Geometric phases in physics

Gen YUE

October 2021

Path integral and geometric phase

For some physical systems described by a Langrangian $\mathcal{L}$, when we integrate out the spacetime and get the action, there may be a part of the action which does not depend on the specific world line, but only depend on it's geometric properties. We put it on exponential part and name it geometric phase or topological phase.

Example 1: Aharonov-Bohm effect
Consider an electron in the electromagnetic field given by a thin solenoid:

$$L = \frac{1}{2} m \boldsymbol{v}^2 + e \boldsymbol{A} \cdot \boldsymbol{v} \tag{1}$$

For a given path $(\boldsymbol{x}_f, t_f; \boldsymbol{x_i}, t_i)$, the action is

$$S = S_0 + e \int \boldsymbol{A} \cdot d\boldsymbol{x} \tag{2}$$

where S_0 is the dynamical term. If the path is a closed loop around the solenoid, $e \int \boldsymbol{A} \cdot d\boldsymbol{x} = e\Phi$, Φ is the flux in the solenoid. For a closed loop, the propagator is given by the path integral:

$$\langle \boldsymbol{x_f} | U(t_f, t_i) | \boldsymbol{x_i} \rangle = \int \mathcal{D}[\boldsymbol{x}] e^{iS[\boldsymbol{x}]} = \sum_n e^{\frac{ine\Phi}{\hbar}} \int_n \mathcal{D}[\boldsymbol{x}] e^{iS_0[\boldsymbol{x}]} \tag{3}$$

define the flux quantum $\phi_0 = \frac{h}{e}$, the phase

$$e^{i2\pi n \frac{\Phi}{\phi_0}} \tag{4}$$

is the geometric phase.

Example 2: Field theory of 1+1d anti-ferro Heisenberg chain

$$H = -J \sum_{<i,j>} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{5}$$

we can use the spin path integral strategy to get the Euclidean action of the system:

$$S_E = \int d^2x \frac{1}{2g} (v_s (\partial_1 \boldsymbol{m})^2 + \frac{1}{v_s} (\partial_2 \boldsymbol{m})^2) + i \frac{\theta}{8\pi} \int d^2x \ \epsilon_{ij} \boldsymbol{m} \cdot (\partial_i \boldsymbol{m} \times \partial_j \boldsymbol{m}) \tag{6}$$

where g, v_s, θ are parameters which can be expressed by the coupling constant J, lattice spacing a and spin s. $\boldsymbol{m}$ is a unit vector, which can be viewed as a map from (compacted) space time S^2 to S^2.

The first term is so called $O(3)$ NLσM, which is dynamical, and the second term is indeed topological. Since the field $\boldsymbol{m}$ is a map from S^2 to S^2, so all field configuration can be classified by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Two field configurations can be differernt in detials, but as long as they belong to the same (homotopic) equivalent class, the second term of the action will give the same value, which is an integer, so called winding number, multiplier of $i\theta$. Such term is defined as θ term, mathematically, it is a mapping degree.

Adiabatic theorem, Berry phase, Chern number

Consider a system which is parameterized by λ, λ changes slowly with time. For any time, the Hamiltonian of the system has a completed spectrum

$$H(\lambda)|\phi_n(\lambda)\rangle = E_n(\lambda)|\phi_n(\lambda)\rangle$$

The adiabatic theorem says that: if we start from an non-degenerate eigenstate $|\phi_n(\lambda(0))\rangle$ at $t = 0$, and let the state evolves over time and make sure the no degeneracy is opened (we say such system is adiabatically protected), then at time t, the state will almost be $|\phi_n(\lambda(t))\rangle$ up to two phase factors:

$$|\psi\rangle = e^{i\gamma} e^{\int dt' - \frac{iE_n(t')}{\hbar}} |\phi_n(\lambda(t))\rangle$$

where

$$\gamma = i \int \mathcal{A}_i \ d\lambda_i$$

is the Berry phase,

$$\mathcal{A}_i = i\langle\phi_n(\lambda)|\partial_{\lambda_i}|\phi_n(\lambda)\rangle \tag{7}$$

is the Berry connection.

If the parameter's trajectory is a closed loop in the parameter space, then by stokes' theorem:

$$\gamma = i \oint \mathcal{A}_i \ d\lambda_i = i \iint \mathcal{F}_{ij} d\lambda_i d\lambda_j \tag{8}$$

where

$$\mathcal{F}_{ij} = \partial_j \mathcal{A}_i - \partial_i \mathcal{A}_j \tag{9}$$

is the Berry flux.

Here emerges $U(1)$ gauge field structure: if we add a local $U(1)$ factor on to the state:

$$|\phi_n(\lambda)\rangle \longrightarrow e^{if(\lambda)}|\phi_n(\lambda)\rangle$$

then

$$\mathcal{A}_i \longrightarrow \mathcal{A}_i - \partial_{\lambda_i} f \tag{10}$$

and $\mathcal{F}_{ij}$ stay the same, so Berry phase is gauge invariant.

Example 1: spin-$\frac{1}{2}$ in magnetic field (exercise in Griffith, the Berry phase is half (since spin is half) of the solid angle given by the curve formed by the variation of magnetic field direction parameters.)

The Berry connection of eigenstate $|\boldsymbol{n},+\rangle$ is

$$\boldsymbol{\mathcal{A}}^{+} = -\frac{sin^2(\theta/2)}{r sin\theta}\hat{\phi}$$

when $\theta = \pi$, it's not well defined. Similarly,

$$\boldsymbol{\mathcal{A}}^{-} = -\frac{cos^2(\theta/2)}{r sin\theta}\hat{\phi}$$

which is not well defined at $\theta = 0$.

Now, if we integrate out on the whole parameter space, the Berry phase must be an integer multiplier of 2π. Such integer is called first Chern number

$$c^1 = \frac{i}{2\pi}\int_{\lambda} \mathcal{F}_{ij} d\lambda_i d\lambda_j \tag{11}$$

For the example given above, the total solid angle for a sphere is 4π, so the (first) Chern number is $c^1(\pm) = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot(\mp 4\pi) = \mp 1$. Note that the sum of the two Chern numbers is 0. (I don't know how to explain it mathematically now, but I think it reveals the vector bundle is somehow trivial...)

Example 2:(2-level Lattice system)

Consider the Hamiltonian $H(k) = h^i(k)\cdot\sigma_i$, where $k \in BZ \simeq T^2$,$h^i : BZ \longrightarrow \mathbb{R}$ is a real sooth function, and $|h| = \sqrt{\sum_i {h^i}^2} \neq 0$ for all k to keep the system gapped. $\frac{\boldsymbol{h}}{|h|} : T^2 \longrightarrow S^2$,

The Berry curvature is then

$$\mathcal{F}(k) = -\frac{i}{2|h|^3}\boldsymbol{h}\cdot(\partial_{k_x}\boldsymbol{h}\times\partial_{k_y}\boldsymbol{h})$$

so the first Chern number is

$$c^1 = \frac{i}{2\pi}\int_{BZ} dk_x dk_y \ \mathcal{F}(k)$$

If you are familiar with winding number, you'll see that this expression is exactly a winding number! So why in this case Chern number is equal to winding number? The answer is that:
1. T^2 and S^2 are both two dimensional manifold, and there is a canonical map from T^2 to S^2.
2. The Chern number on S^2 is 1.

I give a brief explanation here.

Let's first give an exact definition of Chern number:

Given a base manifold M, and let E be the complex vector bundle, suppose the fibre are r dimension, then we can define Chern Class $c_i(E) \in H^{2i}(M;\mathbb{Z})$, the first Chern class is defined as $c_1(E) = \frac{1}{2\pi}\Omega$, where Ω is the curvature 2-form on M. Define the first Chern number:

$$c^1(E) = \int_M c_1(E) \tag{12}$$

Suppose $f : M \longrightarrow N$ with $dim(M) = dim(N)$, if E is a complex vector bundle on N, then f^*E is a complex vector bundle on M, and satisfies

$$c_1(f^*E) = f^*c_1(E)$$

Now integrate on M, we get the relation between Chern number:

$$c^1(f^*E) = deg(f) \cdot c^1(E)$$

Then it's clear why the Chern number is a mapping degree in our example: there is a complex vector bundle on S^2, where the base manifold is parameter space of $\frac{\boldsymbol{h}}{|h|}$,i.e. S^2, and at each point we assign a 2-d Hilbert space which is spanned by the eigenstates of the local Hamiltonian $H(h)$. Such structure is just the same as the spin in magnetic field, whose Chern number is 1. There is also a complex vector bundle on T^2, where the base manifold is the parameter space of k which is just BZ, and we assign a 2-d Hilbert space at each k, which is spanned by the eigenstates of local Hamiltonian $H(k)$. Since $\frac{\boldsymbol{h}}{|h|} : T^2 \longrightarrow S^2$, so the complex vector bundle on T^2 is just the pullback bundle of on S^2, so numerically, the Chern number here on T^2 is equal to a mapping degree.

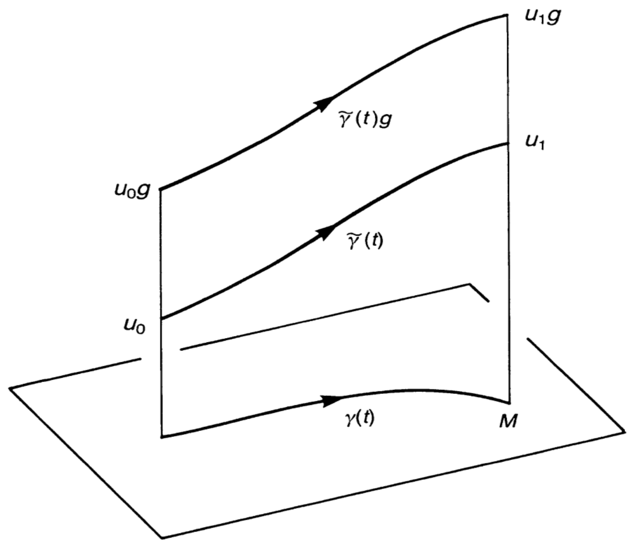
Why connection is called connection?(not completed yet...)

In vector bundle, we use connection to relates the vector at different places. In Principal bundle, the story is quite similar: we want to relate fibers from different points at base manifold.

Through the definition of connection (pullback of connection one form on principal bundle), we can define the horizontal lift of a curve γ in base manifold, which is a curve $\tilde{\gamma}$ in the principal bundle, the point is said to be horizontally transported when moving on $\tilde{\gamma}$. $\tilde{\gamma}(t)$ is defined as:

$$\tilde{\gamma}(t) = s_i(\gamma(t))\mathcal{P}exp(-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i,\mu}dx^{\mu}) \tag{13}$$

where s_i is the local section, $\mathcal{P}$ is the path ordering operator, since for general gauge group, $\mathcal{A}$s are matrices and don't commute with each other.



The factor $\mathcal{P}exp(-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i,\mu}dx^{\mu})$ is the generalized Berry phase.