Introduction to Reshetikhin–Turaev knot polynomials

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Slides available at: shana-y-li.github.io/Notes/RT_talk.pdf

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2 The Reshetikhin–Turaev functor

3 Applications and Implications

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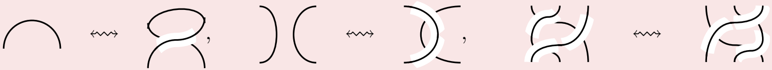
Theorem (Reidemeister)

Two knots are isotopic if and only if their diagrams are related by a finite sequence of Reidemeister moves.



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- Oriented links and tangles are equivalent classes modulo orientation-preserving isotopies.

Category of oriented tangles:

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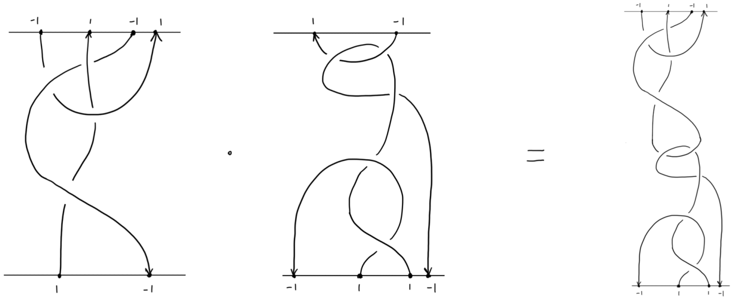

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Ingredients:

- A vector space V over a field $\mathbb{F}$.
- An R-matrix $R \in \mathrm{Aut}(V \otimes V)$.
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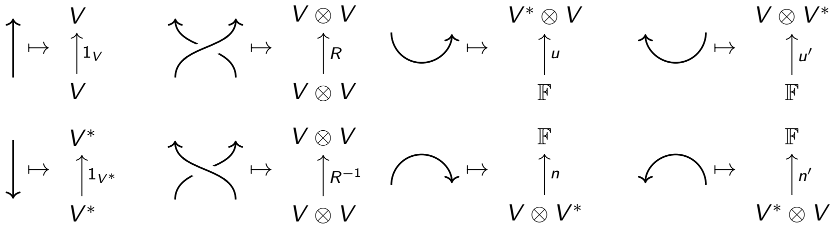
Sources of such ingredients: Representations of ribbon Hopf algebras

$$\rho \colon \mathfrak{A} \to \mathrm{End}(V).$$

Reshetikhin–Turaev functor: oriented tangles $\mapsto$ vector spaces



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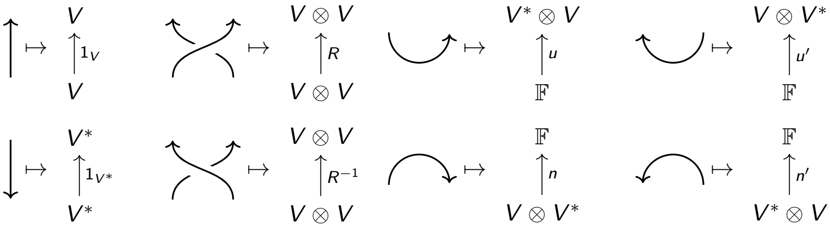
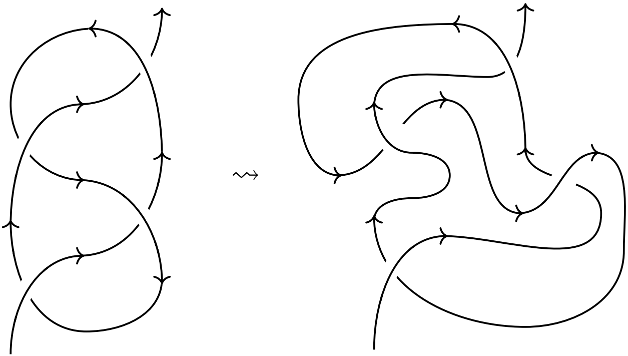


$$u(1) = \sum_i e_i^* \otimes h(e_i), \quad u'(1) = \sum_i e_i \otimes e_i^*,$$

$$n(x \otimes f) = f(h^{-1}(x)), \quad n'(f \otimes x) = f(x)$$

Reshetikhin–Turaev: oriented tangles $\mapsto$ vector spaces

Example: the 4_1 knot



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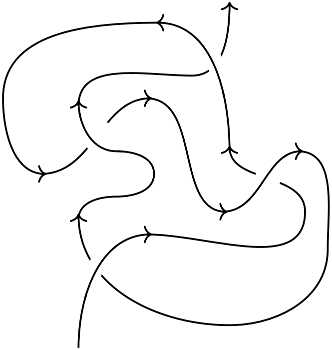
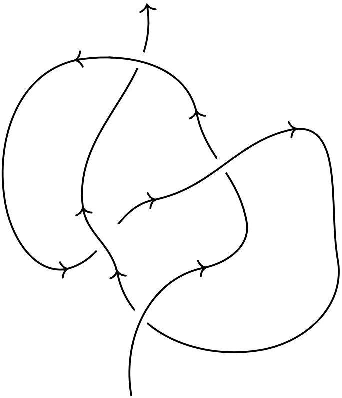


$\rightsquigarrow$



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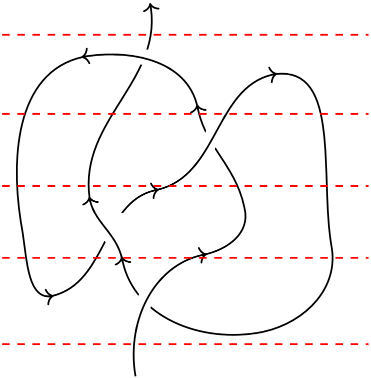

$\mapsto$

$$
\begin{array}{ll}
V & \\
\uparrow & \left(\left(n' \otimes 1_V\right) \circ \left(1_V \otimes R^{-1}\right)\right) \otimes n \\
V^{\otimes 5} & \\
\uparrow & 1_V \otimes 1_V \otimes R \otimes 1_V \\
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\uparrow & 1_V \otimes R^{-1} \otimes 1_V \otimes 1_V \\
V^{\otimes 5} & \\
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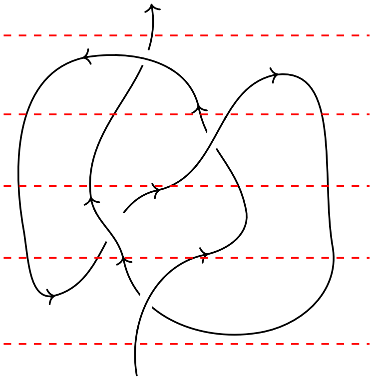

$$\mapsto \quad \begin{array}{cr} V & \\ \uparrow & \left(\left(n' \otimes 1_V\right) \circ \left(1_V \otimes R^{-1}\right)\right) \otimes n \\ V^{\otimes 5} & \\ \uparrow & 1_V \otimes 1_V \otimes R \otimes 1_V \\ V^{\otimes 5} & \\ \uparrow & 1_V \otimes R^{-1} \otimes 1_V \otimes 1_V \\ V^{\otimes 5} & \\ \uparrow & u \otimes \left((R \otimes 1_V) \circ \left(1_V \otimes u'\right)\right) \\ V & \end{array}$$

Fact

The endomorphism on V we obtain is a scalar multiple of 1_V. Plugging indeterminants in R and h gives polynomial invariants.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ (and h) become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)_{e_i \otimes e_j}^{e_k \otimes e_l}$.

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To compute the eigenvalue of the End(V)-valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1, \cdots, a_{2c-1} \in \mathcal{B} \\ a_0 = a_{2c} = 1}} \underbrace{\left(R^{\pm 1}\right)_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots \left(R^{\pm 1}\right)_{a_{2c-3} \otimes a_{2c-2}}^{a_{2c-1} \otimes a_{2c}}}_{\text{a product of length } c},$$

where c is the number of crossings of the knot. This sum is the so called *state sum*.

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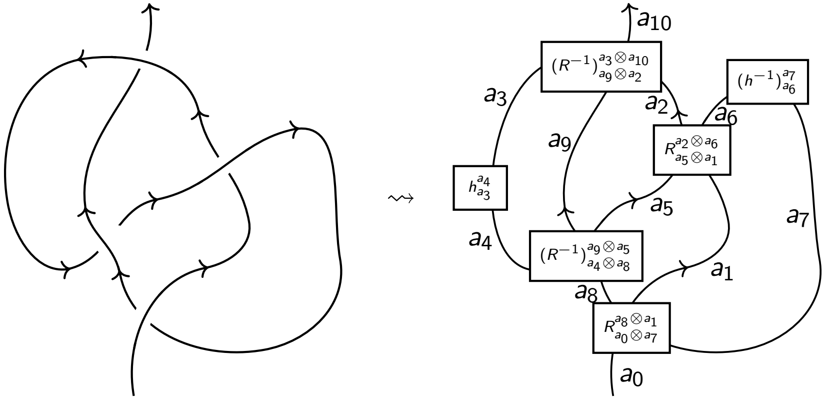
Therefore, it requires

$$c \cdot (\dim V)^{2c-1}$$

multiplications and additions to compute the eigenvalue.

Example: the 4_1 knot again



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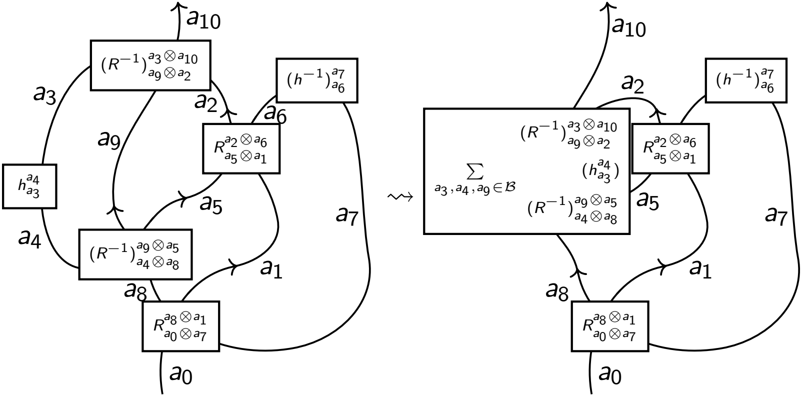

(L. 2026)

Computing Reshetikhin–Turaev knot polynomials via the presented method is

$$O((\dim V)^w c^{k+2} (\log c)^3),$$

where k is the number of variables in the polynomial and w grows at worst $O(\sqrt{c})$.

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Algorithms with time complexity of the form

$$O(f(i) \cdot p(n))$$

for an input of size n where p is a polynomial and f is an arbitrary computable function is called *fixed-parameter tractable with respect to* i.

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Proposition

A knot is the unknot if and only if its genus is 0.

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Definition (co-NP)

A decision problem lies in co-NP if its negation lies in NP.

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A decision problem is NP-hard if there is a polynomial-time reduction from any NP problem to it such that the problem has a positive answer when and only when the given NP problem also has a positive answer.

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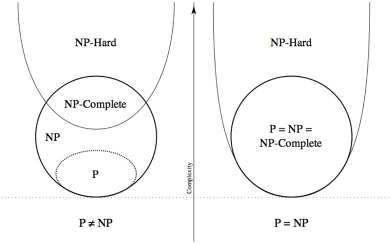

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Definition (NP-intermediate)

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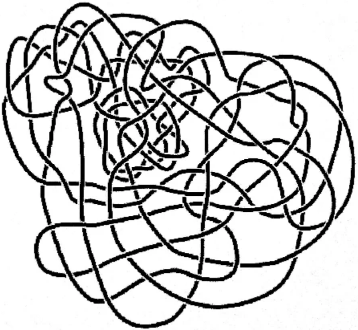
Mesmay, Rieck, Sedgewick & Tancer, 2018

Deciding if a diagram of the unknot can be related to the unknot by at most k Reidemeister moves is NP-hard.

Haken, 1961

The knot diagram below represents the unknot.



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Proposition

If the above conjecture is true, then there are algorithms for the top two items respectively.

Definition (Conway mutation)

A knot K is *Conway mutant* to another knot K' if there exists a 3-ball B whose boundary intersects K exactly four times such that digging B out, rotating and filling it back to switch pairs of the four intersections results in K'.

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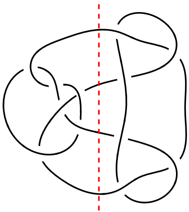
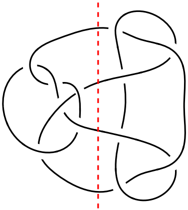


$12n364$



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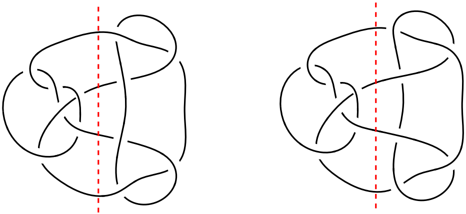

$12n364$ $\overline{12n365}$

Conway mutation preserves many invariants, but not the genus.

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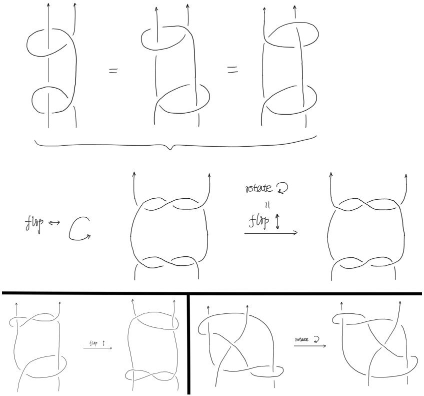

Figure: 3 of the 7 special tangles.