On the Cayley-persistence algebra

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This is a joint work with
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Contents

Persistent homology

- In 2005, G. Carlsson, A. Zomorodian et al. introduced the persistent homology for extracting topological features.

Figure 1: Gunnar Carlsson, Stanford University

- In 2009, G. Carlsson, A. Zomorodian et al. introduced the multidimensional (or multi-parameter) persistent homology to deal with multi-parameter filtration of simplicial complexes.

Persistent homology

- In 2010, G. Carlsson and V. D. Silva developed the theory of zigzag persistence for studying the persistence of topological features across a family of spaces or point-cloud data sets.
- In 2011, the persistent cohomology (or copersistence) was introduced to identify candidates for significant circle-structures in the data.

Persistent homology

We compare the above outstanding variants of persistence as follows.

Different persistences	Gradings	The corresponding spaces
The usual persistence	$\mathbb{Z}$ (or $\mathbb{R}$)	$\mathbb{Z}$-graded vector spaces
Multidimensional persistence	$\mathbb{Z}^n$	$\mathbb{Z}^n$-graded vector spaces
Zigzag persistence	Zigzag sequence	Zigzag diagrams of vector spaces
Copersistence	$\mathbb{Z}$	$\mathbb{Z}$-graded dual vector spaces

Persistence module

Usually, a persistence module is a functor $\mathcal{F} : \mathrm{cat}(\mathbb{Z}, \leq) \to \mathbf{Vec_k}$ from the category $\mathrm{cat}(\mathbb{Z}, \leq)$ to the category of $\mathbf{k}$-linear spaces. Let $\mathcal{K} : \mathrm{cat}(\mathbb{Z}, \leq) \to \mathbf{Simp}$ be a persistence simplicial complex, that is, a filtration of simplicial complexes satisfies

(i) For each $i \in \mathbb{Z}$, $\mathcal{K}_i$ is a simplicial complex. For any integers $i \leq j$, there is a morphism of simplicial complexes $f_{i,j} : \mathcal{K}_i \to \mathcal{K}_j$.

(ii) For $i \leq j \leq k$, we have $f_{j,k} \circ f_{i,j} = f_{i,k}$.

Persistence module

One has that the functor

$$H_*(\mathcal{K};\mathbf{k}) : \mathrm{cat}(\mathbb{Z},\leq) \to \mathbf{Vec_k}, \quad i \mapsto H_*(\mathcal{K}_i;\mathbf{k})$$

is a persistence module.
For $i \leq j$, the (i,j)-persistent homology is defined by

$$H_*^{i,j} = \mathrm{im}(H_*(\mathcal{K}_i;\mathbf{k}) \to H_*(\mathcal{K}_j;\mathbf{k})).$$

Persistence module

Let $\mathbf{H} = \bigoplus_{i\in\mathbb{Z}} H_*^i$ be a graded $\mathbf{k}$-linear space. The morphism $t : H_*(\mathcal{K}_i;\mathbf{k}) \to H_*(\mathcal{K}_{i+1};\mathbf{k})$ induces a morphism

$$t : \mathbf{H} \to \mathbf{H}$$

of degree 1. Consider the polynomial ring $\mathbf{k}[t]$. For any $f(t) = \sum_{k=0}^{n} a_k t^k \in \mathbf{k}[t]$, we have a morphism

$$f(t) : \mathbf{H} \to \mathbf{H}$$

given by $f(t)(\alpha) = \sum_{k=0}^{n} a_k \cdot \overbrace{t \circ t \circ \cdots \circ t}^{k}(\alpha)$. This shows that $\mathbf{H}$ is a graded left $\mathbf{k}[t]$-module given by

$$\mathbf{k}[t] \times \mathbf{H} \to \mathbf{H}, \quad (f(t), \alpha) \mapsto f(t)(\alpha).$$

Persistence module

Theorem (A. Zomorodian & G. Carlsson)

If $\mathbf{H}$ is a finitely generated $\mathbf{k}[t]$-module, then we have a finite direct decomposition

$$\mathbf{H} \cong \left(\bigoplus_{i=1}^{k} \mathbf{k}[t] \cdot e_i^{b_i} \right) \oplus \left(\bigoplus_{j=1}^{l} \frac{\mathbf{k}[t]}{\mathbf{k}[t] \cdot t^{s_j}} \cdot \varepsilon_j^{r_j} \right),$$

where $e_i^{b_i}, \varepsilon_j^{r_j}$ are generators of degree b_i, r_j, respectively.

Note that $\mathbf{k}[t]$ is a principal ideal domain (PID). The proof of this decomposition mainly depends on the structure theorem of finitely generated modules over a PID.

Group grading and Cayley digraph

In this talk, we consider the persistence based on group grading. To give persistence on a group, we recall the Cayley digraph, which endows a group with a direction in some sense.

Definition

Let G be a group and S be a subset of G. A Cayley digraph $\mathrm{Cay}(G, S)$ is a digraph with the elements of G as vertices and the pairs $(a, b) \in G \times G$ satisfying $ba^{-1} \in S$ as arcs.

Examples of digraphs

- $(\mathbb{R}^n, \mathbb{R}^n_+)$ is a Cayley digraph. Here, $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$.
- $(\mathbb{Z}^n, \mathbb{Z}^n_+)$ is a Cayley digraph. Here, $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$.
- $(\mathbb{Z}/p, \mathbb{Z}/p)$ is a Cayley digraph.
- Let $G = F[x_1, \ldots, x_n]$ be a free group generated by $x_1, \ldots, x_n$, and let $S = F^+[x_1, \ldots, x_n]$ be the free monoid generated by $x_1, \ldots, x_n$. Then (G, S) is a Cayley digraph.

Cayley persistence

We may regard a Cayley digraph $\mathrm{Cay}(G, S)$ as a category, denoted by $\mathrm{cat}(\mathrm{Cay}(G, S))$, with the vertices as objects and directed paths as morphisms.
If S is a subset of G, let $\langle S \rangle$ be the monoid generated by S. Then we have

$$\mathrm{cat}(\mathrm{Cay}(G, S)) = \mathrm{cat}(\mathrm{Cay}(G, \langle S \rangle)).$$

Cayley persistence

Let $\mathfrak{C}$ be a category. A Cayley-persistence object is a functor $\mathcal{F} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathfrak{C}$ from the category $\mathrm{cat}(\mathrm{Cay}(G,S))$ to $\mathfrak{C}$.

Dually, the Cayley-copersistence object is a contravariant functor $\mathcal{F} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathfrak{C}$ from the category $\mathrm{cat}(\mathrm{Cay}(G,S))$ to the category $\mathfrak{C}$.

Cayley persistence

Let $l_x : a \to xa$ be a G-graded map for $a \in G, x \in S$ in the category $\mathrm{cat}(\mathrm{Cay}(G, S))$. Then we have a morphism

$$\mathcal{F}(l_x) : \mathcal{F}_a \to \mathcal{F}_{xa}, \quad a \in G, x \in S$$

in category $\mathfrak{C}$. Moreover, the set $\{\mathcal{F}(l_x)\}_{x \in S}$ can be regarded as a monoid with composition as multiplication, denoted by $\mathcal{L}_S$.

Cayley persistence

- In 2010, G. Carlsson and A. Zomorodian considered classification of the multidimensional persistence module.
- In this talk, we have a Cayley-persistence structure of modules based on the finitely generated module.

Cayley persistence

Theorem (Wanying Bi, Jingyan Li, Jian Liu & Jie Wu)

Let $M_{\mathcal{F}}$ be a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module and S be a finitely generated monoid. Then we have a finite direct sum decomposition

$$M_{\mathcal{F}} \cong \bigoplus_{i=1}^{k} \mathbf{k}[\mathcal{L}_S] \cdot e_i^{x_i} \oplus \left(\left(\bigoplus_{j=1}^{l} \mathbf{k}[\mathcal{L}_S] \cdot \varepsilon_j^{y_j} \right) / N \right)$$

for some k, l, where $e_i^{x_i} \in \mathcal{F}_{x_i}, \varepsilon_j^{y_j} \in \mathcal{F}_{y_j}$ and N is a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module generated by the elements of the form $\mathcal{F}(l_y)\varepsilon_t^{y_t} - \mathcal{F}(l_{yy_t y_s^{-1}})\varepsilon_s^{y_s}$ for some $1 \leq s, t \leq l, y \in S$.

Cayley persistence

Corollary

Let $\mathcal{F} : \text{cat}(\text{Cay}(G, S)) \to \mathbf{Vec}_{\mathbf{k}}$ be a Cayley-persistence $\mathbf{k}$-linear space. If $\mathcal{F}_G = \bigoplus_{x \in G} \mathcal{F}_x$ is a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module and S is a finitely generated monoid, then we have a finite direct sum decomposition

$$\mathcal{F}_G \cong \bigoplus_{i=1}^{k} \mathbf{k}[\mathcal{L}_S] \cdot e_i^{x_i} \oplus \left(\left(\bigoplus_{j=1}^{l} \mathbf{k}[\mathcal{L}_S] \cdot \varepsilon_j^{y_j} \right) / N \right)$$

for some k, l, where $e_i^{x_i} \in \mathcal{F}_{x_i}, \varepsilon_j^{y_j} \in \mathcal{F}_{y_j}$ and N is a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module generated by the elements of the form $\mathcal{F}(l_y)\varepsilon_t^{y_t} - \mathcal{F}(l_{y y_t y_s^{-1}})\varepsilon_s^{y_s}$ or $\mathcal{F}(l_y)\varepsilon_t^{y_t}$ for some $1 \leq s, t \leq l, y \in S$.

Cayley persistence

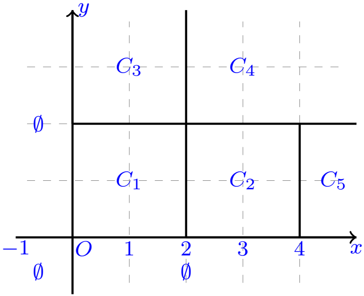
The condition that $M_{\mathcal{F}}$ is a finitely generated $R_{\mathcal{F}}$-module ensures the existence of the "born time" of the generators. The condition that S is a finitely generated monoid makes it possible for us to obtain the finiteness of the time of disappearance and meeting of generators, that is, N is a finitely generated $R_{\mathcal{F}}$-module.

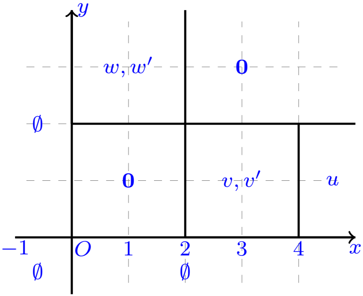
Cayley persistence

Note that $e_i^{x_i} \in \mathcal{F}_{x_1}, \varepsilon_j^{y_j} \in \mathcal{F}_{y_j}$ represent the elements coming into being at x_i, y_j, respectively. A generator $\mathcal{F}(l_y)\varepsilon_t^{y_t} - \mathcal{F}(l_{yy_t y_s^{-1}})\varepsilon_s^{y_s}$ of N shows the information that two elements $\varepsilon_t^{y_t}, \varepsilon_s^{y_s}$ become the same one at the time yy_t. We denote $y_{st} = yy_t$. Then the information of N can be represented by all the triples $(\varepsilon_s^{y_s}, \varepsilon_t^{y_t}, y_{st})$, which means the two generators appearing at y_s, y_t and meeting at y_{st}.

A example

Let $\mathcal{F} : (\mathbb{R}^2, \mathbb{R}^2_+) \to \mathbf{Vec_k}$ be a Cayley-persistence vector space, where $\mathbf{Vec_k}$ is a category of vector spaces. Let $\mathbf{V}_v = \mathbf{k}v \oplus \mathbf{k}v'$, $\mathbf{V}_w = \mathbf{k}w \oplus \mathbf{k}w'$, and let $\mathbf{0}$ be the null space.





An example

Let $C_1 = \{(x,y) | 0 \le x, y < 2\}$, $C_2 = \{(x,y) | 2 \le x < 4, 0 \le y < 2\}$, $C_3 = \{(x,y) | 0 \le x < 2, y \ge 2\}$, $C_4 = \{(x,y) | 0 \le x, y \le 2\}$, $C_5 = \{(x,y) | x \ge 4, 0 \le y < 2\}$. The functor $\mathcal{F}$ is given by

$$\mathcal{F}_P = \begin{cases} \mathbf{V}_v, & P \in C_2; \\ \mathbf{V}_w, & P \in C_3; \\ \mathbf{V}_u, & P \in C_5; \\ \mathbf{0}, & P \in C_1 \cup C_4; \\ \emptyset, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{F}_{P \to Q} = \begin{cases} \text{id}, & P, Q \in C_2, C_3; \\ \varphi, & P \in C_2, Q \in C_5; \\ 0, & P \in C_1 \text{ or } Q \in C_4; \\ \emptyset, & \text{otherwise.} \end{cases}$$

An example

Here, $\varphi : \mathbf{V}_v \to \mathbf{k}u$ is given by $\varphi(v) = \varphi(v') = u$. Then w, w' begin to appear at $(0,2)$ and disappear at $(2,2)$ while v, v' begin to appear at $(2,0)$, meet at $(4,0)$ and disappear at $(2,2)$. A straightforward calculation shows that

$$\mathcal{F}_G \cong \frac{R_{\mathcal{F}} \cdot e_v^{(2,0)} \oplus R_{\mathcal{F}} \cdot e_{v'}^{(2,0)} \oplus R_{\mathcal{F}} \cdot e_w^{(0,2)} \oplus R_{\mathcal{F}} \cdot e_{w'}^{(0,2)} \oplus R_{\mathcal{F}} \cdot e_u^{(4,0)}}{N},$$

where N is an $R_{\mathcal{F}}$-module generated by

$$\mathcal{F}_{l_{(0,2)}} e_v^{(2,0)}, \mathcal{F}_{l_{(0,2)}} e_{v'}^{(2,0)}, \mathcal{F}_{l_{(2,0)}} e_w^{(0,2)}, \mathcal{F}_{(l_{(2,0)}} e_{w'}^{(0,2)},$$
$$\mathcal{F}_{l_{(2,0)}} e_v^{(2,0)} - e_u^{(4,0)}, \mathcal{F}_{l_{(2,0)}} e_{v'}^{(2,0)} - e_u^{(4,0)}.$$

An example

By reduction, we get

$$\mathcal{F}_G \cong \frac{R_{\mathcal{F}} \cdot e_v^{(2,0)} \oplus R_{\mathcal{F}} \cdot e_{v'}^{(2,0)} \oplus R_{\mathcal{F}} \cdot e_w^{(0,2)} \oplus R_{\mathcal{F}} \cdot e_{w'}^{(0,2)}}{N'},$$

where

$$\begin{aligned} N' =& R_{\mathcal{F}} \cdot \mathcal{F}(l_{(0,2)}) e_v^{(2,0)} \oplus R_{\mathcal{F}} \cdot \mathcal{F}(l_{(0,2)}) e_{v'}^{(2,0)} \oplus R_{\mathcal{F}} \cdot \mathcal{F}(l_{(2,0)}) e_w^{(0,2)} \\ &\oplus R_{\mathcal{F}} \cdot \mathcal{F}(l_{(2,0)}) e_{w'}^{(0,2)} \oplus R_{\mathcal{F}} \cdot \mathcal{F}(l_{(2,0)}) (e_v^{(2,0)} - e_{v'}^{(2,0)}). \end{aligned}$$

An example

For the two 1-parameter filtrations given by $y = 0$ and $y = x^3$, we have the following barcode diagrams.

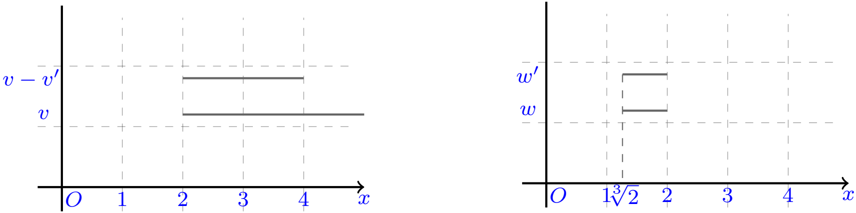


Figure 2: The barcodes of persistence sets obtained by $y = 0$ and $y = x^3$.

An example

We have a collection of triples

$$(e_v^{(2,0)}, \mathbf{0}, (2,2)), (e_{v'}^{(2,0)}, \mathbf{0}, (2,2)), (e_w^{(0,2)}, \mathbf{0}, (2,2)),$$
$$(e_{w'}^{(0,2)}, \mathbf{0}, (2,2)), (e_v^{(2,0)}, e_{v'}^{(2,0)}, (4,0)).$$

This gives a finite set to represent the information of infinite many 1-parameter filtrations in the parameter space. This also shows us the survival spaces rather then some survival intervals of the generators.

The Cayley persistent homology

When a Cayley persistent homology is a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module?

From now on, let $S \subseteq G$ be a monoid such that the identity 1 is the unique invertible element in S. Then the group G can be regarded as a poset with partial order given by $a \leq b$ if $ba^{-1} \in S$.

The Cayley persistent homology

Definition

A Cayley persistence object $\mathcal{F} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathfrak{C}$ is called noetherian if every ordered subset $X \subseteq G$ has an element x such that $\mathcal{F}(l_z) : \mathcal{F}_x \to \mathcal{F}_{zx}$ is the identity morphism for all $z \in S$.

A Cayley persistence object $\mathcal{F} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathfrak{C}$ is called lower bounded if there exists an element $a \in G$ such that $\mathcal{F}_x = \emptyset$ unless $x \geq a$.

The Cayley persistent homology

Theorem (Wanying Bi, Jingyan Li, Jian Liu & Jie Wu)

Let $\mathcal{F} : \mathrm{cat}(\mathrm{Cay}(G, S)) \to \mathbf{Vec_k}$ be a noetherian and lower bounded Cayley-persistence simplicial complex. If S is a finitely generated commutative monoid, then $\mathcal{F}(G)$ is a finitely generated $\mathbf{k}[\mathcal{L}_S]$-module.

The Cayley persistent homology

In practical application, the filtration of simplicial complexes always begins at a given parameter and becomes stable at a finite parameter. The beginning parameter indicates that

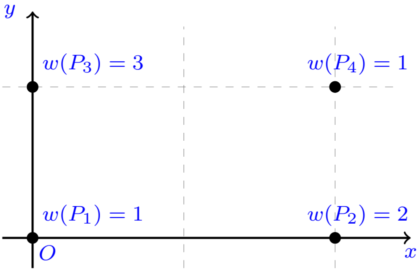
$$\mathcal{K} : \mathrm{cat}(\mathrm{Cay}(\mathbb{Z}^n, \mathbb{Z}^n_{\geq 0})) \to \mathbf{Simp}$$

is lower bounded, while "stable" implies that $\mathcal{K}$ is noetherian.

An example

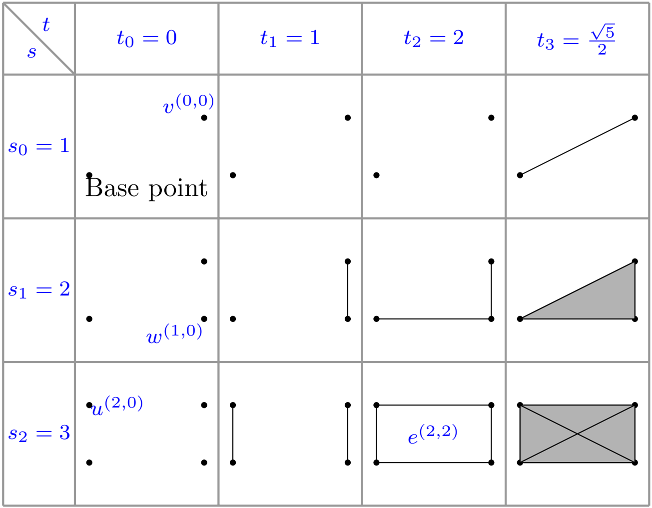
Let $V = \{x_1, x_2, x_3, x_4\}$, where $x_1 = (0,0), x_2 = (2,0), x_3 = (0,1), x_4 = (2,1)$ are points in $\mathbb{R}^2$. Consider the weight function $w : V \to \mathbb{R}$ given by

$$w(P_1) = 1, \quad w(P_2) = 2, \quad w(P_3) = 3, \quad w(P_4) = 1.$$



An example

Then we have a 2-parameter filtration $\mathcal{K}_{s,t}$ given as the following diagram.



An example

We have a finite direct sum decomposition

$$\mathbf{H}(\mathcal{K}) \cong \frac{\mathbf{k}[L_1, L_2] \cdot v^{(0,0)} \oplus \mathbf{k}[L_1, L_2] \cdot w^{(1,0)} \oplus \mathbf{k}[L_1, L_2] \cdot u^{(2,0)} \oplus \mathbf{k}[L_1, L_2] \cdot e^{(2,2)}}{N},$$

where N is a $\mathbf{k}[L_1, L_2]$-module generated by

$$L_2^3 v^{(0,0)}, \quad L_2 w^{(1,0)} - L_1 L_2 v^{(0,0)}, \quad L_2^2 w^{(1,0)}, \quad L_2 u^{(2,0)}, \quad L_2 e^{(2,2)}.$$

Here, $L_1 = H_*(\mathcal{K}(l_{(1,0)}); \mathbf{k})$ and $L_2 = H_*(\mathcal{K}(l_{(0,1)}); \mathbf{k})$.

Cayley persistence algebra

Let $\mathcal{K} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathbf{Simp}$ be a Cayley-persistence simplicial complex such that the homology $H_*(\mathcal{K}_a;\mathbf{k})$ is of finite dimension for all $a \in G$. Then the (reduced) cohomology induces a Cayley-copersistence module

$$H^*(\mathcal{K};\mathbf{k}) : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathbf{Vec}_{\mathbf{k}}.$$

Cayley persistence algebra

Let $S \subseteq G$ be a monoid such that the identity element e is the unique invertible element in S. Then the group G can be regarded as a poset with partial order given by $a \leq b$ if $ba^{-1} \in S$. Suppose that the category $\mathrm{cat}(\mathrm{Cay}(G, S))$ has finite product, for example, G is a lattice group. Let $\mathbf{H} = \bigoplus_{a \in G} H^*(\mathcal{K}_a; \mathbf{k})$. The morphism

$$L_x = H^*(\mathcal{K}_{a,xa}; \mathbf{k}) : H^*_{xa} \to H^*_a, \quad a \in G, x \in S.$$

induced by $a \to xa$ gives a right action on $\mathbf{H}$. Note that the set $\mathbb{L}_S = \{L_x\}_{x \in S}$ is a monoid with multiplication $L_x \cdot L_y = L_y \circ L_x$ for $x, y \in S$. Let $\mathbf{k}[\mathbb{L}_S]$ be a monoid ring of $\mathbb{L}_S$ over $\mathbf{k}$, then $\mathbf{H}$ is a right $\mathbf{k}[\mathbb{L}_S]$-module.

Cayley persistence algebra

For $\beta \in H^*_{xa}$ and $\gamma \in H^*_{ya}$, the cup product of β, γ at the time a is defined by

$$\beta \cup_a \gamma = (L_x \beta) \cup (L_y \gamma).$$

Definition

For $b, c \in G$, the persistence-cup product of $\beta \in H^*_b$ and $\gamma \in H^*_c$ is defined by

$$\beta \cdot \gamma = \beta \cup_{b \times c} \gamma.$$

Indeed, the persistence-cup product defined above is a product.

Cayley persistence algebra

Definition

Let R be a commutative ring with unit. An R-module A is an R-twisted algebra if there is a twisted multiplication $\cdot : A \times A \to A$ and a bilinear function $f : R \times R \to R$ satisfying

(i) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ for any $\alpha, \beta, \gamma \in A$.

(ii) $(\lambda\alpha) \cdot (\mu\beta) = f(\lambda, \mu)(\alpha \cdot \beta)$ for any $\lambda, \mu \in R, \alpha, \beta \in A$.

Cayley persistence algebra

Definition

Let $R = \bigoplus_{a \in G} R_a$ be a G-graded ring. A G-graded R-twisted algebra $A = \bigoplus_{a \in G} A_a$ is a G-graded R-module with twisted multiplication $\cdot$ and bilinear functions $f_{a,b} : R \times R \to R, a, b \in G$ satisfying

(i) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ for any $\alpha, \beta, \gamma \in A$.

(ii) $(\lambda\alpha) \cdot (\mu\beta) = f_{a,b}(\lambda, \mu)(\alpha \cdot \beta)$ for any $\lambda, \mu \in R, \alpha \in A_a, \beta \in A_b$.

Cayley persistence algebra

Theorem (Wanying Bi, Jingyan Li, Jian Liu & Jie Wu)

Let G be an abelian group. The persistence-cup product on $\mathbf{H}$ is uniquely determined by the persistence-cup product of $\mathbf{k}[\mathbb{L}_S]$-module generators. Moreover, $(\mathbf{H}, \cdot)$ is a G-graded $\mathbf{k}[\mathbb{L}_S]$-twisted algebra.

Order group persistence module

Let G be an abelian order group. By endowing S with an interval topology, we have a topological monoid S. There is a decomposition theorem of persistent cohomology for order group grading.

Theorem

Let $\mathbf{H}$ be a finitely generated R-module. Then we have a finite direct sum decomposition

$$\mathbf{H} \cong \bigoplus_{i=1}^{k} e^{i}_{x_i} \cdot R \oplus \bigoplus_{j=1}^{l} \frac{\varepsilon^{j}_{y_j} \cdot R}{\mathbf{k}[I_j]}$$

for some k, l, where $\mathbf{k}[\overline{I_j}] = \varepsilon^{j}_{y_j} \cdot L_{w_j} \cdot R$ and $\overline{I_j}$ is the closure of I_j for some $I_j \subseteq S$, $w_j \in S$, $j = 1, \ldots, l$.

Note that $\mathbf{k}[I_j]$ does not have to be a finitely generated R-module.

The barcode for persistence-cup product

Denote

$$I(\alpha) = \mathrm{supp}(\alpha) = \{x \in G | \alpha_x \neq 0\}$$

the support of α in grading G. We have a description of barcode for the persistence-cup product

Theorem (Wanying Bi, Jingyan Li, Jian Liu, & Jie Wu)

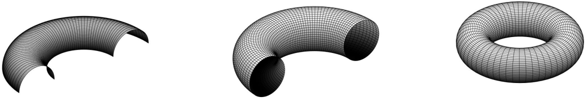
Let G be an abelian order group. Let α, β be two of the generators of $\mathbf{H}$ as R-module. If $\alpha \cup \beta$ is nontrivial, then

$$\min(\sup(I(\alpha)), \sup(I(\beta))) \leq \sup(I(\alpha \cup \beta)),$$
$$\max(\inf(I(\alpha)), \inf(I(\beta))) \leq \inf(I(\alpha \cup \beta)).$$

An example

Consider the filtrations of torus as follows.

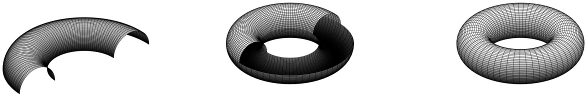
(i) Grow from a rectangle to a cylinder and then to a torus.



We set $\mathcal{K}_0 \simeq pt$, $\mathcal{K}_1 \simeq S^1$, $\mathcal{K}_2 \simeq S^1 \times S^1$, and $\mathcal{K}_3 \simeq S^1 \times S^1$.

An example

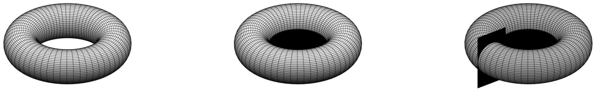
(ii) Grow from a rectangle to an incomplete torus, and finally come into a torus.



We set $\mathcal{K}'_0 \simeq pt$, $\mathcal{K}'_1 \simeq S^1 \vee S^1$, $\mathcal{K}'_2 \simeq S^1 \times S^1$, and $\mathcal{K}'_3 \simeq S^1 \times S^1$.

An example

(iii) Grow from a rectangle to an incomplete torus, and finally come into a torus.



We set $\mathcal{K}_0'' \simeq S^1 \times S^1$, $\mathcal{K}_1'' \simeq S^1 \vee S^2$, $\mathcal{K}_2'' \simeq S^2$, and $K_3'' \simeq S^2$.

An example

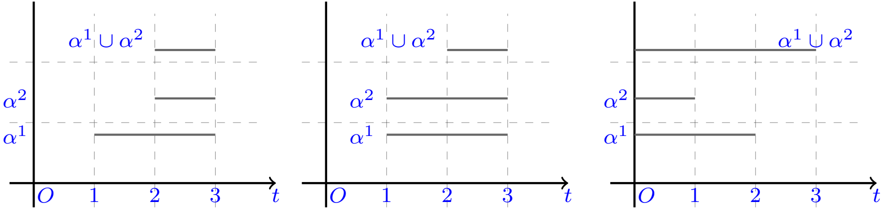


Figure 3: The persistence-cup products on tori of different filtrations.

Persistence products on manifolds

In this talk, all the manifolds considered are assumed to be compact orientable n-manifold without boundary.

Let G be an abelian group, and let $S \subseteq G$ be a monoid such that the identity 1 is the unique invertible element in S. Let $\mathcal{M} : \mathrm{cat}(\mathrm{Cay}(G,S)) \to \mathbf{Mani}$ be a Cayley-persistence manifold. The homology and cohomology considered are unreduced

Persistence products on manifolds

Recall that the Poincaré duality

$$D : H^p(M;\mathbf{k}) \to H_{n-p}(M;\mathbf{k})$$

defined by $D(\alpha) = \omega \cap \alpha$ is an isomorphism for all p. Here, ω is the fundamental class in $H_n(M;\mathbf{k})$.

Persistence products on manifolds

Let

$$\begin{aligned} H_*^{a,b} &= \mathrm{im}(H_*^a(\mathcal{M};\mathbf{k}) \to H_*^b(\mathcal{M};\mathbf{k})), \\ H_{a,b}^* &= \mathrm{im}(H_b^*(\mathcal{M};\mathbf{k}) \to H_a^*(\mathcal{M};\mathbf{k})), \\ P_{a,b}^* &= \mathrm{im}(H_b^*(\mathcal{M};\mathbf{k}) \times H_b^*(\mathcal{M};\mathbf{k}) \xrightarrow{\cup} H_b^*(\mathcal{M};\mathbf{k}) \to H_a^*(\mathcal{M};\mathbf{k})) \end{aligned}$$

be the (a,b)-persistent homology, cohomology, and cup-space, respectively.

Persistence products on manifolds

Theorem (Wanying Bi, Jingyan Li, Jian Liu, & Jie Wu)

For $a, b \in G$ with $ba^{-1} \in S$, let $f_*^{a,b} : H_*(\mathcal{M}_a) \to H_*(\mathcal{M}_b)$ be a map induced by $\mathcal{M}_a \to \mathcal{M}_b$ and $f_n^{a,b}(\omega^a) = \lambda^{a,b}\omega^b, \lambda^{a,b} \in \mathbf{k}$.

(i) If $\lambda^{a,b} \neq 0$, then the map

$$D^{a,b} = f_*^{a,b} \circ D : H_{a,b}^p \to H_{n-p}^{a,b}$$

is an isomorphism for all p. Moreover, we have $\beta_{a,b}^p = \beta_{a,b}^{n-p}$.

(ii) If $\lambda^{a,b} = 0$, then the (a, b)-persistent cup-space $P_{a,b}^n = 0$. Moreover, we have

$$\beta_p^{a,b} + \beta_{n-p}^{a,b} \leq \beta_p^a, \quad \beta_{a,b}^p + \beta_{a,b}^{n-p} \leq \beta_a^p.$$

For more details, please refer to

Wanying Bi, Jingyan Li, Jian Liu, and Jie Wu. On the Cayley-persistence algebra[J]. arXiv preprint arXiv:2205.10796, 2022.

Thank you !