Characteristic Element and Rockhlin's Theorem

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We now have the Stiefel-Whitney class $w_i(M) \in H^i(M;\mathbb{Z}_2)$ of M in hand. Recall that the tangent bundle TM of M can be trivialized over the 2-skeleton of M if and only if $w_2(M) = 0$.

Definition 1. *A* ***characteristic surface*** *of M is an oriented surface Σ embedded in M such that $[\Sigma] \in H_2(M;\mathbb{Z})$ is (the Poincare dual to) an integral lift $w \in H^2(M;\mathbb{Z})$ of the class $w_2(M)$.*

The class w is called a ***characteristic element****.*

Characteristic elements are not unique: for any $\gamma \in H_2(M,\mathbb{Z})$, adding 2γ to w does not affect its reduction in $H_2(M,\mathbb{Z}_2)$.

Another proof of Wu's formula. We have already seen Wu's Formula:

Theorem 1. *Let M be a simply-connected 4-manifold. An oriented surface Σ is characteristic if and only if $\sigma \cdot S = S \cdot S$ (mod 2) for all oriented surfaces S inside M.*

We will give a more concrete proof of this, although a lot of facts have to be admitted without proof.

Definition 2. *Let M be a smooth n-manifold, $v \in \Gamma(T_M)$ be a tangent vector field of M. Assume all zeros of v are isolated. Then at each $x \in M$ with $v(x) = 0$, there is an n-ball $D \ni x$ such that $v|_{D-\{x\}}$ is nowhere-zero.*

Define a map $d : \partial D = S^{n-1} \to S^{n-1}$, $d(x) = \frac{v(x)}{\|v(x)\|}$. The ***index of*** *$v$* ***at the zero*** *$x$ is defined to be the degree of d.*

Lemma 1 (Poincare-Hopf)**.** *Keep the notion of the previous definition, then*

$$\sum_i \text{index}(i) = \chi(M)$$

where the sum is taken over all zeros of v and $\chi(M)$ is the Euler characteristic of M.

Definition 3. *Keep all the notations above. At a zero x of v, one can choose a neighborhood $U \cong \mathbb{R}^n$ of x such that $TM|_U$ is trivial. Then $v|_U$ can be identified as a map $\mathbb{R}^n \to \mathbb{R}^n$. If the derivative dv_x at x has nonzero determinant, then we call the zero x* ***non-degenerate****, otherwise call it* ***degenerate****.*

One should check that the degeneracy does not depend on the choice of U.

Lemma 2. *If $v \in \Gamma(M)$ has all zeros isolated and non-degenerate, then each zero has index ± 1.*

Proof of theorem 1 (not very rigorous). Let $\tau \in \Gamma(T_S)$ be a tangent vector field of S, $v \in \Gamma(N_{S/M})$ be a normal vector field of S. By assuming τ and v generic, we suppose that they vanish only at non-degenerate isolated points of S. We may pertube them a little such that they are zero at different point of S.

Pick a vector field $\tilde{\tau} \in \Gamma(T_S)$ that is orthonormal to τ and a vector field $\tilde{v} \in \Gamma(N_{S/M})$ orthonormal to v. Then $\tilde{\tau}$, $\tilde{v}$ are zero only possibly on the zeros of τ, v, respectively. Then the vector field $\tilde{\tau} + \tilde{v}$ is nowhere vanishing on S.

The 3-frame $\{\tau, v, \tilde{\tau} + \tilde{v}\}$ can be completed to a full 4-frame $\{\tau, v, \tilde{\tau} + \tilde{v}, u\}$ of T_M whenever none of τ, v vanishes, then TM can be trivialized on S. At each isolated zero x, there is a disk D such that $\tau|_{D-\{x\}}$ is nowhere vanish. The map $f : \partial D \to SO(4)$ given by mapping each x to the orthonormal matrix $[\tau(x), v(x), (\tilde{\tau} + \tilde{v})(x), u(x)]$, defines a cycle in $SO(4)$. It's not hard to see that this full 4-frame can be extended to the zero if and only if this cycle is trivial in $SO(4)$. That is, if and only if $f = 0 \in \pi_1(SO(4)) = \mathbb{Z}_2$.

Therefore, the obstruction extending the 3-frame comes entirely from the zeros of τ and v. That is,

$$\text{obstruction} = \#\{\text{zeros of } \tau\} + \#\{\text{zeros of } v\} \pmod{2}$$

By the genericity assumption and lemma 1, lemma 2, the first term on the right is equal to $\chi(S)$ modulo 2, which is 0 since the Euler characteristic of a closed oriented surface is even. The second term vanishes only when the tangent space of S is equal to the tangent space of M, that is, at the self-intersection points of S. Thus we have

$$\text{obstruction} = 0 + S \cdot S$$

On the other hand, the obstruction trivializing is measured by the second Stiefel-Whitney class $w_2(M)$ operating on S, which is $\Sigma \cdot S$, implemented by a characteristic surface.

Therefore, we have

$$\text{obstruction} = S \cdot S = \Sigma \cdot S.$$

Conversely, if an integral class $w \in H^2(M; \mathbb{Z})$ satisfying $w(S) = S \cdot S$ for any oriented surface S, then by definition its modulo 2 reduction in $H^2(M; \mathbb{Z}_2)$ must be $w_2(T_M)$. □

Wu's theorem describes the relation between the characteristic surface and the intersection form. By the properties of the intersection form, such characteristic surface do exist:

Theorem 2. *Let M be a 4-manifold, then there exist an integral class $w \in H_2(M; \mathbb{Z})$ such that $w \cdot x = x \cdot x$ (mod 2) for every $x \in H_2(M; \mathbb{Z})$.*

Proof. Let $Q'_M : H_2(M; \mathbb{Z}_2) \times H_2(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the modulo-2 reduction of the intersection form Q_M of M. Define $f : H_2(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ by $x \mapsto Q'_M(x, x)$. This is linear, since $Q'_M(x + y, x + y) = x \cdot x + 2x \cdot y + y \cdot y \equiv x \cdot x + y \cdot y = Q'_M(x, x) + Q'_M(y, y) \pmod{2}$. By the unimodularity of Q'_M, there is an element $\Sigma' \in H_2(M; \mathbb{Z}_2)$ such that $\Sigma' \cdot x = f(x) = x \cdot x$ for any x. Since we can assume $H_2(M; \mathbb{Z})$ is free, $H_2(M; \mathbb{Z}_2) = H_2(M; \mathbb{Z}) \oplus \mathbb{Z}_2$, there is an integral lift $\Sigma \in H_2(M; \mathbb{Z})$ of Σ'. This completes the proof. □

Remark. *The existence of characteristic elements is equivalent to the existence of spin*$^{\mathbb{C}}$ *structures, which play an essential role in Seiberg-Witten theory. (I know nothing about this.)*

Rokhlin's theorem We introduce the interesting Rokhlin's theorem and several consequences of it without proof. For the proofs readers may refer to [1] section 4.4.

Lemma 3 (Van der Blij's lemma)**.** *For every characteristic element w we must have*

$$\text{sign} Q_M = w \cdot w \pmod{8}$$

In particular, if M has a spin structure, then we can always choose $w = 0$, so its signature must be a multiple of 8. Moreover,

Theorem 3 (Rockhlin's theorem)**.** *If M^4 is smooth with $w_2(M) = 0$, then we must have* $\text{sign} Q_M = 0 \pmod{16}$.

Corollary 1. *E_8 cannot be the intersection form of a smooth manifold. In particular, the E_8-manifold M_{E_8} does not admit any smooth strucutre.*

Proof. We know that E_8 is an even form, so $w \cdot x = x \cdot x = 0 \pmod 2$ for any $x \in H_2(M_{E_8}; Z_2)$. Since the second Stiefel-Whitney class is determined by its operation on surfaces, we have $w_2(M) = 0$. By Rokhlin's theorem, M_{E_8} cannot have any smooth structure. □

Recall that $H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is of signature 0, and that signature is additive over direct product.

Corollary 2. *Suppose M is smooth without 2-torsion (for example, when M is simply-connected). If the intersection form of M is of the form*

$$Q_M = \bigoplus \pm m E_8 \bigoplus nH$$

then m must be even.

Proof. When M has no 2-torsion, even intersection form implies $w_2(M) = 0$. Then by Rokhlin's theorem, $\text{sign} Q_M = 0 \pmod{16}$. Then m must be even. □

Remark. *The absence of 2-torsion is essential. The complex Enriques surface, which is doubly covered by K_3, has intersection form $-E_8 \bigotimes H$. Its fundamental group is $\pi_1 = \mathbb{Z}_2$. The 2-torsion of the fundamental group allows the intersection form to be even without w vanishing.*

Rokhlin's theorem is a very fundamental result in topology. It is closely related to Kirby-Siebenmann invariant, governing the existence of smooth structures on a manifold. It can be further generalized:

Theorem 4 (M.Kervaire & J.Milnor)**.** *Let M be a simply-connected 4-manifold. If Σ is a* ***characteristic sphere*** *of M, then*

$$\text{sign} M = \Sigma \cdot \Sigma \pmod{16}$$

This result can be used to determine which characteristic elements cannot be represented by an embedded sphere. If Σ is not a sphere, this is true after adding some correction term:

$$\mathrm{sign} M = \Sigma \cdot \Sigma + 8\mathrm{Arf}(M, \Sigma) \pmod{16}$$

The $\mathrm{Arf}(M, \sigma)$ is the correction term, which is 0 for Σ a sphere. We will not introduce further about this. Interested readers can refer to [1] chapter 11.

References

[1] Alexandru Scorpan. *The Wild World of 4-Manifolds*. American Mathematical Society, 2005.