On the equivariant KU_G local sphere for finite abelian groups

Yingxin Li

June 24, 2026

1. Background
2. $L_{KU_G/p}S_G$ as a homotopy fiber
3. A computation for $p = 2$
4. Relation with equivariant Morava K-theory

Background

The J-homomorphism is a map

$$J\colon \pi_r O \longrightarrow \pi_r S,$$

from the homotopy groups of the infinite orthogonal group to the homotopy groups of the sphere spectrum.

For any odd prime p, write

$$\mathrm{Im}(J)_r = \mathrm{Im}\big(\pi_r O \to \pi_r S \to \pi_r S_p^\wedge\big)$$

for the image of the p-complete J-homomorphism. For $r > 0$, the map $S_p^\wedge \to L_{KU/p} S$ induces an isomorphism

$$\mathrm{Im}(J)_r \cong \pi_r L_{KU/p} S.$$

Hence, the image of J is essentially the first chromatic layer of the sphere spectrum.

Background

The J-homomorphism is a map

$$J\colon \pi_r O \longrightarrow \pi_r S,$$

from the homotopy groups of the infinite orthogonal group to the homotopy groups of the sphere spectrum.

For any odd prime p, write

$$\mathrm{Im}(J)_r = \mathrm{Im}\big(\pi_r O \to \pi_r S \to \pi_r S_p^{\wedge}\big)$$

for the image of the p-complete J-homomorphism. For $r > 0$, the map $S_p^{\wedge} \to L_{KU/p} S$ induces an isomorphism

$$\mathrm{Im}(J)_r \cong \pi_r L_{KU/p} S.$$

Hence, the image of J is essentially the first chromatic layer of the sphere spectrum.

Background

The J-homomorphism is a map

$$J\colon \pi_r O \longrightarrow \pi_r S,$$

from the homotopy groups of the infinite orthogonal group to the homotopy groups of the sphere spectrum.

For any odd prime p, write

$$\mathrm{Im}(J)_r = \mathrm{Im}\big(\pi_r O \to \pi_r S \to \pi_r S_p^\wedge\big)$$

for the image of the p-complete J-homomorphism. For $r > 0$, the map $S_p^\wedge \to L_{KU/p}S$ induces an isomorphism

$$\mathrm{Im}(J)_r \cong \pi_r L_{KU/p}S.$$

Hence, the image of J is essentially the first chromatic layer of the sphere spectrum.

Background

Let G be a finite group. Complex (resp. real) K theory admits a natural equivariant refinement, denoted by KU_G (resp. KO_G), which is defined by equivalence classes of virtual equivariant complex (resp. real) bundles.

The equivariant J-homomorphism

$$J_G\colon \pi_r KO_G \longrightarrow \pi_{r-1} S_G$$

was defined by Segal, analogously to the classical case. Here S_G is the G-equivariant sphere spectrum.

For a G-spectrum X, we have

- $\underline{\pi}_n(X)(G/H) = [G/H_+ \wedge S^n, X]_G \cong \pi_n(X^H)$ for any subgroup $H \subset G$;
- $RO(G)$-graded homotopy groups $\pi_V(X) \cong [S^V, X]_G$ for any G-representation V.

We want to study the image of G-equivariant J-homomorphism via the homotopy of KU_G-local sphere.

Background

Let G be a finite group. Complex (resp. real) K theory admits a natural equivariant refinement, denoted by KU_G (resp. KO_G), which is defined by equivalence classes of virtual equivariant complex (resp. real) bundles.

The equivariant J-homomorphism

$$J_G\colon \pi_r KO_G \longrightarrow \pi_{r-1} S_G$$

was defined by Segal, analogously to the classical case. Here S_G is the G-equivariant sphere spectrum.

For a G-spectrum X, we have

- $\underline{\pi}_n(X)(G/H) = [G/H_+ \wedge S^n, X]_G \cong \pi_n(X^H)$ for any subgroup $H \subset G$;
- $RO(G)$-graded homotopy groups $\pi_V(X) \cong [S^V, X]_G$ for any G-representation V.

We want to study the image of G-equivariant J-homomorphism via the homotopy of KU_G-local sphere.

Background

Let G be a finite group. Complex (resp. real) K theory admits a natural equivariant refinement, denoted by KU_G (resp. KO_G), which is defined by equivalence classes of virtual equivariant complex (resp. real) bundles.

The equivariant J-homomorphism

$$J_G\colon \pi_r KO_G \longrightarrow \pi_{r-1} S_G$$

was defined by Segal, analogously to the classical case. Here S_G is the G-equivariant sphere spectrum.

For a G-spectrum X, we have

- $\underline{\pi}_n(X)(G/H) = [G/H_+ \wedge S^n, X]_G \cong \pi_n(X^H)$ for any subgroup $H \subset G$;
- $RO(G)$-graded homotopy groups $\pi_V(X) \cong [S^V, X]_G$ for any G-representation V.

We want to study the image of G-equivariant J-homomorphism via the homotopy of KU_G-local sphere.

Background

Let G be a finite group. Complex (resp. real) K theory admits a natural equivariant refinement, denoted by KU_G (resp. KO_G), which is defined by equivalence classes of virtual equivariant complex (resp. real) bundles.

The equivariant J-homomorphism

$$J_G \colon \pi_r KO_G \longrightarrow \pi_{r-1} S_G$$

was defined by Segal, analogously to the classical case. Here S_G is the G-equivariant sphere spectrum.

For a G-spectrum X, we have

- $\underline{\pi}_n(X)(G/H) = [G/H_+ \wedge S^n, X]_G \cong \pi_n(X^H)$ for any subgroup $H \subset G$;
- $RO(G)$-graded homotopy groups $\pi_V(X) \cong [S^V, X]_G$ for any G-representation V.

We want to study the image of G-equivariant J-homomorphism via the homotopy of KU_G-local sphere.

Bousfield localization

Let $E \in Sp^G$.

- A G spectrum X is E-acyclic if $E \wedge X \simeq *$.
- A map $f : X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology, i.e., $E \wedge X \simeq E \wedge Y$.
- A G-spectrum Y is E-local if for each E-acyclic G-spectrum X, $[X, Y]_G = 0$.

Definition

$f : X \to L_E X$ is a E localization if f is an E-equivalence such that $L_E X$ is E-local.

- $L_{S/p} X \simeq X_p^\wedge$.
- $L_{S \otimes \mathbb{Z}_{(p)}} X \simeq X \otimes \mathbb{Z}_{(p)}$.

Let $E, F \in Sp^G$, for any $X \in Sp^G$, if $E \wedge X \simeq * \Leftrightarrow F \wedge X \simeq *$, then $L_E \simeq L_F$, and we write $\langle E \rangle = \langle F \rangle$.

Bousfield localization

Let $E \in Sp^G$.

- A G spectrum X is E-acyclic if $E \wedge X \simeq *$.
- A map $f : X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology, i.e., $E \wedge X \simeq E \wedge Y$.
- A G-spectrum Y is E-local if for each E-acyclic G-spectrum X, $[X, Y]_G = 0$.

Definition

$f : X \to L_E X$ is a E localization if f is an E-equivalence such that $L_E X$ is E-local.

- $L_{S/p} X \simeq X^{\wedge}_p$.
- $L_{S \otimes \mathbb{Z}_{(p)}} X \simeq X \otimes \mathbb{Z}_{(p)}$.

Let $E, F \in Sp^G$, for any $X \in Sp^G$, if $E \wedge X \simeq * \Leftrightarrow F \wedge X \simeq *$, then $L_E \simeq L_F$, and we write $\langle E \rangle = \langle F \rangle$.

Bousfield localization

Let $E \in Sp^G$.

- A G spectrum X is E-acyclic if $E \wedge X \simeq *$.
- A map $f : X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology, i.e., $E \wedge X \simeq E \wedge Y$.
- A G-spectrum Y is E-local if for each E-acyclic G-spectrum X, $[X, Y]_G = 0$.

Definition

$f : X \to L_E X$ is a E localization if f is an E-equivalence such that $L_E X$ is E-local.

- $L_{S/p} X \simeq X_p^\wedge$.
- $L_{S \otimes \mathbb{Z}_{(p)}} X \simeq X \otimes \mathbb{Z}_{(p)}$.

Let $E, F \in Sp^G$, for any $X \in Sp^G$, if $E \wedge X \simeq * \Leftrightarrow F \wedge X \simeq *$, then $L_E \simeq L_F$, and we write $\langle E \rangle = \langle F \rangle$.

Bousfield localization

Let $E \in Sp^G$.

- A G spectrum X is E-acyclic if $E \wedge X \simeq *$.
- A map $f : X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology, i.e., $E \wedge X \simeq E \wedge Y$.
- A G-spectrum Y is E-local if for each E-acyclic G-spectrum X, $[X, Y]_G = 0$.

Definition

$f : X \to L_E X$ is a E localization if f is an E-equivalence such that $L_E X$ is E-local.

- $L_{S/p} X \simeq X_p^{\wedge}$.
- $L_{S \otimes \mathbb{Z}_{(p)}} X \simeq X \otimes \mathbb{Z}_{(p)}$.

Let $E, F \in Sp^G$, for any $X \in Sp^G$, if $E \wedge X \simeq * \Leftrightarrow F \wedge X \simeq *$, then $L_E \simeq L_F$, and we write $\langle E \rangle = \langle F \rangle$.

Background

- [French 09'] gave a splitting $KO_G(X)^\wedge_p \cong W_G(X) \times WO^\perp_G(X)$ if none of the prime divisors of $|G|$ are congruent to 1 modulo p, such that $WO^\perp_G(X)$ does not contribute to the image of J.
- [Balderrama 22'] computed $\underline{\pi}_\bigstar L_{KU_{C_2}/2} S_{C_2}$.
- [Carawan-Field-Guillou-Mehrle-Stapleton 23'] computed $\underline{\pi}_* L_{KU_G} S_G$ for finite p-groups ($p \neq 2$) via the fiber sequence

$$L_{KU_G/p} S_G \longrightarrow (KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p.$$

- [Balderrama 24'] constructed an $RO(G)$-graded equivariant J-homomorphism.

Background

- [French 09'] gave a splitting $KO_G(X)^\wedge_p \cong W_G(X) \times WO^\perp_G(X)$ if none of the prime divisors of $|G|$ are congruent to 1 modulo p, such that $WO^\perp_G(X)$ does not contribute to the image of J.
- [Balderrama 22'] computed $\underline{\pi}_\bigstar L_{KU_{C_2}/2} S_{C_2}$.
- [Carawan-Field-Guillou-Mehrle-Stapleton 23'] computed $\underline{\pi}_* L_{KU_G} S_G$ for finite p-groups ($p \neq 2$) via the fiber sequence

$$L_{KU_G/p} S_G \longrightarrow (KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p.$$

- [Balderrama 24'] constructed an $RO(G)$-graded equivariant J-homomorphism.

Background

- [French 09'] gave a splitting $KO_G(X)^{\wedge}_p \cong W_G(X) \times WO^{\perp}_G(X)$ if none of the prime divisors of $|G|$ are congruent to 1 modulo p, such that $WO^{\perp}_G(X)$ does not contribute to the image of J.
- [Balderrama 22'] computed $\underline{\pi}_{\bigstar} L_{KU_{C_2}/2} S_{C_2}$.
- [Carawan-Field-Guillou-Mehrle-Stapleton 23'] computed $\underline{\pi}_* L_{KU_G} S_G$ for finite p-groups ($p \neq 2$) via the fiber sequence

$$L_{KU_G/p} S_G \longrightarrow (KU_G)^{\wedge}_p \xrightarrow{\psi^g - 1} (KU_G)^{\wedge}_p.$$

- [Balderrama 24'] constructed an $RO(G)$-graded equivariant J-homomorphism.

Background

- [French 09'] gave a splitting $KO_G(X)^\wedge_p \cong W_G(X) \times WO^\perp_G(X)$ if none of the prime divisors of $|G|$ are congruent to 1 modulo p, such that $WO^\perp_G(X)$ does not contribute to the image of J.
- [Balderrama 22'] computed $\underline{\pi}_\bigstar L_{KU_{C_2}/2} S_{C_2}$.
- [Carawan-Field-Guillou-Mehrle-Stapleton 23'] computed $\underline{\pi}_* L_{KU_G} S_G$ for finite p-groups ($p \neq 2$) via the fiber sequence

$$L_{KU_G/p} S_G \longrightarrow (KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p.$$

- [Balderrama 24'] constructed an $RO(G)$-graded equivariant J-homomorphism.

We compute $\underline{\pi}_* L_{KU_G} S_G$ for finite abelian groups.

- When G is not a p-group, the fiber of

$$(KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p$$

  is not the KU_G/p-local sphere.

- When G is a finite 2-group, nontrivial extension problems arise in determining the homotopy G-Mackey functors.

We compute $\underline{\pi}_* L_{KU_G} S_G$ for finite abelian groups.

- When G is not a p-group, the fiber of

$$(KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p$$

  is not the KU_G/p-local sphere.
- When G is a finite 2-group, nontrivial extension problems arise in determining the homotopy G-Mackey functors.

We compute $\underline{\pi}_* L_{KU_G} S_G$ for finite abelian groups.

- When G is not a p-group, the fiber of

  $$(KU_G)^\wedge_p \xrightarrow{\psi^g - 1} (KU_G)^\wedge_p$$

  is not the KU_G/p-local sphere.

- When G is a finite 2-group, nontrivial extension problems arise in determining the homotopy G-Mackey functors.

Geometric fixed point

Let $\mathcal{F}$ be a family of subgroup of G, let $E\mathcal{F}$ and $\widetilde{E\mathcal{F}}$ be G-spaces such that

$$(E\mathcal{F})^H = \begin{cases} pt & \text{if } H \in \mathcal{F} \\ \emptyset & \text{otherwise} \end{cases}, \quad (\widetilde{E\mathcal{F}})^H = \begin{cases} pt & \text{if } H \in \mathcal{F} \\ S^0 & \text{otherwise} \end{cases}.$$

Definition

Let $\mathcal{F}$ be the family of all closed proper subgroups of G, the G-geometric fixed point is defined by $\Phi^G(E) := (E \wedge \widetilde{E\mathcal{F}})^G$.

- $f : E \to F$ is a weak equivalence if and only if $\Phi^H(f)$ is an equivalence of spectra for all subgroups H.
- $\Phi^G(E \wedge F) \simeq \Phi^G(E) \wedge \Phi^G(F)$ but $(E \wedge F)^G \not\simeq E^G \wedge F^G$.
- For any finite group G and normal subgroup N, a G/N-spectrum X can be regarded as a G-spectrum via $G \to G/N$; this is denoted by $\mathrm{Inf}_{G/N}^G X$. $\Phi^N \mathrm{Inf}_{G/N}^G(X) \simeq X$ but $(\mathrm{Inf}_{G/N}^G X)^N \not\simeq X$.
- $\Phi^H KU_G \simeq \begin{cases} KU[\frac{1}{n}, \zeta_n] & H \cong C_n \\ * & otherwise \end{cases}$

- $f : E \to F$ is a weak equivalence if and only if $\Phi^H(f)$ is an equivalence of spectra for all subgroups H.
- $\Phi^G(E \wedge F) \simeq \Phi^G(E) \wedge \Phi^G(F)$ but $(E \wedge F)^G \not\simeq E^G \wedge F^G$.
- For any finite group G and normal subgroup N, a G/N-spectrum X can be regarded as a G-spectrum via $G \to G/N$; this is denoted by $\mathrm{Inf}^G_{G/N} X$. $\Phi^N \mathrm{Inf}^G_{G/N}(X) \simeq X$ but $(\mathrm{Inf}^G_{G/N} X)^N \not\simeq X$.
- $\Phi^H KU_G \simeq \begin{cases} KU[\frac{1}{n}, \zeta_n] & H \cong C_n \\ * & otherwise \end{cases}$

- $f : E \to F$ is a weak equivalence if and only if $\Phi^H(f)$ is an equivalence of spectra for all subgroups H.
- $\Phi^G(E \wedge F) \simeq \Phi^G(E) \wedge \Phi^G(F)$ but $(E \wedge F)^G \not\simeq E^G \wedge F^G$.
- For any finite group G and normal subgroup N, a G/N-spectrum X can be regarded as a G-spectrum via $G \to G/N$; this is denoted by $\mathrm{Inf}_{G/N}^G X$. $\Phi^N \mathrm{Inf}_{G/N}^G(X) \simeq X$ but $(\mathrm{Inf}_{G/N}^G X)^N \not\simeq X$.
- $\Phi^H KU_G \simeq \begin{cases} KU[\frac{1}{n}, \zeta_n] & H \cong C_n \\ * & otherwise \end{cases}$

$L_{KU_G/p}S_G$ as a homotopy fiber

Theorem

Let G be a finite nilpotent group, and let Cyc be the family of all cyclic subgroups of G. For any prime p, let N_p be the Sylow p-subgroup of G, and let g be a topological generator of $\mathbb{Z}_p^\times/\{\pm 1\}$. Then for any finite G-spectrum X, there is a fiber sequence

$$L_{KU_G/p}X \longrightarrow (ECyc_+ \wedge \operatorname{Inf}^G_{N_p} KO_{N_p} \wedge X)^\wedge_p \xrightarrow{\psi^g - 1} (ECyc_+ \wedge \operatorname{Inf}^G_{N_p} KO_{N_p} \wedge X)^\wedge_p.$$

Sketch of proof:

- $\langle KU_G/p\rangle = \langle ECyc_+ \wedge \operatorname{Inf}^G_{N_p} KO_{N_p}/p\rangle$.
- $(ECyc_+ \wedge \operatorname{Inf}^G_{N_p} KO_{N_p} \wedge X)^\wedge_p$ is $ECyc_+ \wedge \operatorname{Inf}^G_{N_p} KO_{N_p}/p$ local.
- $X \to \operatorname{fib}(\psi^g - 1)$ is a KU_G/p equivalence.

$L_{KU_G/p}S_G$ as a homotopy fiber

Theorem

Let G be a finite nilpotent group, and let Cyc be the family of all cyclic subgroups of G. For any prime p, let N_p be the Sylow p-subgroup of G, and let g be a topological generator of $\mathbb{Z}_p^\times/\{\pm 1\}$. Then for any finite G-spectrum X, there is a fiber sequence

$$L_{KU_G/p}X \longrightarrow (ECyc_+ \wedge \operatorname{Inf}_{N_p}^G KO_{N_p} \wedge X)_p^\wedge \xrightarrow{\psi^g - 1} (ECyc_+ \wedge \operatorname{Inf}_{N_p}^G KO_{N_p} \wedge X)_p^\wedge.$$

Sketch of proof:

- $\langle KU_G/p \rangle = \langle ECyc_+ \wedge \operatorname{Inf}_{N_p}^G KO_{N_p}/p \rangle$.
- $(ECyc_+ \wedge \operatorname{Inf}_{N_p}^G KO_{N_p} \wedge X)_p^\wedge$ is $ECyc_+ \wedge \operatorname{Inf}_{N_p}^G KO_{N_p}/p$ local.
- $X \to \operatorname{fib}(\psi^g - 1)$ is a KU_G/p equivalence.

Corollary

$$L_{KU_G/p}S_G \simeq (ECyc_+ \wedge \mathrm{Inf}^G_{N_p} L_{KU_{N_p}/p}S_{N_p})^\wedge_p.$$

Let G be a finite nilpotent group with Sylow p-subgroup N_p, and let N be the subgroup of G such that $G \cong N_p \times N$. Defining the following N-Mackey functor:

- $\underline{A}_N(N/K) = A(K)$.
- $\underline{J}_N(N/K) = J(K) := \ker(A(K) \to RU(K))$.
- $\underline{A/J}_N := \underline{A}_N/\underline{J}_N$.

Proposition

For any N_p-spectrum E such that E is S/p-equivalent to $ECyc_+ \wedge E$, we have an isomorphism between G-Mackey functor

$$\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p.$$

Corollary

$$L_{KU_G/p} S_G \simeq (ECyc_+ \wedge \mathrm{Inf}^G_{N_p} L_{KU_{N_p}/p} S_{N_p})^\wedge_p.$$

Let G be a finite nilpotent group with Sylow p-subgroup N_p, and let N be the subgroup of G such that $G \cong N_p \times N$. Defining the following N-Mackey functor:

- $\underline{A}_N(N/K) = A(K)$.
- $\underline{J}_N(N/K) = J(K) := \ker(A(K) \to RU(K))$.
- $\underline{A/J}_N := \underline{A}_N / \underline{J}_N$.

Proposition

For any N_p-spectrum E such that E is S/p-equivalent to $ECyc_+ \wedge E$, we have an isomorphism between G-Mackey functor

$$\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p.$$

Corollary

$$L_{KU_G/p}S_G \simeq (ECyc_+ \wedge \mathrm{Inf}^G_{N_p} L_{KU_{N_p}/p}S_{N_p})^{\wedge}_p.$$

Let G be a finite nilpotent group with Sylow p-subgroup N_p, and let N be the subgroup of G such that $G \cong N_p \times N$. Defining the following N-Mackey functor:

- $\underline{A}_N(N/K) = A(K)$.
- $\underline{J}_N(N/K) = J(K) := \ker(A(K) \to RU(K))$.
- $\underline{A/J}_N := \underline{A}_N/\underline{J}_N$.

Proposition

For any N_p-spectrum E such that E is S/p-equivalent to $ECyc_+ \wedge E$, we have an isomorphism between G-Mackey functor

$$\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^{\wedge}_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^{\wedge}_p.$$

Proof of $\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p$.

We compute the G-fixed point of $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p$ via the formula

$$(x \otimes \mathrm{Inf}^G_{G/N} y)^N \simeq x^N \otimes y \in Sp^{G/N}, \quad \forall x \in Sp^G, y \in Sp^{G/N}.$$

- Let X be a pointed N-space. Regard X as a G-space via the quotient map $G \to N$. Then, as an N_p-spectrum,

$$(\Sigma^\infty_G \mathrm{Inf}^G_N X)^N \simeq \mathrm{Inf}^{N_p}_e (\Sigma^\infty_N X)^N.$$

- For any $H \subset G$, let Cyc^H be the family of cyclic subgroups of H. We have equivalences of N_p-spectra

$$\begin{aligned}(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^N &\simeq (ECyc^N_+ \wedge \mathrm{Inf}^G_{N_p}(ECyc^{N_p}_+ \wedge E))^N \\ &\simeq (ECyc^{N_p}_+ \wedge E) \wedge (\Sigma^\infty_G ECyc^N_+)^N.\end{aligned}$$

- Then we have equivalence $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^G \simeq (ECyc^{N_p}_+ \wedge E)^{N_p} \wedge (\Sigma^\infty_N ECyc^N_+)^N$.
- After p-completion, we get $\bigvee_{T \in Cyc}((KU_{N_p})^{N_p})^\wedge_p$.

Proof of $\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p$.

We compute the G-fixed point of $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p$ via the formula

$$(x \otimes \mathrm{Inf}^G_{G/N}\, y)^N \simeq x^N \otimes y \in Sp^{G/N}, \quad \forall x \in Sp^G, y \in Sp^{G/N}.$$

- Let X be a pointed N-space. Regard X as a G-space via the quotient map $G \to N$. Then, as an N_p-spectrum,

$$(\Sigma^\infty_G \mathrm{Inf}^G_N X)^N \simeq \mathrm{Inf}^{N_p}_e (\Sigma^\infty_N X)^N.$$

- For any $H \subset G$, let Cyc^H be the family of cyclic subgroups of H. We have equivalences of N_p-spectra

$$\begin{aligned}(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^N &\simeq (ECyc^N_+ \wedge \mathrm{Inf}^G_{N_p}(ECyc^{N_p}_+ \wedge E))^N \\ &\simeq (ECyc^{N_p}_+ \wedge E) \wedge (\Sigma^\infty_G ECyc^N_+)^N.\end{aligned}$$

- Then we have equivalence $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^G \simeq (ECyc^{N_p}_+ \wedge E)^{N_p} \wedge (\Sigma^\infty_N ECyc^N_+)^N$.
- After p-completion, we get $\bigvee_{T \in Cyc} ((KU_{N_p})^{N_p})^\wedge_p$.

Proof of $\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p$.

We compute the G-fixed point of $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p$ via the formula

$$(x \otimes \mathrm{Inf}^G_{G/N}\, y)^N \simeq x^N \otimes y \in Sp^{G/N}, \quad \forall x \in Sp^G, y \in Sp^{G/N}.$$

- Let X be a pointed N-space. Regard X as a G-space via the quotient map $G \to N$. Then, as an N_p-spectrum,
$$(\Sigma^\infty_G \mathrm{Inf}^G_N X)^N \simeq \mathrm{Inf}^{N_p}_e (\Sigma^\infty_N X)^N.$$
- For any $H \subset G$, let Cyc^H be the family of cyclic subgroups of H. We have equivalences of N_p-spectra
$$\begin{aligned}(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^N &\simeq (ECyc^N_+ \wedge \mathrm{Inf}^G_{N_p}(ECyc^{N_p}_+ \wedge E))^N \\ &\simeq (ECyc^{N_p}_+ \wedge E) \wedge (\Sigma^\infty_G ECyc^N_+)^N.\end{aligned}$$
- Then we have equivalence $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^G \simeq (ECyc^{N_p}_+ \wedge E)^{N_p} \wedge (\Sigma^\infty_N ECyc^N_+)^N$.
- After p-completion, we get $\bigvee_{T \in Cyc}((KU_{N_p})^{N_p})^\wedge_p$.

Proof of $\underline{\pi}_*(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p \cong (\underline{\pi}_* E \otimes \underline{A/J}_N)^\wedge_p$.

We compute the G-fixed point of $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^\wedge_p$ via the formula

$$(x \otimes \mathrm{Inf}^G_{G/N}\, y)^N \simeq x^N \otimes y \in Sp^{G/N}, \quad \forall x \in Sp^G, y \in Sp^{G/N}.$$

- Let X be a pointed N-space. Regard X as a G-space via the quotient map $G \to N$. Then, as an N_p-spectrum,

$$(\Sigma^\infty_G \mathrm{Inf}^G_N X)^N \simeq \mathrm{Inf}^{N_p}_e (\Sigma^\infty_N X)^N.$$

- For any $H \subset G$, let Cyc^H be the family of cyclic subgroups of H. We have equivalences of N_p-spectra

$$\begin{aligned}(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^N &\simeq (ECyc^N_+ \wedge \mathrm{Inf}^G_{N_p}(ECyc^{N_p}_+ \wedge E))^N \\ &\simeq (ECyc^{N_p}_+ \wedge E) \wedge (\Sigma^\infty_G ECyc^N_+)^N.\end{aligned}$$

- Then we have equivalence $(ECyc_+ \wedge \mathrm{Inf}^G_{N_p} E)^G \simeq (ECyc^{N_p}_+ \wedge E)^{N_p} \wedge (\Sigma^\infty_N ECyc^N_+)^N$.
- After p-completion, we get $\bigvee_{T \in Cyc}((KU_{N_p})^{N_p})^\wedge_p$.

$\underline{\pi}_* L_{KU_G/p} S_G$

When $E = L_{KU_{N_p}/p} S_{N_p}$, we have

$$\underline{\pi}_*(L_{KU_G/p} S_G) \cong \underline{\pi}_* L_{KU_{N_p}/p} S_{N_p} \otimes_{\mathbb{Z}_p} (\underline{A/J}_N)^{\wedge}_p.$$

When p is an odd prime, $\underline{\pi}_* L_{KU_{N_p}/p} S_{N_p}$ is computed in [Carawan-Field-Guillou-Mehrle-Stapleton 23'].

Lemma (Carawan-Field-Guillou-Mehrle-Stapleton 23')

Let $\underline{\ker}_p\{k\} := \ker(\underline{\pi}_k KU_{N_p} \xrightarrow{\psi^g - 1} \underline{\pi}_k KU_{N_p})$, *and let* $\underline{\mathrm{coker}}_p\{k\} := \mathrm{coker}(\underline{\pi}_k KU_{N_p} \xrightarrow{\psi^g - 1} \underline{\pi}_k KU_{N_p})$. *Then*

$$\underline{\pi}_n L_{KU_G/p} S_G \cong \begin{cases} (\underline{A/J}_N)^{\wedge}_p \otimes \underline{\ker}_p\{2d\} & n = 2d, \\ (\underline{A/J}_N)^{\wedge}_p \otimes \underline{\mathrm{coker}}_p\{2d\} & n = 2d-1 \end{cases}.$$

$\underline{\pi}_* L_{KU_G/p} S_G$

When $E = L_{KU_{N_p}/p} S_{N_p}$, we have

$$\underline{\pi}_*(L_{KU_G/p} S_G) \cong \underline{\pi}_* L_{KU_{N_p}/p} S_{N_p} \otimes_{\mathbb{Z}_p} (\underline{A/J}_N)^{\wedge}_p.$$

When p is an odd prime, $\underline{\pi}_* L_{KU_{N_p}/p} S_{N_p}$ is computed in [Carawan-Field-Guillou-Mehrle-Stapleton 23'].

Lemma (Carawan-Field-Guillou-Mehrle-Stapleton 23')

Let $\underline{\ker}_p\{k\} := \ker(\underline{\pi}_k KU_{N_p} \xrightarrow{\psi^g - 1} \underline{\pi}_k KU_{N_p})$, *and let* $\underline{\mathrm{coker}}_p\{k\} := \mathrm{coker}(\underline{\pi}_k KU_{N_p} \xrightarrow{\psi^g - 1} \underline{\pi}_k KU_{N_p})$. *Then*

$$\underline{\pi}_n L_{KU_G/p} S_G \cong \begin{cases} (\underline{A/J}_N)^{\wedge}_p \otimes \underline{\ker}_p\{2d\} & n = 2d, \\ (\underline{A/J}_N)^{\wedge}_p \otimes \underline{\mathrm{coker}}_p\{2d\} & n = 2d-1 \end{cases}.$$

1. Background
2. $L_{KU_G/p}S_G$ as a homotopy fiber
3. A computation for $p = 2$
4. Relation with equivariant Morava K-theory

For $p = 2$

When N_2 is abelian, we compute $\underline{\pi}_* L_{KU_{N_2}/2} S_{N_2}$ via the fiber sequence

$$L_{KU_{N_2}/2} S_{N_2} \longrightarrow (KO_{N_2})^{\wedge}_2 \xrightarrow{\psi^g - 1} (KO_{N_2})^{\wedge}_2 .$$

Lemma

Define $\underline{\ker}_2\{k\} := \ker(\underline{\pi}_k KO_{N_2} \xrightarrow{\psi^g - 1} \underline{\pi}_k KO_{N_2})$, *and* $\underline{\operatorname{coker}}_2\{k\} := \operatorname{coker}(\underline{\pi}_k KO_{N_2} \xrightarrow{\psi^g - 1} \underline{\pi}_k KO_{N_2})$. *We have*

$$\underline{\ker}_2\{k\}(N_2/T) = \begin{cases} (R\mathbb{Q}(T))^{\wedge}_2 & k = 0 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta u^d\} & k = 8d+1 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta^2 u^d\} & k = 8d+2 \\ 0 & \textit{otherwise} \end{cases}.$$

For $p = 2$

When N_2 is abelian, we compute $\underline{\pi}_* L_{KU_{N_2}/2} S_{N_2}$ via the fiber sequence

$$L_{KU_{N_2}/2} S_{N_2} \longrightarrow (KO_{N_2})^\wedge_2 \xrightarrow{\psi^g - 1} (KO_{N_2})^\wedge_2 .$$

Lemma

Define $\underline{\ker}_2\{k\} := \ker(\underline{\pi}_k KO_{N_2} \xrightarrow{\psi^g - 1} \underline{\pi}_k KO_{N_2})$*, and* $\underline{\mathrm{coker}}_2\{k\} := \mathrm{coker}(\underline{\pi}_k KO_{N_2} \xrightarrow{\psi^g - 1} \underline{\pi}_k KO_{N_2})$*. We have*

$$\underline{\ker}_2\{k\}(N_2/T) = \begin{cases} (R\mathbb{Q}(T))^\wedge_2 & k = 0 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta u^d\} & k = 8d+1 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta^2 u^d\} & k = 8d+2 \\ 0 & \textit{otherwise} \end{cases} .$$

For $p = 2$

Lemma

$\underline{\text{coker}}_2\{k\}(N_2/T) =$

$$\begin{cases} \bigoplus_{\text{cyclic } T\subset K} \mathbb{Z}_2^{\wedge} & k = 0 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2^{4+\nu_2(d)} \oplus (\bigoplus_{\text{cyclic } C_{2^t}\subset T, t\geq 2} \mathbb{Z}/2^{2+t+\nu_2(d)}) & k = 8d, d \neq 0 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta u^d\} & k = 8d+1 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2\{\eta^2 u^d\} \oplus (\bigoplus_{\text{cyclic } C_{2^t}\subset T, t\geq 2} \mathbb{Z}/2^t) & k = 8d+2 \\ RO(T;\mathbb{R}) \otimes \mathbb{Z}/2^3 \oplus (\bigoplus_{\text{cyclic } C_{2^t}\subset T, t\geq 2} \mathbb{Z}/2^{t+1}) & k = 8d+4 \\ \bigoplus_{\text{cyclic } C_{2^t}\subset T, t\geq 2} \mathbb{Z}/2^t & k = 8d+6 \end{cases},$$

where ν_2 is the 2-adic valuation.

Extension problems

When $k = 0$, we need to study the exact sequence

$$0 \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2 \xrightarrow{i} \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^{\wedge}_{2} \longrightarrow 0.$$

Let $V := C_2 \times C_2$, and let $A, B, C \subset V$ be the three subgroups of order two. Then $J(V) \cong \mathbb{Z}$ is freely generated by

$$X_V = ([V/A] - 1)([V/B] - 1)([V/C] - 1) - 1.$$

It follows that X acts by zero on $R\mathbb{Q}(V)$ and on $\underline{\mathrm{coker}}\{1\}(V/V)$. However, Szymik shows that X is not in the kernel of the unit map

$$A(V) \cong \pi_0^V S_V \to \pi_0 L_{KU_V/2} S_V.$$

So if y is any element of $\pi_0^V L_{KU_V/2} S_V$ lifting the identity of $RQ(V)$, then $X * y$ is a nonzero 2-torsion element. Thus the surjection $\pi_0^V L_{KU_V/2} S_V \to RQ(V)$ does not admit an $A(V)$-linear splitting.

Extension problems

When $k = 0$, we need to study the exact sequence

$$0 \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2 \xrightarrow{i} \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^{\wedge}_2 \longrightarrow 0.$$

Let $V := C_2 \times C_2$, and let $A, B, C \subset V$ be the three subgroups of order two. Then $J(V) \cong \mathbb{Z}$ is freely generated by

$$X_V = ([V/A] - 1)([V/B] - 1)([V/C] - 1) - 1.$$

It follows that X acts by zero on $R\mathbb{Q}(V)$ and on $\underline{\mathrm{coker}}\{1\}(V/V)$. However, Szymik shows that X is not in the kernel of the unit map

$$A(V) \cong \pi_0^V S_V \to \pi_0 L_{KU_V/2} S_V.$$

So if y is any element of $\pi_0^V L_{KU_V/2} S_V$ lifting the identity of $RQ(V)$, then $X * y$ is a nonzero 2-torsion element. Thus the surjection $\pi_0^V L_{KU_V/2} S_V \to RQ(V)$ does not admit an $A(V)$-linear splitting.

Extension problems

When $k = 0$, we need to study the exact sequence

$$0 \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2 \xrightarrow{i} \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^{\wedge}_{2} \longrightarrow 0.$$

Let $V := C_2 \times C_2$, and let $A, B, C \subset V$ be the three subgroups of order two. Then $J(V) \cong \mathbb{Z}$ is freely generated by

$$X_V = ([V/A] - 1)([V/B] - 1)([V/C] - 1) - 1.$$

It follows that X acts by zero on $R\mathbb{Q}(V)$ and on $\underline{\mathrm{coker}}\{1\}(V/V)$. However, Szymik shows that X is not in the kernel of the unit map

$$A(V) \cong \pi_0^V S_V \to \pi_0 L_{KU_V/2} S_V.$$

So if y is any element of $\pi_0^V L_{KU_V/2} S_V$ lifting the identity of $RQ(V)$, then $X * y$ is a nonzero 2-torsion element. Thus the surjection $\pi_0^V L_{KU_V/2} S_V \to RQ(V)$ does not admit an $A(V)$-linear splitting.

Extension problems

When $k = 0$, we need to study the exact sequence

$$0 \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2 \xrightarrow{i} \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^{\wedge}_2 \longrightarrow 0.$$

Let $V := C_2 \times C_2$, and let $A, B, C \subset V$ be the three subgroups of order two. Then $J(V) \cong \mathbb{Z}$ is freely generated by

$$X_V = ([V/A] - 1)([V/B] - 1)([V/C] - 1) - 1.$$

It follows that X acts by zero on $R\mathbb{Q}(V)$ and on $\underline{\mathrm{coker}}\{1\}(V/V)$. However, Szymik shows that X is not in the kernel of the unit map

$$A(V) \cong \pi_0^V S_V \to \pi_0 L_{KU_V/2} S_V.$$

So if y is any element of $\pi_0^V L_{KU_V/2} S_V$ lifting the identity of $RQ(V)$, then $X * y$ is a nonzero 2-torsion element. Thus the surjection $\pi_0^V L_{KU_V/2} S_V \to RQ(V)$ does not admit an $A(V)$-linear splitting.

Since N_2 is a 2-group, $\underline{R\mathbb{Q}}_{N_2} \cong \underline{A/J}_{N_2}$, and the Hurewicz map

$$(\underline{A}_{N_2})^{\wedge}_2 \to \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^{\wedge}_2$$

induces a morphism of Mackey functors

$$\theta_{N_2} : \underline{J}_{N_2} \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2.$$

Lemma

With notations as above, there is a natural isomorphism of Mackey functors

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})^{\wedge}_2 \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in (\underline{J}_{N_2})^{\wedge}_2\}}.$$

Lemma

When $N_2 \cong V$, let ρ_V be the regular real V-representation. Then

$$\theta_V(X_V) = \eta \cdot \rho_V \in RO(V;\mathbb{R})/2 \cdot \eta.$$

Since N_2 is a 2-group, $\underline{R\mathbb{Q}}_{N_2} \cong \underline{A/J}_{N_2}$, and the Hurewicz map

$$(\underline{A}_{N_2})^\wedge_2 \to \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^\wedge_2$$

induces a morphism of Mackey functors

$$\theta_{N_2} : \underline{J}_{N_2} \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2.$$

Lemma

With notations as above, there is a natural isomorphism of Mackey functors

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})^\wedge_2 \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in (\underline{J}_{N_2})^\wedge_2\}}.$$

Lemma

When $N_2 \cong V$, let ρ_V be the regular real V-representation. Then

$$\theta_V(X_V) = \eta \cdot \rho_V \in RO(V;\mathbb{R})/2 \cdot \eta.$$

Since N_2 is a 2-group, $\underline{R\mathbb{Q}}_{N_2} \cong \underline{A/J}_{N_2}$, and the Hurewicz map

$$(\underline{A}_{N_2})^\wedge_2 \to \underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \xrightarrow{\pi} (\underline{R\mathbb{Q}}_{N_2})^\wedge_2$$

induces a morphism of Mackey functors

$$\theta_{N_2} : \underline{J}_{N_2} \longrightarrow \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2.$$

Lemma

With notations as above, there is a natural isomorphism of Mackey functors

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})^\wedge_2 \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in (\underline{J}_{N_2})^\wedge_2\}}.$$

Lemma

When $N_2 \cong V$, let ρ_V be the regular real V-representation. Then

$$\theta_V(X_V) = \eta \cdot \rho_V \in RO(V;\mathbb{R})/2 \cdot \eta.$$

Lemma (Tornehave 84', Bouc 06', Bartel-Dokchitser 15')

For every subquotient $K/L \cong C_2 \times C_2$ of N_2, $J(N_2)$ is generated by $X_{K,L} := \mathrm{tr}_K^{N_2}(X_{K/L})$.

Proposition

Let N_2 be a finite abelian 2-group. There is an isomorphism of N_2-Mackey functors

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})^\wedge_2 \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j-\theta_{N_2}(j) : j \in (\underline{J}_{N_2})^\wedge_2\}}.$$

The map θ_{N_2} is determined on the generators $X_{K,L}$ by

$$\theta_{N_2}(X_{K,L}) = \eta \sum_{\substack{\chi : N_2 \to \{\pm 1\} \\ L \subseteq \ker(\chi)}} \chi.$$

Indeed, this determines the Hurewicz image of $A(N_2) \to \pi_0^{N_2} L_{KU_{N_2}/2} S_{N_2}$ for finite abelian 2-groups.

Lemma (Tornehave 84', Bouc 06', Bartel-Dokchitser 15')

For every subquotient $K/L \cong C_2 \times C_2$ of N_2, $J(N_2)$ is generated by $X_{K,L} := \mathrm{tr}_K^{N_2}(X_{K/L})$.

Proposition

Let N_2 be a finite abelian 2-group. There is an isomorphism of N_2-Mackey functors

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})_2^\wedge \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in (\underline{J}_{N_2})_2^\wedge\}}.$$

The map θ_{N_2} is determined on the generators $X_{K,L}$ by

$$\theta_{N_2}(X_{K,L}) = \eta \sum_{\substack{\chi : N_2 \to \{\pm 1\} \\ L \subseteq \ker(\chi)}} \chi.$$

Indeed, this determines the Hurewicz image of $A(N_2) \to \pi_0^{N_2} L_{KU_{N_2}/2} S_{N_2}$ for finite abelian 2-groups.

Lemma (Tornehave 84', Bouc 06', Bartel-Dokchitser 15')

For every subquotient $K/L \cong C_2 \times C_2$ *of* N_2, $J(N_2)$ *is generated by* $X_{K,L} := \mathrm{tr}_K^{N_2}(X_{K/L})$.

Proposition

Let N_2 *be a finite abelian* 2*-group. There is an isomorphism of* N_2*-Mackey functors*

$$\underline{\pi}_0 L_{KU_{N_2}/2} S_{N_2} \cong \frac{(\underline{A}_{N_2})_2^{\wedge} \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in (\underline{J}_{N_2})_2^{\wedge}\}}.$$

The map θ_{N_2} *is determined on the generators* $X_{K,L}$ *by*

$$\theta_{N_2}(X_{K,L}) = \eta \sum_{\substack{\chi : N_2 \to \{\pm 1\} \\ L \subseteq \ker(\chi)}} \chi.$$

Indeed, this determines the Hurewicz image of $A(N_2) \to \pi_0^{N_2} L_{KU_{N_2}/2} S_{N_2}$ for finite abelian 2-groups.

Similarly, when $k = 8d + 1$, we have the following extension

$$\underline{\pi}_{8d+1} L_{KU_{N_2}/2} S_{N_2} \cong \frac{\underline{A}_{N_2}/2 \oplus \underline{\mathrm{coker}}_2\{8d+2\}}{\{r - \theta_{8d+1}(r) : r \in I_{8d+1}\}}.$$

Here θ_{8d+1} is determined by the case $N_2 = C_4$ and $N_2 = C_2 \times C_2$.

$\underline{\pi}_* L_{KU_G} S_G$

We recover $L_{KU_G} S_G$ via the arithmetic square

$$\begin{tikzcd}
L_{KU_G} S_G \arrow[r] \arrow[d] & \prod_p L_{KU_G/p} S_G \arrow[d] \\
L_{KU_G \otimes \mathbb{Q}} S_G \arrow[r] & (\prod_p L_{KU_G/p} S_G)_{\mathbb{Q}}.
\end{tikzcd}$$

Theorem

Let G be a finite abelian group, and let N_p be its Sylow p-subgroup. We have

$$\underline{\pi}_k L_{KU_G} S_G \cong \begin{cases} \underline{A/J}_{G/N_2} \otimes \frac{\underline{A}_{N_2} \oplus \underline{RO(-;\mathbb{R})}_{N_2}\{\eta\}/2}{\{j - \theta_{N_2}(j) : j \in \underline{J}_{N_2}\}} & k = 0 \\ 0 & k = -1 \\ \mathbb{Q}/\mathbb{Z} \otimes (\prod_p \underline{\text{coker}}_p\{0\} \otimes \underline{A/J}_{G/N_p}) & k = -2 \\ \prod_p \underline{\pi}_k L_{KU_G/p} S_G & \textit{otherwise} \end{cases}$$

1. Background
2. $L_{KU_G/p}S_G$ as a homotopy fiber
3. A computation for $p = 2$
4. Relation with equivariant Morava K-theory

A viewpoint from equivariant chromatic homotopy theory

Let G be a finite abelian group. For $H \subset G$, let

$$K(H,1) := G/H_+ \wedge K(1) \wedge \widetilde{E\mathcal{F}_{H\not\subset}} \in Sp^G_{(p)}.$$

Then $K(H,1)^*(X) \cong K(1)^*(\Phi^H X)$ for $X \in Sp^G$.

Theorem

Let G be a finite abelian group, let Cyc be the family of all cyclic subgroups of G, and let N_p be the Sylow p-subgroup of G. For any prime p and any G-spectrum X, there is an equivalence of G-spectra

$$L_{KU_G/p}X \simeq L_{\bigvee_{H\in Cyc, H\cap N_p=e} K(H,1)}X \simeq \bigvee_{H\in Cyc, H\cap N_p=e} L_{K(H,1)}X.$$

A viewpoint from equivariant chromatic homotopy theory

Let G be a finite abelian group. For $H \subset G$, let

$$K(H,1) := G/H_+ \wedge K(1) \wedge \widetilde{E\mathcal{F}_{H\not\subset}} \in Sp^G_{(p)}.$$

Then $K(H,1)^*(X) \cong K(1)^*(\Phi^H X)$ for $X \in Sp^G$.

Theorem

Let G be a finite abelian group, let Cyc be the family of all cyclic subgroups of G, and let N_p be the Sylow p-subgroup of G. For any prime p and any G-spectrum X, there is an equivalence of G-spectra

$$L_{KU_G/p}X \simeq L_{\bigvee_{H\in Cyc, H\cap N_p=e} K(H,1)}X \simeq \bigvee_{H\in Cyc, H\cap N_p=e} L_{K(H,1)}X.$$

Sketch of proof

- KU_G/p is Bousfield equivalent to $\bigvee_{H \in Cyc, H \cap N_p = \emptyset} K(H,1)$.
- $L_{K(H,1)}X \simeq \mathrm{Inf}^{G}_{G/H} F(EG/H_+, L_{K(1)}\Phi^H X) \wedge \widetilde{E\mathcal{F}_{H \not\subset}}$.
- Let I be a set of subgroups of G, and let H be a subgroup of G such that for all $N \in I$, H is not a subgroup of N. Then for all $X \in Sp^G$, there is a pullback square of G-spectra

$$\begin{tikzcd}
L_{\bigvee_{N \in I \cup \{H\}} K(N,1)}X \arrow[r] \arrow[d] & L_{K(H,1)}X \arrow[d] \\
L_{\bigvee_{N \in I} K(N,1)}X \arrow[r] & L_{K(H,1)}L_{\bigvee_{N \in I} K(N,1)}X.
\end{tikzcd}$$

- $L_{K(H_1,1)}L_{K(H_2,1)}X \simeq *$ for all $H_1 \neq H_2$ with $H_1 \cap N_p = e$.

Sketch of proof

- KU_G/p is Bousfield equivalent to $\bigvee_{H\in Cyc, H\cap N_p=\emptyset} K(H,1)$.
- $L_{K(H,1)}X \simeq \mathrm{Inf}^G_{G/H} F(EG/H_+, L_{K(1)}\Phi^H X) \wedge \widetilde{E\mathcal{F}_{H\not\subset}}$.
- Let I be a set of subgroups of G, and let H be a subgroup of G such that for all $N \in I$, H is not a subgroup of N. Then for all $X \in Sp^G$, there is a pullback square of G-spectra

$$\begin{tikzcd}
L_{\bigvee_{N\in I\cup\{H\}} K(N,1)}X \arrow[r] \arrow[d] & L_{K(H,1)}X \arrow[d] \\
L_{\bigvee_{N\in I} K(N,1)}X \arrow[r] & L_{K(H,1)}L_{\bigvee_{N\in I} K(N,1)}X.
\end{tikzcd}$$

- $L_{K(H_1,1)}L_{K(H_2,1)}X \simeq *$ for all $H_1 \neq H_2$ with $H_1 \cap N_p = e$.

Sketch of proof

- KU_G/p is Bousfield equivalent to $\bigvee_{H\in Cyc, H\cap N_p=\emptyset} K(H,1)$.
- $L_{K(H,1)}X \simeq \mathrm{Inf}^G_{G/H} F(EG/H_+, L_{K(1)}\Phi^H X) \wedge \widetilde{E\mathcal{F}_{H\not\subset}}$.
- Let I be a set of subgroups of G, and let H be a subgroup of G such that for all $N \in I$, H is not a subgroup of N. Then for all $X \in Sp^G$, there is a pullback square of G-spectra

$$\begin{tikzcd}
L_{\bigvee_{N\in I\cup\{H\}} K(N,1)}X \arrow[r] \arrow[d] & L_{K(H,1)}X \arrow[d] \\
L_{\bigvee_{N\in I} K(N,1)}X \arrow[r] & L_{K(H,1)}L_{\bigvee_{N\in I} K(N,1)}X.
\end{tikzcd}$$

- $L_{K(H_1,1)}L_{K(H_2,1)}X \simeq *$ for all $H_1 \neq H_2$ with $H_1 \cap N_p = e$.

Sketch of proof

- KU_G/p is Bousfield equivalent to $\bigvee_{H\in Cyc, H\cap N_p=\emptyset} K(H,1)$.
- $L_{K(H,1)}X \simeq \mathrm{Inf}^G_{G/H} F(EG/H_+, L_{K(1)}\Phi^H X) \wedge \widetilde{E\mathcal{F}_{H\not\subset}}$.
- Let I be a set of subgroups of G, and let H be a subgroup of G such that for all $N \in I$, H is not a subgroup of N. Then for all $X \in Sp^G$, there is a pullback square of G-spectra

$$\begin{tikzcd}
L_{\bigvee_{N\in I\cup\{H\}} K(N,1)}X \arrow[r] \arrow[d] & L_{K(H,1)}X \arrow[d] \\
L_{\bigvee_{N\in I} K(N,1)}X \arrow[r] & L_{K(H,1)}L_{\bigvee_{N\in I} K(N,1)}X.
\end{tikzcd}$$

- $L_{K(H_1,1)}L_{K(H_2,1)}X \simeq *$ for all $H_1 \neq H_2$ with $H_1 \cap N_p = e$.

$RO(G)$-graded homotopy

Corollary

For any $V \in RO(G)$,

$$\underline{\pi}_V L_{KU_G/p} S_G \cong \bigoplus_{H \in Cyc, p \nmid |H|} \underline{\pi}_{n_{V,H}} L_{KU_{N_p}/p} S_{N_p},$$

where $n_{V,H}$ is the dimension of V^H.

Sketch of proof: For any $V \in RO(G)$,

$$\begin{aligned}\pi^G_V L_{K(H,1)} S_G &\cong \pi^G_V \operatorname{Inf}^G_{G/H} F(EG/H_+, L_{K(1)}S)[\mathcal{F}^{-1}_{H \not\subset}] \\ &\cong \pi^G_{V^H} F(EG/H_+, L_{K(1)}S) \cong L_{K(1)} S^{-n_{V,H}}(BG/H) \\ &\cong L_{K(1)} S^{-n_{V,H}}(BN_p) \cong \pi^G_{n_{V,H}} L_{KU_{N_p}/p} S_{N_p}.\end{aligned}$$

Thanks!