Wall's theorems on h-cobordisms & 4-manifolds

Lectured by Pengcheng Li

Based on Wall's two papers in 1964:

[1] Diffeomorphisms of 4-manifolds

[2] On simply-connected 4-mfds.

- Convention: M_i here always means simply-connected, closed, oriented, smooth 4-mfds.
  $\hookrightarrow w_1(M_i)=0$

Review. 1. Freedman's topological 4-dim h-cobordism thm.

Let (W, M_1, M_2) be an h-cobordism of simply connected mfds.

Then $W \cong M_1 \times [0,1]$.
homeomorphic

In general, $W \not\cong M_1\times[0,1]$ because we cannot embed D^2 into M^4, or equivalently, we cannot eliminate self-intersections of immersed Whitney disks (2-handles).

2. Intersection form. $Q_M: H_2M \times H_2M \to \mathbb{Z}$

① $(\alpha.\beta) \mapsto \alpha^*(\beta)$, α^* = the PD of α, $\alpha = \alpha^* \cap [M]$, cap product.

$\pi_2 M \cong H_2M$

② $(\alpha,\beta) \mapsto S_\alpha \cdot S_\beta$ = the intersection number of the embedded surfaces S_α, S_β.

$Q_M: H^2M \times H^2M \to \mathbb{Z}$

③ $(\alpha^*, \beta^*) \mapsto \langle \alpha^* \cup \beta^*, [M]\rangle = \langle \alpha^*, \beta^* \cap [M]\rangle = \alpha^*(\beta)$.

So ① $\Leftrightarrow$ ③ $\Leftrightarrow$ ②.

Let $\{e_i\}_{i=1}^n$ be a basis of $H^2(M)$, $a_{ij} = (e_i \cup e_j)[M]$. Then $Q_M \leftrightarrow [a_{ij}]_{n\times n}$, which is still denoted by Q_M.

Basic properties :

① bilinear, symmetric & unimodular.

the matrix Q_M is nonsingular, $\det Q_M = \pm 1$.

or equivalently, $H^2M \xrightarrow{\cong} \mathrm{Hom}(H^2M, \mathbb{Z})$

$x \longmapsto Q_x = Q_M(x, -)$

② $Q_{\overline{M}} = -Q_M$, $Q_{M\#N} \cong Q_M \oplus Q_N$.

③ $\mathrm{rank}\, Q_M = \dim H_2 M = b_2(M)$

signature $\sigma(M) = \mathrm{sign}\, Q_M = b_2^+(M;\mathbb{R}) - b_2^-(M;\mathbb{R})$.

- Q_M can always be diagonalizable over $\mathbb{R}$ or $\mathbb{Q}$ $\longrightarrow$ $Q_M(\mathbb{R}) = \mathrm{diag}\{\lambda_1, \cdots \lambda_n\}$

$b_2^+(M;\mathbb{R})$ = the number of positive entries among $\lambda_1, \cdots \lambda_n$

$b_2^-(M;\mathbb{R})$ = $\cdots\cdots$ negative $\cdots\cdots\cdots$.

$\sigma(\overline{M}) = -\sigma(M)$, $\sigma(M\#N) = \sigma(M) + \sigma(N)$

④ Parity (even or odd)

Q_M is called <u>even</u> if $Q_M(e_i, e_i) = a_{ii}$ are all even.

Q_M is called <u>odd</u> if $\exists\, \alpha \in H$ s.t. $Q_M(\alpha, \alpha)$ is odd.

Examples. $Q_{\mathbb{CP}^2} = [1]$. $Q_{S^2\times S^2} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$, $Q_{S^2\tilde{\times}S^2} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$. $Q_{S^2\times S^2}$, $Q_{S^2\tilde{\times}S^2}$ are both indefinite but of different parities.

Fact: $S^2\tilde{\times}S^2 \approx \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

$$\begin{array}{ccc} S^2 & \longrightarrow & S^2\tilde{\times}S^2 \\ & & \downarrow \\ & & S^2 \end{array}$$

S^2-bundle over S^2

$[S^2, BSO(3)] \cong \pi_1 SO(3) \cong \mathbb{Z}/2$.

- $\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ is not diagonalizable over $\mathbb{Z}$.

  Thm (Donaldson, 1983) If a unimodular quadratic form Q is definite, then Q is diagonalizable over $\mathbb{Z}$.

3. Milnor, Whitehead Thm. $Q_{M_1} \cong Q_{M_2} \iff M_1 \simeq M_2$ homotopic equi.
   1958

4. Thm. $M_1 \sim_{cob} M_2$, i.e. $[M_1] = [M_2] \in \Omega_4 \iff \sigma(M_1) = \sigma(M_2)$.

   Cor. $\forall\, [M] \in \Omega_4$, we have $[M] = [\#k\, \mathbb{CP}^2]$ or $[M] = [\#l\, \overline{\mathbb{CP}^2}]$.
   that is, $\Omega_4 \cong \mathbb{Z}[\mathbb{CP}^2]$.

   Recall that $\Omega_0 \cong \mathbb{Z}$, $\Omega_1 = \Omega_2 = 0$: $S^1 = \partial D^2$, $\Sigma_g = \partial M_g$, the connected sum of g solid torus.

   $\Omega_3 = 0$ (Rohlin 1951: every closed M^3 bounds)

5. Thm: $\sigma(M) = 0 \iff M^4 = \partial W^5$.
   (Rohlin)

Lem 1 (Wall, [2]) Let $f: S^1 \to M$ be an embedding. Then $\exists$ an isotopy $h_t: S^1 \to M$ s.t. $h_i = f$, $i=0,1$.

Lem 2 (Wall, [2]) If Q_M is odd, then $M \# (S^2 \times S^2) \approx M \# (S^2 \tilde{\times} S^2)$.

Thm 1: (Wall, [2]) If $M_1 \sim_{h\text{-}cob} M_2$, then $M_1 \# k(S^2 \times S^2) \approx M_2 \# k(S^2 \times S^2)$ for some $k \geq 1$.

proof of Thm 1.

Given an h-cobordism W between M_1 and M_2, then a nice Morse function on W which yields a handle decomposition of W. Similar to the process of Smale's proof of higher dimensional h-cobordism thm, we have the sub-handlebody $N = M_1 \cup_{f_k} k(D_i^2 \times D^3)$.

Firstly we assume $k=1$. Let $f: S^1 \times D^3 \to M_1$ be an embedding and let $H: S^1 \times I \to M_1$ be an isotopy with $H(x,i) = f(x,0)$, $i=1,2$. By the isotopy extension thm, H extends to an isotopy $\bar{H}$ of $\bar{M}_1$.

By the tubular nbhd thm, we may assume that $h = \bar{H}(-,1): f(S^1 \times D^3) \to f(S^1 \times D^3)$ is a bundle map.

Spherical modification: $N = (M_1 \setminus f(S^1 \times D^3)^\circ) \cup_{f(S^1 \times S^2)} D^2 \times S^2$

$$\rightsquigarrow N' = (M_1 \setminus hf(S^1 \times D^3)^\circ) \cup_{hf(S^1 \times S^2)} D^2 \times S^2.$$

It's clearly $(h, 1_{D^2 \times S^2})$ defines a diffeomorphism $N \xrightarrow{\approx} N'$.

Since M_1 is 1-conn., $f(S^1\times 0)$ bounds an embedded $D^2\subseteq M$, which lies in the interior of an embedded $D^4\subseteq M$.
$\underset{*}{\overset{\shortparallel}{\ }}$

We have a chain of diffeomorphisms:

$$N\approx (M_1\setminus \mathring{D}^4)\cup_{S^3} D^2\times D^2 \cup_{f(S^1\times S^2)} D^2\times S^2$$
$$\approx (M_1\setminus \mathring{D}^4)\cup_{S^3} (D^2\times S^2\setminus \mathring{D}^4)\cup_{f(S^1\times S^2)} D^2\times S^2$$
$$\approx (M_1\setminus \mathring{D}^4)\cup_{S^3} [(D^2\times S^2\cup_{f(S^1\times S^2)} D^2\times S^2)\setminus \mathring{D}^4]$$
$$\approx M_1\# S^2\times S^2 \text{ or } M_1\# S^2\tilde{\times} S^2.$$

By induction we get $N\approx M_1\# k(S^2\times S^2)$ or $N\approx M_1\# k(S^2\tilde{\times} S^2)$.

Similarly $N\approx M_2\# k(S^2\times S^2)$ or $N\approx M_2\# k(S^2\tilde{\times} S^2)$.

If Q_M is odd, by Lem 2, $M_1\# k(S^2\times S^2)\approx M_2\# k(S^2\tilde{\times} S^2)\approx M_2\# k(S^2\times S^2)$.

If Q_M is even, then $w_2(M)=0$ and hence $w_2(W)=0$

Since N has trivial normal bundle in W, $w_2(N)=0 \Rightarrow Q_N$ is also even

$Q_{S^2\tilde{\times} S^2}$ is odd $\Longrightarrow N\approx M_1\# k(S^2\times S^2)\approx M_2\# k(S^2\times S^2)$. $\square$

Here we use the fact that Q_M is even $\Leftrightarrow w_2(M)=0$ (Spinor).

$\Leftrightarrow$ normal bundle is trivial

Lemma 3 If $\sigma(M) = \operatorname{sign} Q_M = 0$, $w_2(M) = 0$, then $M = \partial V^5$ and $w_2(V) = 0$.
(Wall, [2])
(If $w_2(V^5) \neq 0$, then $w_2(M) = i^* w_2(V^5) \neq 0$, $i: M \to V$).

Lefschetz duality: Suppose $\partial W^n = M_1 \amalg M_2$, then $H^i(W, M_1) \cong H_{n-i}(W, M_2)$.
($M_i = \phi$ is allowed, when $M_1 = M_2 = \phi$, the duality becomes the Poincaré duality)

Thm 2 (Thm 1, [2]) Suppose M^4 satisfies $\sigma(M) = 0$, then $M = \partial V^5$ and $V^5 \simeq V_m S^2$.

proof. We have $M = \partial V^5$ by Rohlin's thm. Since $\partial V = M \neq \phi$, $H_5(V) \cong H^0(V, M) = 0$.
Moreover, by lem 3, $w_2(M) = 0 \Rightarrow w_2(V) = 0$.

- Step 1. Killing $\pi_1(V)$. Suppose $f: S^1 \hookrightarrow V$ represents a generator of $\pi_1(V)$.

Take a tubular nbhd $f(S^1 \times D^4)$

$$V' = (V \setminus f(S^1 \times D^4)^\circ) \cup_{S^1 \times S^3} D^2 \times S^3.$$

Then $\partial V' = \partial V = M$ and $0 = [f] \in \pi_1(V')$; note that $w_2(M) = 0 \Rightarrow w_2(V') = 0$.
By such modifications we finally get a new 5-mfd V with $\pi_1 V = 0$.

(By step 1, $\pi_1(V) = 0 \Rightarrow H_1(V) = 0 = H^1(V) \cong H_4(V)$ and hence V only has H_2, H_3)

- ## Step 2. Killing $H_2(V,M)$

Consider the homology exact seq for the pair (V,M).

$$0\to H_3(V)\to H_3(V,M)\to H_2(M)\xrightarrow{i_*} H_2(V)\xrightarrow{j_*} H_2(V,M)\to 0$$

Case 1. $H_2(V,M)$ is infinite

Choose $x\in H_2(V,M)$, $|x|=\infty$ and let $y\in H_2(V)$, $y\overset{j_*}{\mapsto} x$.

If $w_2(V)(y)\neq 0$, then $w_2(V)\neq 0 \Rightarrow w_2(M)\neq 0$.

Choose $z\in H_2(M)$ satisfying $w_2(M)(z)\neq 0$.

After replacing y by $y+i_*(z)$, we now can assume $w_2(V)(y)=0$.

Representing $y\in H_2(V)\cong \pi_2(V)$ by an embedding sphere and since $w_2(V)(y)=0$, the image of y has trivial normal bundle, which ensures the existence of an embedding $S^2\times D^3\hookrightarrow V$.

Let $X=V\setminus(S^2\times D^3)^\circ$ and $V'=X\cup_{S^2\times S^2}(D^3\times S^2)$

Then $\partial V'=\partial V=M$ and Milnor showed that this spherical modification doesn't change w_2:

$$w_2(M)=0\Rightarrow w_2(V')=0.$$

Consider the homology exact seq for the triple (V,X,M):

$$\text{(S)}:\quad \begin{tikzcd} H_3(V,M) \arrow[r, "\alpha"] \arrow[d, phantom, "\cong" rotate=90] & H_3(V,X) \arrow[r, "\beta"] \arrow[d, phantom, "\cong" rotate=90] & H_2(X,M) \arrow[r, "\gamma"] & H_2(V,M) \arrow[r] \arrow[d, phantom, "\ni" rotate=90] & H_2(V,X) \arrow[d, phantom, "\cong" rotate=90] \\ H^2(V) \arrow[r, "\cdot y"] & \mathbb{Z} & & x & 0 \end{tikzcd}$$

These two isomorphisms follow from the homeomorphism: $V/X\cong S^2\times D^3/S^2\times S^2$.

By the Lefschetz duality, $H_3(V,M) \cong H^2(V)$ and the homomorphism α corresponds to the intersection with y and since $|y|=\infty$, we see ~~that~~ $\operatorname{coker}\alpha$ is finite $\Rightarrow$ $\operatorname{rank} H_2(V,M) = \operatorname{rank} H_2(X,M)$

Consider the exact seq (S') with V' in place of V, ~~then~~ $\operatorname{Im}\beta' = \gamma^{-1}(x)$ has infinite order.

Thus $\operatorname{rank} H_2(V',M) = \operatorname{rank} H_2(X,M)-1 = \operatorname{rank} H_2(V,M)-1$.

By induction we reduce $\operatorname{rank} H_2(V;M)=0$.

<u>Case 2. $H_2(V,M)\neq 0$ is finite.</u>

We may assume $H_2(V)$ is infinite.

$$\begin{array}{ccccccc} H_3(V,M) & \xrightarrow{\partial} & H_2M & \xrightarrow{\alpha} & H_2V & \longrightarrow H_2(V,M) \longrightarrow 0 \\ \wr\| & & \Big\downarrow\Big\| & & & \\ H^2(V) & & & & & \\ \wr\| & & & & & \\ \operatorname{Hom}(H_2V,\mathbb{Z}) & \xrightarrow{\alpha^*} & \operatorname{Hom}(H_2M,\mathbb{Z}) & & & \end{array}$$

We have $\operatorname{rank}\partial = \operatorname{rank}\alpha^* = \operatorname{rank}\alpha = \dim H_2M - \dim \operatorname{Ker}\alpha = \dim H_2M - \dim \operatorname{Im}\partial = \dim H_2M - \operatorname{rank}\partial$

$\Rightarrow$ $\operatorname{rank}\partial = \frac{1}{2}\dim H_2M$.

Since $H_2(V,M)$ is finite, $\dim H_2V = \operatorname{rank}\partial = \frac{1}{2}\dim H_2M$.

- We ~~temporarily~~ assume ~~that~~ $\dim H_2M \geq 4$ and hence $\dim H_2V \geq 2$.

  This is the only case we shall deal with.

<u>Claim</u> : There exists $y \in H_2(V)$ satisfying:

(i) $w_2(V)(y) = 0$ (ii) the image of y in $H_2(V,M)$ is nonzero : $H_2(V) \longrightarrow H_2(V,M) \to 0$.

(iii) y is indivisable and hence $\langle y \rangle \subseteq H_2(V)$ is a direct summand.

proof of the claim. If $w_2(M)=0$, then $w_2(V)=0$, (i) clearly holds.

If each indivisable element of $H_2(V)$ has zero image in $H_2(V,M)$, then the homo $H_2(V) \to H_2(V,M)$ is zero, contradicts.

Hence $w_2(V) \neq 0$, $w_2(M) \neq 0$.

Since $\dim H_2 V \geq 2$, we can choose a basis $\{x_i\}$ of $H_2(V)_{free}$ — the free part of $H_2(V)$ s.t.

$$w_2(V)(x_i) = 0, \quad i > 1.$$

If every indivisable element of $H_2(V)$ on which $w_2(V)$ vanishes has zero image in $H_2(V,M)$, then the exact seq. $H_2 M \xrightarrow{i_*} H_2 V \to H_2(V,M) \to 0$ implies that

$$w_2(V)\, i_*(H_2 M) = w_2(M)\,(H_2 M) = 0 \Rightarrow w_2(M) = 0, \text{ contradicts.}$$

Thus the claim is proved.

Make a spherical modification as did in Case 1, starting with y.

In the Seq. (S), since y is indivisable, α is onto and we get $H_2(X,M) \cong H_2(V,M)$

Similarly by the seq. (S') we deduce $|H_2(V',M)| < |H_2(V,M)|$ and by induction we get $H_2(V',M) = 0$, or equivalently, $H^2(V',M) = H_3(V') = 0$.

Therefore we obtain a simply-conn. 5-mfd W whose only nontrivial homology group is $H_2(W) \cong H^3(W,M) \cong \mathrm{Hom}(H_3(W,M), \mathbb{Z})$ free.

The 2nd Skeleton $V^{(2)} = \vee_m S^2 \hookrightarrow V$ is then a homology equivalence.

By the Whitehead thm, it is a homotopy equivalence: $V \simeq \vee_m S^2$. $\square$

- Review of some homotopy theory:

- Hurewicz Thm.

Let X be a based CW-complex. The <u>Hurewicz homomorphism</u>

$$h_n^X : \pi_n(X) \longrightarrow H_n(X)$$

is defined by $h_n([f]) = f_*(\iota_n)$, where $\iota_n = [S^n] \in H_n(S^n)$ is the fundamental class.

- naturality:

$$\begin{tikzcd} \pi_n X \arrow[r, "h_n^Y"] \arrow[d, "f_\#"] & H_n X \arrow[d, "f_*"] \\ \pi_n Y \arrow[r, "h_n^X"] & H_n Y \end{tikzcd}$$

$h_n : \pi_n(-) \longrightarrow H_n(-)$ can be viewed as a natural transformation.

Thm. If X is $(n-1)$-connected $(n \geq 2)$, then $h_n : \pi_n X \longrightarrow H_n X$ is an isomorphism. and $h_{n+1} : \pi_{n+1} X \longrightarrow H_{n+1} X$ is surjective.

$\pi_i X = 0$ for $i \leq n-1$

If $n=1$, $h_1 : \pi_1 X \longrightarrow H_1 X$ is surjective with kernal $[\pi_1 X, \pi_1 X]$:

$$\frac{\pi_1 X}{[\pi_1 X, \pi_1 X]} \cong H_1 X.$$

• Whitehead Thm

Let $f : X \to Y$ be a map between simply connected (or simple) CW-cpxes.

TFAE:

(i) f is a homotopy equivalence: $\pi_i(X) \xrightarrow{f_\#} \pi_i(Y)$ for all i.

(ii) f is a homology equivalence: $H_i(X) \xrightarrow[\cong]{f_*} H_i(Y)$ for all i

Where $M_f = Y \cup_f X \times I$ is the mapping cylinder.
$(x,0) \sim f(x)$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\ & M_f \arrow[ur, "r"', "\simeq"] & \end{tikzcd}$$

$f \simeq r \circ \iota$, where
r is a htp equi. and
ι is an including map
($X \subseteq M_f$ as a subcomplex).

(iii) $H_i(M_f, X) = 0$ for all i.

Lemma 4 Let $M = \partial W^5$, W, M simply conn. and $W \simeq V_m S^2$.

Then (1) W admits a handlebody $H = D^5 \times D^0 \cup_m D_i^2 \times D^3$ as a deformation retract.

(2) The closure $C = \overline{W \setminus H}$ gives an h-cobordism of M to ∂H.

proof. (1) firstly imbeds D^5 in W and imbeds $D_i^2 \times D^3$, which represents the generators of $H_2(W) \cong \oplus_m H_2(S^2)$.

The boundaries of $D_i^2 \times D^3$ are attached to D^5 and the interiors of D_i^2 are disjoint with D^5.

H: 

The resulting handlebody H is of type $(5, m, 2)$.

The inclusion $H \hookrightarrow W$ is a homology equivalence of simply-conn. spaces

and hence a homotopy equivalence.

(2) $\partial C = \partial H \sqcup \partial W = \partial H \cup M$, C is a simply-connected 5-mfd.

By the excision thm of homology: $0 = H_i(W, H) \cong H_i(C, \partial H)$.

Hence $\partial H \hookrightarrow C$ is a homology/homotopy equivalence.

By the Lefschetz thm, $H_i(C, M) \cong H^{5-i}(C, \partial H) = 0$ and hence

$M \hookrightarrow C$ is a homology/homotopy equi. $\square$.

Wall's thm on diffeomorphisms.

Let M be a 1-conn, closed oriented 4-mfd. Suppose that either

(i) Q_M is indefinite, or

(ii) $\dim H_2(M) \leq 8$

Then $\mathrm{Diffeo}(M \# S^2 \times S^2) \twoheadrightarrow \mathrm{Aut}(Q_{M \# S^2 \times S^2}) = \{ g \in \mathrm{Aut}(H_2(M \# S^2 \times S^2)) \mid Q_M = Q_M(g \times g) \}$.

$\varphi \longmapsto \varphi_*$

Cor. $M = \# k(S^2 \times S^2) \# \ell(S^2 \tilde{\times} S^2) \# i(\mathbb{C}P^2) \# j(\overline{\mathbb{C}P^2})$ ✓.

Thm 2 (Wall, 1964) Let M_1, M_2 be two simply conn. 4-mfds with $Q_{M_1} \xrightarrow[\cong]{\alpha} Q_{M_2}$.

Then $M_1 \sim_{hcob} M_2$

Cor. Let M_1, M_2 be as in the above thm. Then

(i) $\exists\, k \in \mathbb{Z}_{\geq 1}$ such that $M_1 \# k(S^2 \times S^2) \cong M_2 \# k(S^2 \times S^2)$.

(ii) (Freedman) $M_1 \cong M_2$.

proof. Form $N = M_1 \# \overline{M_2}$, then $\operatorname{Sign} Q_N = \operatorname{Sign} Q_{M_1} - \operatorname{sign} Q_{M_2} = 0$.

Case 1. $\dim H_2 M_1 \geq 2$. By Thm 1 and Lemma, $N \sim_{hcob} \partial V$, V a handlebody of type $(5, m, 2)$.

Fact: V is determined by m and whether $w_2(V) = 0$ or not:

V is a boundary sum of m handlebodies of type $(5, 1, 2)$:

$$V = V_1 + \cdots + V_m$$

Each V_i is a D^3-bundle over S^2; since $[S^2, BSO(3)] \cong \pi_1 SO(3) \cong \mathbb{Z}/2$, there are only two such bundles.

$S^2 \times D^3$, $S^2 \tilde{\times} D^3$. whose boundaries are $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$, respectively.

$\therefore$ $N \sim_{hcob} \partial V \sim_{hcob} \# k S^2 \times S^2 \# l\, S^2 \tilde{\times} S^2$.

claim 1. there exists an automorphism $T \in \mathrm{Aut}\, Q_N = \mathrm{Aut}(Q_{\partial V})$ such ~~that~~ that

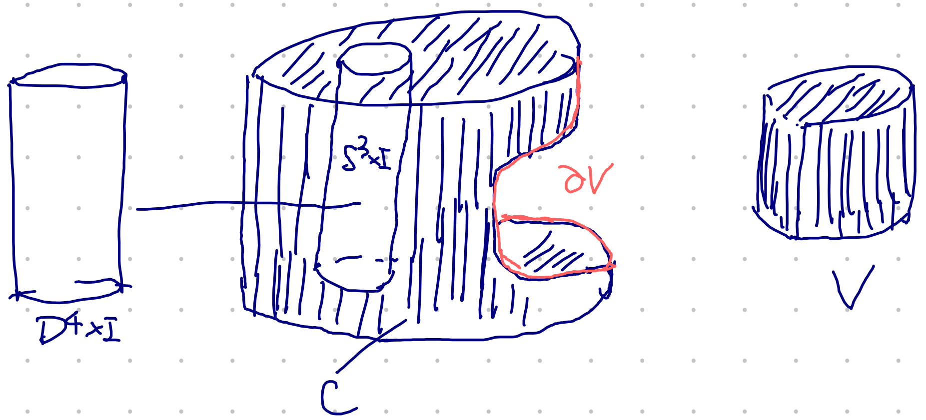
the kernel $L = \mathrm{Ker}(H_2(\partial V) \to H_2(V))$ has the form:

$$K = \{(x, y) \in H_2(\partial V) \cong H_2(N) \cong H_2(M_1) \oplus H_2(\overline{M_2}) \mid \alpha x = y\}, \text{ where } \alpha : Q_{M_1} \xrightarrow{\cong} Q_{M_2}.$$

$K \cong H_2(M_1)$ is free abelian of half rank of $H_2(\partial V)$.

By Wall's thm on diffeomorphism, $\exists$ a diffeomorphism φ ~~of~~ of ∂V st $\varphi_* = T$.

$$0 \to K \hookrightarrow H_2(\partial V) \xrightarrow{i_*} H_2(W)$$



$(V, M_1 \# \overline{M_2}, \partial V)$ an h-cobordism

$M_1 \# \overline{M_2} = (M_1 \setminus \mathring{D}^4) \cup S^3 \times I \cup (M_2 \setminus \mathring{D}^4)$

$R := C \cup D^4 \times I \cup_{\partial V} V$, a simply-conn. 5-mfd with $\partial R = M_1 \amalg M_2$.

Note that R has nontrivial homology groups only in dimension 0, 2, 4.
and the attachment of $D^4 \times I$ doesn't affect the isomorphism $H_2(M_1) \oplus H_2(M_2) \cong H_2(\partial V)$

Attaching V along ∂V: MV seq (Mayer-Vietories):

$$0 \to K \to H_2(\partial V) \xrightarrow{(i_*, \cong)} H_2(V) \oplus H_2(C) \longrightarrow H_2(R) \to 0$$

Since $H_2(\partial V) \xrightarrow{\cong} H_2(C)$, ($\partial V \to C$ is a htp. equi.), we get $H_2(R) \cong \operatorname{coker} i_* \cong H_2(M_i)$.

Since $H_2 M_i \cap K = \{(0,0)\}$, $H_2 M_i \to H_2 R$ is an isomorphism, $i = 1, 2$.

$\Rightarrow H_k(R, M_i) = 0$ for $k \leq 2$ and $i = 1, 2$

$\Longrightarrow H_{5-k}(R, M_i) \cong H^k(R, M_{3-i}) = 0$

Thus R is an h-cobordism.

Case 2 If $\dim H_2 N = 2$ and $\operatorname{sign} Q_N = 0$, then $Q_N \cong Q_{S^2 \times S^2}$ or $Q_N \cong Q_{S^2 \tilde{\times} S^2}$.

$\Longrightarrow N \sim_{hcob} S^2 \times S^2$ or $N \sim_{hcob} S^2 \tilde{\times} S^2$.

Filling in V by $D^3 \times S^2$ or $D^3 \tilde{\times} S^2$, then $\partial V' = N$

The arguments in the case 1 is now valid.

$\square$