Mayer-Vietoris Sequence of de Rham Cohomology

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1 A Cheatsheet on Differential Forms

Learn differential forms in 5 minutes!

1. A smooth k-form is a smooth section of $\Lambda^k T^*M$.
2. $\Lambda^k T^*M$ is the smooth vector bundle obtained by gluing together $\Lambda^k T_p^*M$ at each p using a really cool trivialization that encodes the global information of the manifold.
3. $\Lambda^k T_p^*M \cong (\Lambda^k T_pM)^* := \mathrm{Hom}_{\mathsf{Vec}_{\mathbb{R}}}(\Lambda^k T_pM, \mathbb{R})$ which is the space of k-anti-symmetric linear form on T_pM.
4. $T_pM := \{$ the linear space of all directional derivatives at point $p\}$ is the tangent space at p.

Other tools in the toolbox:

1. Pullback of differential form ω along smooth map $F : M \to N$:
$$(F^*\omega)_p(v_1, \dots, v_k) := \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$
in which $F^* : \Omega^*(N) \to \Omega^*(M)$, $(F^*\omega)_p : (\Lambda^k T_pM)^*$ and $v_1, \dots, v_k \in T_pM$.
2. $dF_p : T_pM \to T_pN$ is the differential of smooth map F at p, defined by
$$dF_p(v)(f) := v(f \circ F).$$
3. $\Lambda_{i\in I}^k dx_i$ is a handful basis of $\Lambda^k T_p^*M$ when a chart $\phi : p \mapsto (x_1, \dots, x_n)$ is specified.

2 Mayer-Vietories Sequence of De Rham Cohomology

De Rham cohomology is supposed to be some kind of cohomology theory, so the operator $\Omega^*(\bullet)$ we have met before should also be a contravariant functor. In fact, the pullback is in charge of turning a morphism $f : M \to N$ to $f^* : \Omega(N) \to \Omega(M)$, completing the information of a contravariant functor.

We make it precise that $\Omega^*(\bullet)$ is a functor from the category of smooth manifolds to the category of *chain complexes* (or differential graded algebra, since we may consider some operation between elements in the chain making it more natural to consider the chain groups as a whole structure). To fit beautifully into the framework of category theory, we need to check that F^* is qualified as a morphism of chain complexes, that is, it commutes with differential operator

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ {\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}$$

This requirement is quite useful afterwards.

Let M be a smooth manifold which can be decomposed into open subsets U and V such that $U \cap V \neq \varnothing$. Consider the following diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{j_U} & U \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle i_U} \\ V & \xrightarrow{i_V} & U \cup V \end{array}$$

which turns into

$$\begin{array}{ccc} \Omega^*(U \cup V) & \xrightarrow{i_U^*} & \Omega^*(U) \\ {\scriptstyle i_V^*}\downarrow & & \downarrow{\scriptstyle j_U^*} \\ \Omega^*(V) & \xrightarrow{j_V^*} & \Omega^*(U \cap V) \end{array} \tag{1}$$

after applying the functor $\Omega^*(\bullet)$. It may tempt us into considering the following sequence

$$\Omega^*(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V) \ .$$

We have the following claim:

Proposition 2.1. *The sequence of chain complexes*

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V) \longrightarrow 0$$

is exact.

Proof. First we check the exactness at $\Omega^*(U \cup V)$. For any $\omega \in \Omega^*(M)$, suppose $i_U^*(\omega) = 0$. The intuition is, $i_U^*(\omega)$ may be understood to be some kind of "restriction" $\omega|_U$ of ω as smooth function. Concretely, we have

$$0 = i_U^*(\omega)_p = \omega_p \circ \Lambda^k(\mathrm{d}i_U)_p. \tag{2}$$

for any $p \in U$ by expansion of notations. By the local nature of tangent spaces, the inclusion $i_U : U \hookrightarrow M$ induces isomorphism $(\mathrm{d}i_U)_p : T_pU \to T_pM$ for all p, and taking their wedge product will yield another isomorphism $\Lambda^k T_pU \to \Lambda^k T_pM$. Therefore we have $\omega|_U = 0$. Similarly, $\omega|_V = 0$. Since $M = U \cup V$, ω is zero everywhere.

Similarly, suppose $\omega \in \Omega(U)$ and $\eta \in \Omega(V)$ satisfy $j_U(\omega) = j_V(\eta)$. Informally it means $\omega|_{U\cup V} = \eta|_{U\cup V}$, which means ω and η can be glued together to form a global section on M. To fulfill this dream, we need to deal with these tedious definitions. Drawing diagrams will be useful to avoid getting lost. The assumption gives us

$$\omega_p \circ \Lambda^k(\mathrm{d}j_U)_p = \eta_p \circ \Lambda^k(\mathrm{d}j_V)_p \tag{3}$$

for each $p \in U \cap V$.

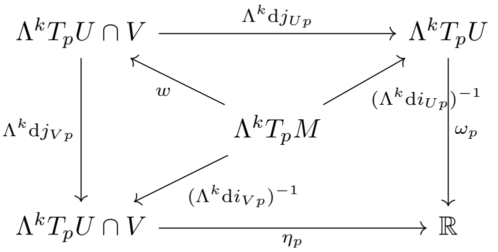
Now construct $\tau \in \Omega^k(M)$ to be

$$p \in M \mapsto \begin{cases} \omega_p \circ (\Lambda^k \mathrm{d}j_{Up})^{-1} & p \in U \\ \eta_p \circ (\Lambda^k \mathrm{d}j_{Vp})^{-1} & p \in V \end{cases} \tag{4}$$

and clearly such τ satisfy $i_U^*(\tau) = \omega$ and $i_V^*(\tau) = \eta$. To make this well-defined, we need to check that for $p \in U \cap V$,

$$\omega_p \circ (\Lambda^k \mathrm{d}i_{Up})^{-1} = \eta_p \circ (\Lambda^k \mathrm{d}i_{Vp})^{-1}. \tag{5}$$

In fact, we have the diagram



(6)

in which w is (the lift of) the inverse of witness of commutativity in diagram (1). There is

$$\omega_p \circ (\Lambda^k di_{Up})^{-1} = \omega_p \circ \Lambda^k(\mathrm{d}j_U)_p \circ \text{witness} = \eta_p \circ \Lambda^k(\mathrm{d}j_V)_p \circ \text{witness} = \eta_p \circ (\Lambda^k di_{Vp})^{-1} \tag{7}$$

for $p \in U \cap V$. Note: we are allowed to perform such kind of acrobatics thanks to the fact that inclusions induce isomorphisms between tangent spaces, which is then due to the idea that tangent space is some kind of *local* object.

On the other hand, a form $\omega \in \Omega^*(M)$ traveling all the way through $\Omega^*(U) \oplus \Omega^*(V)$ into $\Omega^*(U \cap V)$ will annihilate since

$$(j_U^* \circ i_U^*)(\omega) = (j_U \circ i_U)^*(\omega) = (j_V \circ i_V)^*(\omega) = (j_V^* \circ i_V^*)(\omega) \tag{8}$$

by the functorality of pullback. So the exactness is also satisfied at $\Omega^*(U) \oplus \Omega^*(V)$.

Remark: the two steps that we have done on $\Omega^*(U) \oplus \Omega^*(V)$ actually checks that the contravariant functor $\Omega^*(\bullet)$ is actually a *sheaf* of differential graded algebra on the space M if we consider these steps on arbitrary open cover instead of just U and V.

Finally we check the same thing at $\Omega^*(U \cap V)$. The seemingly redundant magic mentioned before, partition of unity, is used at this point. We first use the simple case of $M = \mathbb{R}$ to demonstrate the ideal. Pick $f \in \Omega^0(\mathbb{R})$, which is a smooth function on $\mathbb{R}$. We need to find a pair of functions defined on U and V respectively whose difference on $U \cap V$ happens to be f. Since U and V form an open cover of M, there is a partition of unity $\{\rho_U, \rho_V\}$ subordinate to this cover. We can see that $\rho_V \cdot f$ is roughly actually a function on U, since ρ_V is annihilating on $U - U \cap V$ so it doesn't matter whatever f evaluates to over U, even if f is undefined. And here's the trick:

$$(\rho_U f) - (-\rho_V f) = f \tag{9}$$

holds everywhere on $U \cap V$.

Now for general manifolds, consider $\omega \in \Omega^k(U \cap V)$. Defining $\tau \in \Omega^k(U)$ as

$$p \mapsto \begin{cases} \rho_V(p) \cdot \omega_p \circ \Lambda^k \mathrm{d}j_{Up} & p \in U \cap V \\ 0 & \text{otherwise.} \end{cases}$$

and $\eta \in \Omega^k(V)$ likewise will do the trick. □

The short exact sequence of the graded differential algebra mentioned above is actually a short exact sequence of chain complexes since the horizontal induced maps cannot change the degree of differential forms and these squares commutes.

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^{q+1}(M) & \longrightarrow & \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) & \longrightarrow & \Omega^{q+1}(U \cap V) & \longrightarrow & 0 \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & \Omega^{q}(M) & \longrightarrow & \Omega^{q}(U) \oplus \Omega^{q}(V) & \longrightarrow & \Omega^{q}(U \cap V) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

By snake lemma such short exact sequence induce a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^q(M) \longrightarrow H^q(U) \oplus H^q(V) \longrightarrow H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(M) \longrightarrow H^{q+1}(U) \oplus H^{q+1}(V) \longrightarrow H^{q+1}(U \cap V) \longrightarrow \cdots$$

and this is the main result of this lecture. For some purpose (i.e. when we regard the cohomology chain as a graded differential algebra around its "cup product" (that is, exterior product)) we need to find out what is the generator of each group. This can be done by careful diagram chasing. It turns out that the coboundary operator is given by

$$d^*[\omega]_p = \begin{cases} [-d(\rho_V(p)\omega_p \circ \Lambda^k \mathrm{d}i_{Up})] & p \in U, \\ [d(\rho_U(p)\omega)_p \circ \Lambda^k \mathrm{d}i_{Vp})] & p \in V. \end{cases} \tag{10}$$

which is actually a series of operation:

$$\Omega^q(U \cap V) \xrightarrow{\text{extend}} \Omega^q(U) \oplus \Omega^q(V) \xrightarrow{d} \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) \xrightarrow{\text{glue}} \Omega^q(M) \tag{11}$$

Having been tortured by writing these $\Lambda^k(\mathrm{d}i_U)_p$ things, we allow ourselves to abuse the notation a little bit to identify T_pM and T_pU the same thing based on the fact that they are actually isomorphic.

Example 2.1 (Cohomology of S^1). Decompose the circle S^1 into left and right hemispheres called U and V. Then we get the following exact sequence

$$\begin{aligned}
0 &\longrightarrow H^0(S^1) = \mathbb{R} \longrightarrow H^0(U) \oplus H^0(V) = \mathbb{R} \oplus \mathbb{R} \longrightarrow H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R} \\
&\xrightarrow{d^*} H^1(S^1) \longrightarrow H^1(U) \oplus H^1(V) = 0 \longrightarrow H^1(U \cap V) = 0 \\
&\xrightarrow{d^*} 0 \longrightarrow 0 \longrightarrow \cdots
\end{aligned}$$

in which $H^0(S^1) \cong H^0(U) \cong H^0(V) \cong \mathbb{R}$ is easy to see (and this is a common property of a qualified cohomology theory); $H^1(U) \cong H^1(V) \cong H^1(U \cap V) = 0$ is pretty intuitive since these spaces are "flat". Therefore we get the following

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(S^1) \longrightarrow 0$$

and it is quite lucky that we can calculate $H^1(S^1) = \mathbb{R}$ directly from exactness without looking into the details of these connecting arrows.

We are particularly interested in determine the generator of $H^1(S^1)$. It is clear that $H^1(S^1)$ is generated by $d^*([\alpha])$, in which α is a non-exact form i.e. α takes different constant value on upper and lower piece of $U \cap V$. We pick α to be constantly 1 on the upper and 0 on the lower piece. Then

$$d^*[\alpha] = \begin{cases} [-\rho_V(p)d(\alpha)_p] & p \in U, \\ [\rho_U(p)d(\alpha)_p] & p \in V. \end{cases} \tag{12}$$

since these two parts agree on their overlap, the representative of $d^*[\alpha]$ is forced to be supported on the upper piece of $U \cap V$. It turns out that it is a bump 1-form supported on upper $U \cap V$.

3 Mayer-Vietories Sequence of De Rham Cohomology with Compact Support

We can build the Mayer-Vietoris sequence of de Rham cohomology with compact supports almost alongside that of ordinary de Rham cohomology but with some subtle points to deal with. We cannot still use pullbacks to define the functor since compactness of a form is not preserved in general. To fix this, we restrict our consideration to an appropriate subset of smooth maps, inclusions. We use the following definition:

Definition 3.1 (Pushout of compactly supported form along inclusion)**.** If $j : U \hookrightarrow M$ is the inclusion of open set U in manifold M, then $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ is the map which "extends" a form on U by zero. Such extension $j_*\omega$ is surely smooth since the form is compactly supported. In fact, if a point of M is inside $\text{Supp} j_*\omega$, then p is inside U and has a neighbor on which ω is smooth; if not, then ω is constantly zero on a neighborhood around p, thus it is still smooth. (This demonstrates the benefits of defining support to be the *closure* of the non-vanishing points.) (Letting an open set be the "soft cover" of some closed set is also a usual trick to avoid some harsh edge cases from topology.)

Although also entitled with the name "cohomology", de Rham cohomology with compact support is actually *not* a qualified cohomology theory in the sense of Eilenberg–Steenrod axioms, for it does not satisfy homotopy invariance *and it is even a covariant functor!* However, as we will see soon, this kind of "cohomology" actually fits into the place of homology when we talk about pairings in the context of Poincáre's duality.

Then for the sequence of inclusions that we considered before,

$$M \leftarrow U \coprod V \leftleftarrows U \cap V$$

we apply $\Omega_c^*(\bullet)$ and get

$$\begin{gathered} \Omega_c^*(M) \xleftarrow{\ +\ } \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{\ \delta\ } \Omega_c^*(U \cap V) \\ \\ (-j_{U*}\omega, j_{V*}\omega) \longleftarrow \omega \end{gathered} \tag{13}$$

Proposition 3.1. *The Mayer-Vietoris sequence of forms with compact support*

$$0 \longleftarrow \Omega_c^*(M) \longleftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longleftarrow \Omega_c^*(U \cap V) \longleftarrow 0 \tag{14}$$

is exact.

Proof. The exactness at $\Omega_c^*(U \cap V)$ and $\Omega_c^*(U) \oplus \Omega_c^*(V)$ is trivial. For $\Omega_c^*(M)$, we restrict the form on M to U and V respectively and use the same bump-function trick to handle the troublesome overlapping area. ☐

This short exact sequence induce a long exact sequence of cohomology groups (looks similar but totally different!)

$$\begin{array}{c} \hookrightarrow H_c^{q+1}(U \cap V) \longrightarrow H_c^{q+1}(U) \oplus H_c^{q+1}(V) \longrightarrow H_c^{q+1}(M) \longrightarrow \cdots \\ \cdots \longrightarrow H_c^q(U \cap V) \longrightarrow H_c^q(U) \oplus H_c^q(V) \longrightarrow H_c^q(M) \xrightarrow{\ d_*\ } \end{array}$$

in which

$$d_*[\omega] = \begin{cases} [-d(\rho_V\omega)] & p \in U \\ [d(\rho_U\omega)] & p \in V \end{cases} \tag{15}$$

Example 3.1. (Compactly supported De Rham cohomology of S^1) Cover S^1 with U and V as before.

$$
\begin{array}{l}
\hookrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\
\hookrightarrow H_c^1(U \cap V) = \mathbb{R} \oplus \mathbb{R} \longrightarrow H_c^1(U) \oplus H^1(V) = \mathbb{R} \oplus \mathbb{R} \longrightarrow H_c^1(S^1) = \mathbb{R} \xrightarrow{d^*} \\
0 \longrightarrow H_c^0(U \cap V) = 0 \longrightarrow H_c^0(U) \oplus H_c^0(V) = 0 \longrightarrow H^0(S^1) = \mathbb{R} \xrightarrow{d^*}
\end{array}
$$

It is clear that $H_c^0(U \cap V)$ and $H_c^0(U) \oplus H_c^0(V) = 0$ since the only compactly supported constant function on them is the zero function. However since S^1 is compact itself, any constant function is still valid. Thus $H_c^0(S^1) = \mathbb{R}$. Now we can deduce that $H_c^1(S^1) = \mathbb{R}$ by playing with dimensions of linear maps.