On the Hochschild (Co)homology of a Koszul Algebra

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Outline

Associative Algebra

Definition: Graded Associative Algebra

- A (unital) associative algebra A is a vector space over $\mathbf{k}$ endowed with an associative binary multiplication

  $$\mu_A : A \otimes A \to A$$

  which is compatible with the vector space structure of A.
- If A is a graded vector space: $A \cong \bigoplus_{n\geq 0} A_n$, and the multiplication μ_A is compatible with the grading: $\mu_A(A_n \otimes A_m) \subseteq A_{m+n}$, then A is called a **graded associative algebra**.

Free Algebra

The free associative algebra over a vector space V is defined to be

$$T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$$

whose product is given by the concatenation of tensors. We have

$$\mathrm{Hom}_{\mathrm{Vect}}(V, B) \cong \mathrm{Hom}_{\mathrm{Ass}}(T(V), B)$$

for any associative algebra B.

- $T(V)$ has a grading: $T(V)_n := V^{\otimes n}$.
- $T(V)$ is generated by $T(V)_1 = V$.

$\mathbf{k}$ as an A-Module

Assumption: The grading of A is connected, i.e., $A_0 = \mathbf{k}$.
Consequence: An augmentation map $\epsilon : A \to \mathbf{k}$, making $\mathbf{k}$ a left A-module.

Free Resolution of $\mathbf{k}$ over $A = T(V)$

Let $V = \mathbf{k}\{v_1, \cdots, v_n\}$, denote by $A = T(V) =: \mathbf{k}\langle v_1, \cdots, v_n\rangle$.
Then the sequence

$$0 \to A \otimes \mathbf{k}\{v_1, \cdots, v_n\} \to A \to \mathbf{k} \to 0$$

is exact, giving a free resolution of $\mathbf{k}$ over A.

Observation: The map is the sum of multiplications by degree 1 elements $v_1, \cdots, v_n$.

Koszul Algebras

A graded connected algebra A is called **Koszul** if $\mathbf{k}$ admits a free resolution over A, of which all the entries of the matrices are of degree 1. Free resolutions of this kind are called **linear**.

Koszul Algebras

Examples

- The free algebra $T(V)$ is Koszul.
- If the generators of A is not of degree 1, then A is not Koszul.
- What if $A = T(V)/I$, where I is a two-sided ideal of $T(V)$?
  - If I is not generated by quadratic elements, then A is not Koszul.
  - If I is, is A Koszul?

Koszul Algebras

Example

The polynomial ring $S := \mathbf{k}[x, y] = \mathbf{k}\langle x, y\rangle/(x\otimes y - y\otimes x)$ is Koszul. In fact,

$$0 \to S \otimes \mathbf{k}\{xy - yx\} \to S \otimes \mathbf{k}\{x, y\} \to S \to \mathbf{k} \to 0$$

is a linear free resolution.

A Non-Koszul Algebra

Non-Example

Let $B = \mathbf{k}\langle x, y\rangle/(x^2, yx + y^2)$.
Observe: $y^3 = y \cdot x^2 - (yx + y^2) \cdot x + y \cdot (yx + y^2) = 0 \in B$.
The natural complex

$$\cdots \to A \otimes \mathbf{k}\{x^2, y(x+y)\} \to A \otimes \mathbf{k}\{x, y\} \to A \to \mathbf{k}$$

has a non-linear kernel.

Q: What if things like $y^3 \in (x^2, yx + y^2)$ do not happen?
A: Then the algebra is Koszul, called a **PBW algebra**.

Gröbner Basis

Definition.

Let $B = \mathbf{k}\langle x_1, \ldots, x_n \rangle / I$ be a **k**-algebra.

- There exists a "good" order on monomials.
- For $f \in I$, let $\mathrm{lt}(f)$ be the greatest term of f, called the **leading term** of f. For $J \subseteq I$, let $\mathrm{lt}(J) := \{f \in J \mid \mathrm{lt}(f)\}$.

A subset $\mathcal{G}$ of I is called a **Gröbner basis** of I if $\mathrm{lt}(\mathcal{G})$ geneartes $\mathrm{lt}(I)$ as an ideal.

PBW Algebras

Definition.

$A = T(V)/I$ is called a PBW algebra if I admits a quadratic Gröbner basis.

Example.

The previous algebra $B = \mathbf{k}\langle x, y\rangle/(x^2, y(x+y))$ has $I = (x^2, y(x+y))$ and $y^3 \in I$.
$\mathrm{lt}(y^3) = y^3 \notin \mathrm{lt}(x^2, y(x+y))$. So $\{x^2, y(x+y)\}$ is not a Gröbner basis of I, and B is not a PBW algebra.

Theorem (Priddy).

A PBW algebra is Koszul.

The Algebra of Interest

$A = \mathbf{k}\langle x, y, z\rangle/(x^2 + yx, xz, zy)$.

Proposition (Dotsenko, Roy Chowdhury, 2016).

A Gröbner basis of the algebra $A = \mathbf{k}\langle x, y, z\rangle/(x^2 + yx, xz, zy)$ with the order $x < y < z$ of generators, is given by

$$\mathcal{G} = \{xy^n x + y^{n+1}x, xz, zy\}_{n \geq 0}$$

The element $xyx + y^2x = 0 \in A$. However, A is Koszul: the sequence

$$0 \to A \to A^{\oplus 2} \to A^{\oplus 3} \to A^{\oplus 3} \to A \to \mathbf{k} \to 0$$

is exact.

The Algebra of Interest

Conjecture (Polishchuk, Positselski, 2005)

If a Koszul algebra has a finite global dimension n, then its number of generators is at least n.

Proposition (Serre, 1956; Auslander, Buchsbaum, 1957)

Let C be a **commutative** local ring, then C is regular if and only if $\operatorname{gl}\dim C < \infty$. In that case, $\operatorname{gl}\dim C$ is equal to the dimension of the tangent space at the maximal ideal.

Failure in Non-commutative World

Proposition (Iyudu, Shkarin, 2016)

$A = \mathbf{k}\langle x, y, z\rangle/(x^2 + yx, xz, zy)$ is a counter-example:

- A is **Koszul**;
- We have $\operatorname{gl}\dim(A) = 4$ but $\dim A_1 = 3$.

- The completion $\hat{A} = \mathbf{k}\langle\langle x, y, z\rangle\rangle/(x^2 + yx, xz, zy)$ of A is a local ring, generated by **3** elements.
- However, we have $\operatorname{gl}\dim \hat{A} = \operatorname{gl}\dim A = \mathbf{4}$.
- This implies that a non-commutative analogue of the Auslander-Buchsbaum-Serre theorem is unlikely to hold.
- We are interested in the full Tamarkin–Tsygan calculus of A.

Outline

1. Koszul Algebra
2. Tamarkin–Tsygan Calculus
3. Computation

Hochschild (Co)homology

- Let A be an associative algebra over a field $\mathbf{k}$.
- Let $A^e := A \otimes A^{op}$, where A^{op} is the opposite algebra of A.

Definition: A-bimodule

An A-bimodule is a left A^e-module.

- A admits a natural A-bimodule structure: $(b \otimes c) \cdot a = bac$.

Definition: Hochschild (Co)homology

- The Hochschild Homology group of A is defined to be $\mathrm{HH}_*(A) := \mathrm{Tor}_*^{A^e}(A, A)$;
- The Hochschild Cohomology group of A is defined to be $\mathrm{HH}^*(A) := \mathrm{Ext}_{A^e}^*(A, A)$.

Hochschild (Co)chain

Proposition: Two-sided Bar Complex.

Let $B_n(A) = A \otimes A^{\otimes n} \otimes A$, with A^e acting on the sides, and

$$\begin{array}{rcl} d_n : B_n & \to & B_{n-1} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} & \mapsto & \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \end{array}$$

Then $(B_\bullet, d_\bullet)$ is a free resolution of A over A^e, called the **Two-sided Bar Complex** of A.

Corollary.

- $\mathrm{HH}_*(A) = \mathrm{H}_*(A \otimes_{A^e} B_\bullet)$, call $C_\bullet := A \otimes_{A^e} B_\bullet$ the **Hochschild chain complex** of A;
- $\mathrm{HH}^*(A) = \mathrm{H}^*(\mathrm{Hom}_{A^e}(B_\bullet, A))$, call $C^\bullet := \mathrm{Hom}_{A^e}(B_\bullet, A)$ the **Hochschild cochain complex** of A.

Hochschild Cochain

We have $C^n = \mathrm{Hom}_{A^e}(B_n, A) \cong \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$. For $f \in C^n$,

$$\begin{aligned} d(f)(a_1 \otimes \cdots \otimes a_{n+1}) \quad &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \textstyle\sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n) \\ &+ (-1)^n f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned}$$

HH^0

- If $a \in C^0$, then $d(a)(b) = ba - ab$.
- $\mathrm{HH}^0(A) = \{a \in A \mid ba = ab \text{ for all } b \in A\}$.

Interpretation in Degree 1

HH^1

- If $\delta \in C^1$, then $d(\delta)(a \otimes b) = a\delta(b) - \delta(ab) + \delta(a)b$.
- So $d(\delta) = 0 \iff \delta : A \to A$ is a derivation (satisfies the Leibniz rule).
- $Z^1(A) \cong \mathrm{Der}(A)$. In particular, the image from C^0, $\{[b, -] : A \to A \mid [b, a] = ba - ab\}$ is a subset of $\mathrm{Der}(A)$, called the **inner derivations**.
- $\mathrm{HH}^1(A) = \mathrm{Der}(A)/(\text{inner derivations}) =$ "outer derivations".

Interpretation in Degree 2

- An extension of A is an exact sequence
$$0 \to I \to E \to A \to 0.$$
- If $I^2 = 0$, it is a **square zero extension.**
- Trivial extension: $E = A \oplus I$, such that $(a, i) * (b, j) = (ab, aj + ib)$
- Nontrivial extension: $(a, i) * (b, j) = (ab, aj + ib + \mathbf{f}(\mathbf{a}, \mathbf{b}))$ for some $f \in C^2$.
- The product is associative $\iff d(f) = 0$.
- Two products are isomorphic $\iff$ the corresponding f's are homologous.
- $\mathrm{HH}^2(A) \cong \{$isomorphic classes of square zero extensions of $A\}$.

Cup Product

For any $n, m \in \mathbb{N}$, we have a bilinear product

$$\cup_{n,m} : C^n \otimes C^m \to C^{n+m}$$

given by

$$(f \cup g)(1 \otimes a_1 \otimes \cdots \otimes a_{n+m} \otimes 1) = (-1)^{mn}$$
$$f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \cdot g(1 \otimes a_{n+1} \otimes \cdots \otimes a_m \otimes 1)$$

. $\cup$ is called the cup product on the Hochschild cochain.

Proposition.

The cup product induces a well-defined associative and graded-commutative product on the cohomology level, which we denote by the same symbol:

$$\cup : \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(A) \to \mathrm{HH}^*(A)$$

Cap Product

- $C_n = A \otimes_{A^e} B_n = A \otimes_{A^e} A \otimes \overline{A}^{\otimes n} \otimes A \cong A \otimes_{A^e} \otimes (A^e \otimes \overline{A}^{\otimes n}) = A \otimes \overline{A}^{\otimes n}$.
- $C^n = \mathrm{Hom}_{A^e}(B_n, A) = \mathrm{Hom}_{A^e}(A^e \otimes \overline{A}^{\otimes n}, A) \cong \mathrm{Hom}(\overline{A}^{\otimes n}, A)$.

Proposition-Definition.

The bilinear map

$$\begin{array}{rcl} \cap : \mathrm{HH}_n(A) \otimes \mathrm{HH}^m(A) & \to & \mathrm{HH}_{n-m}(A) \\ (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \otimes f & \mapsto & (-1)^{m(n-m)} \\ & & a_0 f(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n \end{array}$$

is well-defined, called the **cap product**.

- $\mathrm{HH}_*(A)$ is a right $\mathrm{HH}^*(A)$-module.

Gerstenhaber Bracket

Proposition-Definition

- For $f \in C^n$, $g \in C^m$, define $f \circ g \in C^{n+m-1}$ to be

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{n+m-1})$$
$$:= \sum_{i=1}^{m} (-1)^{(m-1)(i-1)}$$
$$f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{m+n-1}$$

- The **Gerstenhaber bracket**, given by the graded-commutator of $\circ$,

$$\begin{array}{rcl} [-,-] : \mathrm{HH}^n(A) \otimes \mathrm{HH}^m(A) & \rightarrow & \mathrm{HH}^{m+n-1}(A) \\ f \otimes g & \mapsto & f \circ g - (-1)^{(m-1)(n-1)} g \circ f \end{array}$$

is a well-defined graded Lie product.

Connes' Differential

Connes' Differential

The map defined by

$$\begin{array}{rcl} B_n : C_n & \to & C_{n+1} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n & \mapsto & \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i \end{array}$$

induces a well-defined differential on Hochschild homology. We have

- $B : \mathrm{HH}_n(A) \to \mathrm{HH}_{n+1}(A)$ satisfying $B^2 = 0$;
- $[Bi_f - (-1)^p i_f B, i_g] = i_{[f,g]}$ where $f \in \mathrm{HH}^p(A)$, $g \in \mathrm{HH}^q(A)$, $i_f = f \cap - : \mathrm{HH}_*(A) \to \mathrm{HH}_{*-p}$ is the cap product with f.

The Tamarkin–Tsygan Calculus

A **Tamarkin–Tsygan Calculus** consists of the following data:

- Two graded vector spaces $(\mathrm{HH}_*, \mathrm{HH}^*)$;
- A cup product $\cup : \mathrm{HH}^* \otimes \mathrm{HH}^* \to \mathrm{HH}^*$ of degree 0;
- A cap product $\cap : \mathrm{HH}^p \otimes \mathrm{HH}_n \to \mathrm{HH}_{n-p}$;
- A Gerstenhaber bracket $[-,-] : \mathrm{HH}^* \otimes \mathrm{HH}^* \to \mathrm{HH}^*$ of degree -1;
- A Connes' differential $B : \mathrm{HH}_* \to \mathrm{HH}_*$ of degree 1;

such that

- $(\mathrm{HH}^*, \cup, [-,-])$ is a Gerstenhaber algebra;
- $(\mathrm{HH}_*, \cap)$ is a module over $(\mathrm{HH}^*, \cup)$ and $(\mathrm{HH}^*, [-,-])$;
- B is a differential satisfying $[Bi_f - (-1)^p i_f B, i_g] = i_{[f,g]}$.

The Tamarkin–Tsygan Calculus: Geometric Origin

Theorem (Hochschild, Kostant, Rosenberg, 1962; Rinehart 1963; Connes 1990; Calaque, Van den Bergh, Rossi, 2010, 2012;)

Let X be an algebraic variety over $\mathbf{k}$, R be the coordinate algebra on X. Then the Hochschild homology and cohomology of R have a structure of Tamarkin–Tsygan calculus, such that

- $\mathrm{HH}_*(R) \cong \Omega^*(X)$: The Hochschild homology is the differential forms on X;
- $\mathrm{HH}^*(R) \cong \mathfrak{X}^*(X)$: The Hochschild cohomology is the multi-vector fields on X;
- The cup product is the wedge product of multi-vector fields;
- The cap product is the contraction of multi-vector fields by differential forms;
- The Connes' differential is the de Rham differential.

The Tamarkin–Tsygan Calculus

- The non-commutative case:

Proposition (Gelfand, Daletskii, Tsygan, 1989; Tamarkin, Tsygan, 2000).

For an associative algebra A, the Hochschild homology and cohomology $(\mathrm{HH}_*(A), \mathrm{HH}^*(A))$ of A, together with the operations defined before, form a Tamarkin–Tsygan calculus, called the Tamarkin–Tsygan calculus of A.

Theorem (Keller, 2003, 2004, 2018; Armenta, Keller, 2017).

The Tamarkin–Tsygan calculus is a derived invariant.

Outline

1. Koszul Algebra
2. Tamarkin–Tsygan Calculus
3. Computation

Minimal Free Resolution

Proposition.

The two-sided Koszul complex $(K_\bullet, d_\bullet)$ of A is the minimal free resolution of A over A^e. We have $K_\bullet = A^e \otimes V_\bullet$,
where

$$\begin{array}{ll} V_0 = \mathbf{k}, & V_3 = \mathbf{k}\{(x+y)xz, xzy\} \\ V_2 = \mathbf{k}\left\{x^2 + yx, xz, zy\right\} & V_4 = \mathbf{k}\{(x+y)xzy\} \\ V_1 = \mathbf{k}\{x, y, z\} & V_n = 0, n > 4 \end{array}$$

and the differentials can be written explicitly.

Example.

$$\begin{array}{ll} d_4 : K_4 \to K_3, & 1 \otimes ((x+y)xzy) \otimes 1 \mapsto (x+y) \otimes xzy \otimes 1 - 1 \otimes ((x+y)xz) \otimes y, \\ & 1 \otimes x \otimes 1 \mapsto x \otimes 1 - 1 \otimes x \\ d_1 : K_1 \to K_0, & 1 \otimes y \otimes 1 \mapsto y \otimes 1 - 1 \otimes y \quad . \\ & 1 \otimes z \otimes 1 \mapsto z \otimes 1 - 1 \otimes z, \end{array}$$

The Hochschild (Co)chain Complex

Let $C_\bullet := A \otimes_{A^e} K_\bullet$ and $C^\bullet := \mathrm{Hom}_{A^e}(K_\bullet, A)$. We have

- $C_n = A \otimes V_n$; for example, $C_2 = A \otimes \mathbf{k}\left\{x^2 + yx, xz, zy\right\}$ and $C_1 = A \otimes \mathbf{k}\{x, y, z\}$. We have $C_2, C_1 \cong_{\mathbf{k}} A^3$.
- $C^\bullet = \mathrm{Hom}(V_n, A)$; for example, $C^4 = \mathrm{Hom}(\mathbf{k}\{(x+y)xzy\}, A) \cong_{\mathbf{k}} A$ and $C^3 = \mathrm{Hom}(\mathbf{k}\{(x+y)xz, xzy\}, A) \cong_{\mathbf{k}} A^2$.

The differentials are given by, for example,

- $$\begin{array}{rcl} d_2 : C_2 & \to & C_1 \\ a \otimes (x^2 + yx) & \mapsto & (ax + xa + ay) \otimes x + xa \otimes y \\ b \otimes xz & \mapsto & bx \otimes z + zb \otimes x \\ c \otimes zy & \mapsto & cz \otimes y + yc \otimes z \end{array}$$
- $$\begin{array}{rcl} d^4 : C^3 & \to & C^4 \\ \begin{pmatrix} (x+y)xz & \mapsto a \\ xzy & \mapsto b \end{pmatrix} & \mapsto & ((x+y)xzy \mapsto (x+y)b + ay) \end{array}$$

Gröbner Basis

Definition.

Let $B = \mathbf{k}\langle x_1, \ldots, x_n\rangle / I$ be a **k**-algebra.

- There exists a “good” order on monomials.
- For $f \in I$, let $\mathrm{lt}(f)$ be the greatest term of f, called the **leading term** of f. For $J \subseteq I$, let $\mathrm{lt}(J) := \{f \in J \mid \mathrm{lt}(f)\}$.

A subset $\mathcal{G}$ of I is called a **Gröbner basis** of I if $\mathrm{lt}(\mathcal{G})$ geneartes $\mathrm{lt}(I)$ as an ideal.

Proposition.

If $\mathcal{G}$ is a Gröbner basis of $I \subseteq \mathbf{k}\langle x_1, \ldots, x_n\rangle$, then $B = \mathbf{k}\langle x_1, \ldots, x_n\rangle / I$ admits a **k**-basis consisting of monomials that are not divisible by the elements in $\mathrm{lt}(\mathcal{G})$.

Gröbner Basis

Proposition (Dotsenko, Roy Chowdhury, 2016).

A Gröbner basis of the algebra $A = \mathbf{k}\langle x, y, z\rangle/(x^2 + yx, xz, zy)$ with the order $x < y < z$ of generators, is given by

$$\mathcal{G} = \{xy^n x + y^{n+1}x, xz, zy\}_{n\geq 0}$$

Corollary.

A has a $\mathbf{k}$-basis consisting of monomials that are not divisible by the elements in $\{xy^n x, xz, zy\}_{n\geq 0}$.

A **k**-basis of A

We partition the **k**-basis resulting from the Gröbner basis $\mathcal{G}$ into the monomials of the following 17 types:

Type 1 : y^{n_0}; Type 2 : $y^{n_0}x$;
Type 3 : $y^{n_0}xy^{n_1}$;
Type 4 : $y^{n_0}xy^{n_1}z^{m_1}x\cdots z^{m_p}$;
Type 5 : $y^{n_0}xy^{n_1}z^{m_1}x\cdots z^{m_p}x$;
Type 6 : $y^{n_0}xy^{n_1}z^{m_1}x\cdots z^{m_p}xy^{n_{p+1}}$;
Type 7 : x;
Type 8 : xy^{n_1};
Type 9 : $xy^{n_1}z^{m_1}x\cdots z^{m_p}$;
Type 10 : $xy^{n_1}z\cdots zx$;
Type 11 : $xy^{n_1}z^{m_1}x\cdots z^{m_p}xy^{n_{p+1}}$;
Type 12 : $y^{n_1}z^{m_1}x\cdots z^{m_p}$;
Type 13 : $y^{n_1}z^{m_1}x\cdots z^{m_p}x$;
Type 14 : $y^{n_1}z\cdots zxy^{n_{p+1}}$;
Type 15 : $z^{m_1}x\cdots z^{m_p}$;
Type 16 : $z^{m_1}x\cdots z^{m_p}x$;
Type 17 : $z^{m_1}x\cdots z^{m_p}xy^{n_{p+1}}$.

where $z^{m_1}x\cdots z^{m_p}$ represents $z^{m_1}xy^{n_2}z^{m_2}x\cdots z^{m_{p-1}}xy^{n_p}z^{m_p}$, with all the powers positive integers.

An Example of Calculation

Suppose we want to calculate $\mathrm{HH}_2(A) = \ker d_2 / \operatorname{im} d_3$.

- Let $\alpha = a \otimes (x^2 + yx) + b \otimes xz + c \otimes zy \in C_2$, where $a, b, c \in A$.
- Write a, b, c as linear combinations of the 17 types of monomials with arbitrary coefficients in $\mathbf{k}$.
- From the equation $d_2(\alpha) = (ax + xa + ay + zb) \otimes x + (xa + cz) \otimes y + (bx + yc) \otimes z = 0$, i.e.,

$$\begin{aligned} ax + xa + ay + zb &= 0, \\ xa + cz &= 0, \\ bx + yc &= 0, \end{aligned}$$

  deduce relations between the coefficients of the terms of a, b, c.
- Compute $\operatorname{im} d_3$, and deduce more relations between the coefficients.
- Conclusion: α is homologous to 0, that is, $\mathrm{HH}_2(A) = 0$.

Result: Homology

Proposition (Chen, R., Yang).

- The Hochschild homology $\mathrm{HH}_n(A)$ of A is 0 for $n \geq 2$.
- $\mathrm{HH}_1(A)$ has a **k**-basis consisting of the following elements:
- $$\alpha_1^1 = x \otimes x, \quad \alpha_1^2(n_0) = y^{n_0} x \otimes x, \quad \alpha_1^3 = 1 \otimes x$$
  $$\beta_1^1(n_0) = y^{n_0} \otimes y, \quad \beta_1^2 = 1 \otimes y$$
  $$\gamma_1^1(n_0) = z^{n_0} \otimes z, \quad \gamma_1^2 = 1 \otimes z$$
- $$\begin{aligned} &\delta_1^{(n_1,\cdots,n_p,m_1\cdots,m_p)} = (\textstyle\sum_{i=1}^p y^{n_i} z^{m_i} \cdots y^{n_p} z^{m_p} x y^{n_1} z^{m_1} \cdots y^{n_{i-1}} z^{m_{i-1}}) \otimes x \\ &+(\textstyle\sum_{i=1}^p \sum_{t=1}^{n_i} y^{n_i - t} z^{m_i} x \cdots y^{n_p} z^{m_p} x y^{n_1} z^{m_1} x \cdots y^{n_{i-1}} z^{m_{i-1}} x y^{t-1}) \otimes y \\ &+(\textstyle\sum_{i=1}^p \sum_{s=1}^{m_i} z^{m_i - s} x \cdots y^{n_p} z^{m_p} x y^{n_1} z^{m_1} x \cdots y^{n_{i-1}} z^{m_{i-1}} x y^{n_i} z^{s-1}) \otimes z \end{aligned}$$

  for $p, n_0, n_1, \cdots, n_p, m_1, \cdots, m_p \in \mathbb{N}^*$.

Result: Homology

Proposition (continue).

- $\mathrm{HH}_0(A)$ has a **k**-basis consisting of the following elements:

$$\alpha_0^1 = x, \quad \alpha_0^2(n_0) = y^{n_0}x, \quad \beta_0(n_0) = y^{n_0}, \quad \gamma_0(n_0) = z^{n_0}$$

- $$\delta_0^{\overline{(n_1,\cdots,n_p,m_1,\cdots,m_p)}} = \overline{y^{n_1}z^{m_1}x\cdots y^{n_p}z^{m_p}x}, \quad \eta_0 = 1$$

  for $n_0, n_1, \cdots, n_p, m_1, \cdots, m_p \in \mathbb{N}^*$, where the bars mean that $\overline{y^{n_1}z^{m_1}x\cdots y^{n_p}z^{m_p}x} = \overline{y^{n_i}z^{m_i}x\cdots y^{n_p}z^{m_p}xy^{n_1}z^{m_1}x\cdots y^{n_{i-1}}z^{m_{i-1}}x}$, for $i = 1, \cdots, p$.

Remark.

In fact, $\mathrm{HH}_0(A) \cong A/[A, A]$. So we have, for example, $\overline{yzxyz} = \overline{zxyzy} = 0$.

Result: Connes' Differential

Since $\mathrm{HH}_n(A) = 0$ for $n \geq 2$, the only nontrivial Connes' differential is $B_0 : \mathrm{HH}_0(A) \to \mathrm{HH}_1(A)$.

Proposition (Chen, R., Yang).

B_0 sends

- α_0^1 to α_1^3: $x \mapsto 1 \otimes x$;
- $\alpha_0^2(n_0)$ to $\alpha_1^2(n_0 - 1)$: $y^{n_0}x \mapsto (y^{n_0-1}x) \otimes x$;
- $\beta_0(n_0)$ to $\beta_1^1(n_0 - 1)$: $y^{n_0} \mapsto y^{n_0-1} \otimes y$;
- $\gamma_0(n_0)$ to $\gamma_1^1(n_0 - 1)$: $z^{n_0} \mapsto z^{n_0-1} \otimes z$;
- $\overline{\delta_0^{(n_1,\cdots,n_p,m_1,\cdots,m_p)}}$ to $\delta_1^{(n_1,\cdots,n_p,m_1,\cdots,m_p)}$;
- η_0 to 0.

Result: Cohomology

Proposition (Chen, R., Yang).

$\mathrm{HH}^1(A)$ has a $\mathbf{k}$-basis consisting of the following elements:

$$A^1 = \left(\begin{array}{c} x \mapsto x \\ y \mapsto y \end{array} \right), B^1 = \left(\begin{array}{ccc} x & \mapsto & y^n x \\ y & \mapsto & -y^n x \end{array} \right), C^1 = \left(\begin{array}{ccc} y & \mapsto & y^{n+1}x - xy^{n+1} \\ z & \mapsto & zxy^n \end{array} \right),$$

$$D^1 = \left(\begin{array}{ccc} y & \mapsto & y^n x - y^i x y^{n-i} \end{array} \right),$$

$$E^1 = \left(\begin{array}{ccc} y & \mapsto & y^n x + y^{n+1} \end{array} \right),$$

$$F^1 = \left(\begin{array}{ccc} y & \mapsto & y^{n_0} x y^{n_1} z^{m_1} x \cdots z^{m_p} x \end{array} \right),$$

$$G^1 = \left(\begin{array}{ccc} y & \mapsto & y^{n_1} z^{m_1} x \cdots z^{m_p} x \end{array} \right),$$

$$H^1 = \left(\begin{array}{ccc} y & \mapsto & y^{n_0} x y^{n_1} z^{m_1} x \cdots y^{n_{p+1}} \end{array} \right),$$

$$I^1 = \left(\begin{array}{ccc} y & \mapsto & y^{n_1} z^{m_1} x \cdots y^{n_{p+1}} \end{array} \right),$$

$$J^1 = \left(\begin{array}{ccc} y & \mapsto & xy^{n_1} z^{m_1} x \cdots y^{n_{p+1}} \\ z & \mapsto & -zxy^{n_1} z^{m_1} x \cdots y^{n_{p+1}-1} \end{array} \right),$$

$$K^1 = \left(\begin{array}{ccc} z & \mapsto & z^{m_1} x y^{n_2} \cdots z^{m_p} \end{array} \right),$$

Result: Cohomology

Proposition (Chen, R., Yang).

- $\mathrm{HH}^2(A)$ has a **k**-basis consisting of the following elements:

$$A^2 = \begin{pmatrix} (x+y)x & \mapsto & x \\ zy & \mapsto & 0 \\ xz & \mapsto & 0 \end{pmatrix}, B^2 = \begin{pmatrix} (x+y)x & \mapsto & z^{m_1}x\cdots z^{m_p}x \\ zy & \mapsto & 0 \\ xz & \mapsto & 0 \end{pmatrix}.$$

  for $n_2\cdots, n_p, m_1, \cdots, m_p \geq 1$.
- $\mathrm{HH}^0(A) = k$, $\mathrm{HH}^3(A) = 0$.

Result: Cohomology

Proposition (Chen, R., Yang)

$\mathrm{HH}^4(A)$ has a **k**-basis consisting of the following elements:

$$\begin{array}{l}
A^4 : (x+y)xzy \mapsto 1, \\
B^4 : \ (x+y)xzy \ \mapsto \ y^{n_0}x \ , \\
C^4 : \ (x+y)xzy \ \mapsto \ y^{n_0}xy^{n_1}z^{m_1}x\cdots z^{m_p} \ , \\
D^4 : \ (x+y)xzy \ \mapsto \ y^{n_0}xy^{n_1}z^{m_1}x\cdots z^{m_p}x \ , \\
E^4 : \ (x+y)xzy \ \mapsto \ xy^{n_1}z^{m_1}x\cdots z^{m_p} \ , \\
F^4 : \ (x+y)xzy \ \mapsto \ xy^{n_1}z^{m_1}x\cdots z^{m_p}x \ , \\
G^4 : \ (x+y)xzy \ \mapsto \ z^{m_1}x\cdots z^{m_p} \ , \\
H^4 : \ (x+y)xzy \ \mapsto \ z^{m_1}x\cdots z^{m_p}x \ .
\end{array}$$

for $n_0, \cdots, n_p, m_1, \cdots, m_p \geq 1$.

Comparison Map

- The cup product, cap product, Gerstenhaber bracket and Connes' differential are defined based on the bar resolution $B_\bullet$.
- We need maps $\iota : K_\bullet \longleftrightarrow B_\bullet : \pi$ to compare these two resolutions.

Proposition (Chen, R., Yang).

$K_\bullet$ is a deformation retract of $B_\bullet$, with comparison maps $\iota : K_\bullet \longleftrightarrow B_\bullet : \pi$ given by, for example,

- $\iota_1(1 \otimes a \otimes 1) = 1 \otimes a \otimes 1, a = x, y, z;$
- $\iota_3(1 \otimes xzy \otimes 1) = 1 \otimes x \otimes z \otimes y \otimes 1;$
- $\iota_4(1 \otimes (x + y)zy \otimes 1) = 1 \otimes (x + y) \otimes x \otimes z \otimes y \otimes 1;$
- $\pi_1(1 \otimes a_1 \cdots a_n \otimes 1) = \sum_{i=1}^{n} a_{1,i-1} \otimes a_i \otimes a_{i+1,n}, \quad a_1, \cdots, a_n \in \{x, y, z\};$
- $\pi_2(1 \otimes ax \otimes zb \otimes 1) = a \otimes xz \otimes b;$
- $\pi_4(1 \otimes axy^{j_1} \otimes y^{k_1-j_1}x \otimes z \otimes yb \otimes 1) = (axy^{k_1-1} + ay^{k_1}) \otimes (x + y)xzy \otimes b;$

Result: Cup Product

Proposition (Chen, R., Yang).

The cup product $\cup : \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(A) \to \mathrm{HH}^*(A)$ of our algebra A is zero.

Example.

- $\cup_{i,j} : \mathrm{HH}^i(A) \otimes \mathrm{HH}^j(A) \to \mathrm{HH}^{i+j}(A)$ is zero for $(i,j) = (1,2), (3,0)$ since $\mathrm{HH}^3(A) = 0$.
-
$$\begin{aligned} &\cup_{2,2}(A^2, A^2) = ((x+y)xzy \mapsto A^2((x+y)x) \cdot A^2(zy)) \\ &= ((x+y)xzy \mapsto x \cdot z^{m_1} x \cdots z^{m_p} x = 0) \end{aligned}$$

Hence $\cup_{2,2}(A^2, A^2) = 0$.

Result: Cap Product

Proposition.

The cap products $\cap_p^q : \mathrm{HH}^p(A) \otimes \mathrm{HH}_q(A) \to \mathrm{HH}_{q-p}$ is nonzero only for $(p, q) = (0, 0), (0, 1)$ or $(1, 1)$ since $\mathrm{HH}_n(A) = 0$ for $n \neq 0, 1$. Moreover, $\cap_1^0$ and $\cap_0^0$ are identities on $\mathrm{HH}_1(A)$ and $\mathrm{HH}_0(A)$ since $\mathrm{HH}^0(A) = \mathbf{k}$.

We omit the full results, which are long but straight-forward calculations.

Example.

- $$A^1 \cap \delta_1^{(n_1,\cdots,n_p,m_1,\cdots,m_p)} = \sum_{i=1}^{p} (n_i + 1)\delta_0^{\overline{(n_1,\cdots,n_p,m_1,\cdots,m_p)}}$$
- $$F^1 \cap \delta_1^{(n_1,\cdots,n_p,m_1,\cdots,m_p)} = 0$$

Gerstenhaber Bracket

The full result of the Gerstenhaber bracket is too long to be shown here.

Example.

- $[A^1, X] = \lambda(X) \cdot X$, for $X = A^1 - K^1$, where $\lambda(X) \in \mathbf{k}$;
- For $B^1(n) = \left(\begin{array}{ccc} x & \mapsto & y^n x \\ y & \mapsto & -y^n x \end{array} \right)$, $B^1(m) = \left(\begin{array}{ccc} x & \mapsto & y^m x \\ y & \mapsto & -y^m x \end{array} \right)$, $[B^1(n), B^1(m)] = (m - n)B^1(m + n)$. Here $B^1(n)$ behaves like the one-dimensional vector field $x^{n+1}\frac{\partial}{\partial x} =: X_n$, with $[X_n, X_m] = (m - n)x^{m+n+1}X_{n+m}$.
- $[B^1, G^1] = -F^1$;
- $[G^1, J^1] = -F^1 + J^1 + G^1$;
- $[A^1, B^4] = A^4$;
- $\cdots$

Thank you!