_G \to B'_G$$

s.t.

$$E'_G \cong f^*(E_G),\ E_G \cong g^*(E'_G)$$

Then

$$f \circ g : B_G \to B_G$$

is a classifying map of E_G to itselfe since

$$(f \circ g)^*(E_G) \cong g^*(f^*(E_G)) \cong g^*(E'_G) \cong E_G$$

By the injectivity of the map $[X, B_G] \to \mathcal{P}_G X$ defined in Theorem A.0.9, we have

$$f \circ g \simeq Id_{B_G}$$

Similarly,

$$g \circ f \simeq Id_{B'_G}$$

Thus, $B_G \simeq B'_G$ □

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