Möbius inversion at the objective level

This is the last time I will be presenting at the SUSTech Graduate Topology Seminar during my master's studies. This note is a tribute to all the teachers and friends who have supported me along the way — we shared a wonderful time together, and I am grateful to you for witnessing my growth.

The convolution of the zeta function and the Möbius function returns to the origin, yet our lives move ever forward ☺

This note is mainly written on a series of papers. For detailed research on decomposition spaces (used to construct incidence coalgebra in this note) and its application in combinatorics, one can read [2],[3],[4]; For the theory of Möbius inversion, one can refer to [5], and the most key point is summarised in the first section of this note; For the detailed version of cardinality (a tool transposes categorified theory to numbers or vector spaces), I recommend [1]; Besides, I also recommend a further reading about negative sets, which is a generalization of negative numbers, in [6].

1 Classical Möbius inversion

Construction 1.1. (**Convolution algebra**) Let (C, Δ, ε) be a coalgebra and (A, m, u) be an algebra. Denote $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$. Define the convolution product on $\mathrm{Lin}(C, A)$ by $(\alpha * \beta)(x) = \sum_{(x)} m\big(\alpha(x_{(1)}), \beta(x_{(2)})\big)$. Then $(\mathrm{Lin}(C, A), *, u\varepsilon)$ forms an algebra, called the convolution algebra.□

Classical generalized Möbius inversion studies which elements of $\mathrm{Lin}(C, A)$ are invertible for convolution.

Example 1.2. $(\mathbb{C}[\mathbb{N}^\times], \Delta, \varepsilon)$ is a coalgebra, where

$$\Delta(n) := \sum_{d|n} d \otimes \frac{n}{d}, \qquad \varepsilon(n) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise} \end{cases}$$

$\mathbb{C}$ is an algebra. Since a linear map is determined on the basis, we have

$$\mathrm{Lin}(\mathbb{C}[\mathbb{N}^\times], \mathbb{C}) = \{f : \mathbb{N}^\times \to \mathbb{C}\}.$$

In the following we also represent $\zeta, \mu \in \mathrm{Lin}(\mathbb{C}[\mathbb{N}^\times], \mathbb{C})$ by maps $\mathbb{N}^\times \to \mathbb{C}$.

Consider

$$\zeta : \mathbb{N}^\times \to \mathbb{C} \text{ definecd by } n \longmapsto 1$$

and

$$\mu : \mathbb{N}^\times \to \mathbb{C} \text{ defined by } n \longmapsto \begin{cases} 0, & \text{if } n \text{ contains a square factor,} \\ (-1)^r, & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

Then one can check $\zeta * \mu = \mu * \zeta = \varepsilon$.. This is the classical Möbius inversion principle. □

Theorem 1.3. If $\phi \in \mathrm{Lin}(C, A)$ sends all group-like elements ($\Delta(x) = x \otimes x$) to 1, then ϕ is convolution invertible. Its inverse ψ is given recursively by

$$\psi = e - \psi * (\phi - e), \qquad e := u\varepsilon.$$

Proof sketch. We only show the construction of ψ. Let

$$\psi_0 = e, \qquad \psi_{n+1} := \psi_n * (\phi - e).$$

Define

$$\psi_{\text{even}} := \sum_{n \text{ even}} \psi_n, \qquad \psi_{\text{odd}} := \sum_{n \text{ odd}} \psi_n,$$

and then set

$$\psi := \psi_{\text{even}} - \psi_{\text{odd}}.$$

□

In the next section we will formulate the above theorem at the objective level, i.e. categorify the classical generalized Möbius inversion theorem.

2 Categorified analogue of the coalgebra (C, Δ, ε)

Coalgebra is a vector space with compatible comultiplication.

Notation. E, B are groupoids. X is a simplicial space, i.e., a functor $\Delta^{op} \to \mathcal{S}$, where $\mathcal{S}$ is the ∞-category of spaces. Write $\ulcorner b \urcorner : 1 \to B$ for the map picking out b.

Definition 2.1. textbf(Homotopy fiber) Given $f : E \to B$ and $b \in B$, define the homotopy fiber E_b by the homotopy pullback square

$$\begin{array}{ccc} E_b & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow f \\ 1 & \xrightarrow{\ulcorner b \urcorner} & B. \end{array}$$

Fact 2.2. For any $f : E \to B$ in $\mathrm{Grpd}_{/B}$, $f \simeq \int^{b \in B} E_b \ulcorner b \urcorner$. □

This suggests that $\mathrm{Grpd}_{/B}$ behaves like a vector space:

- Basis: $\ulcorner b \urcorner : 1 \to B \quad ([b] \in \pi_0 B)$
- Scalar multiplication:$A\ulcorner b \urcorner$ For $A \in \mathrm{Grpd}$, define

$$A\ulcorner b \urcorner := \left[A \to 1 \xrightarrow{\ulcorner b \urcorner} B \right] \in \mathrm{Grpd}_{/B}.$$

- Addition: homotopy sum

□

Warning.

1) We have only defined "scalar multiplication" for $\ulcorner b \urcorner : 1 \to B$, not arbitrary $E \to B$ in $\mathrm{Grpd}_{/B}$.
2) There is no negative sign here; for example $-\ulcorner b \urcorner$ is meaningless. □

Anyway, we obtain a categorified analogue of a vector space. Next we define comultiplication and counit.

We wantto construct

$$\mathrm{Grpd}_{/B} \to \mathrm{Grpd}_{/B} \times \mathrm{Grpd}_{/B}, \qquad \mathrm{Grpd}_{/B} \to \mathrm{Grpd}_{/1}.$$

Step 1. Using

$$\mathrm{Grpd}_{/B} \times \mathrm{Grpd}_{/B} \cong \mathrm{Grpd}_{/B \times B}, \qquad \mathrm{Grpd}_{/1} \cong \mathrm{Grpd},$$

Step 2. A span $A \xleftarrow{f} C \xrightarrow{g} D$ induces a functor

$$\mathrm{Grpd}_{/A} \xrightarrow{f^*} \mathrm{Grpd}_{/C} \xrightarrow{g_!} \mathrm{Grpd}_{/D}.$$

$$\begin{array}{ccccccc} & & E' & \longrightarrow & E & & \\ & & \downarrow & & \downarrow & & \\ E \to A & \longmapsto & C & \underset{f}{\longrightarrow} & A & \longmapsto & E' \to C \to D \end{array}$$

Remark 2.3. Since a classical coalgebra structure is bilinear, the functor $\mathrm{Grpd}_{/B} \to \mathrm{Grpd}_{/B \times B}$ should be represented by a span, which is the analogue of linear maps in category theory. □

Step 3. It is difficult to define a coalgebra structure on $\mathrm{Grpd}_{/B}$ for arbitrary $B \in \mathrm{Grpd}$. But the simplicial space will provide enough data for constructing. Let $X : \Delta^{\mathrm{op}} \to \mathcal{S}$ be a simplicial space.

Remark 2.4. By definition, X is an ∞-groupoid. But in many cases we only use simplicial groupoids, viewing an ordinary groupoid as an ∞-groupoid. □

If $B = X_1$, then we have a span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_1 \text{ which induces } \Delta : \mathrm{Grpd}_{/X_1} \to \mathrm{Grpd}_{/X_1 \times X_1}$$

and also a span

$$X_1 \xleftarrow{s_0} X_0 \to 1 \text{ which induces } \varepsilon : \mathrm{Grpd}_{/X_1} \to \mathrm{Grpd}_{/1}$$

Step 4. We require Δ and ε to be coassociative and counital. These properties hold when X is a *decomposition space.*

The coalgebra $(\mathrm{Grpd}_{/X_1}, \Delta, \varepsilon)$ is called the *incidence coalgebra* of the decomposition space X.

We take associativity for example: We require the following square of slice groupoids to commute:

$$\begin{array}{ccc} \mathrm{Grpd}_{/X_1} & \xrightarrow{\Delta} & \mathrm{Grpd}_{/X_1\times X_1} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{Grpd}_{/X_1\times X_1} & \xrightarrow[\mathrm{id}\otimes\Delta]{} & \mathrm{Grpd}_{/X_1\times X_1\times X_1}. \end{array}$$

At the level of spans, one is tempted to look at

$$\begin{array}{ccccc} X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2,d_0)} & X_1\times X_1 \\ {\scriptstyle d_1}\uparrow & & & & \uparrow{\scriptstyle d_1\times\mathrm{id}} \\ X_2 & & & & X_2\times X_1 \\ {\scriptstyle (d_2,d_0)}\downarrow & & & & \downarrow{\scriptstyle (d_2,d_0)\times\mathrm{id}} \\ X_1\times X_1 & \xleftarrow[\mathrm{id}\times d_1]{} & X_1\times X_2 & \xrightarrow[\mathrm{id}\times(d_2,d_0)]{} & X_1\times X_1\times X_1 \end{array}$$

However, in this diagram, the notion of commutativity is vacuous.

Fact 2.5. For any homotopy pullback square

$$\begin{array}{ccc} H & \xrightarrow{q} & E \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ G & \xrightarrow[f]{} & B, \end{array}$$

the functors

$$q_!p^*, g^*f_! : \mathrm{Grpd}_{/G} \to \mathrm{Grpd}_{/E}$$

are naturally homotopy equivalent. $\square$

Instead, decompose the above diagram into four smaller squares:

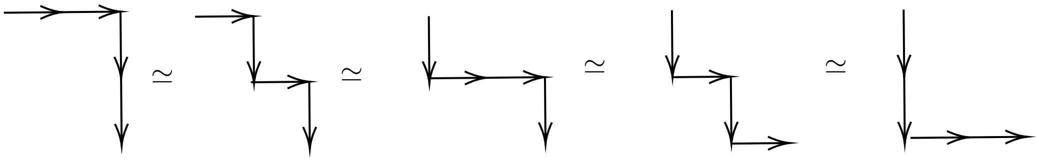
$$\begin{array}{ccccc} X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2,d_0)} & X_1\times X_1 \\ {\scriptstyle d_1}\uparrow & (1) & {\scriptstyle d_1}\uparrow & (2) & \uparrow{\scriptstyle d_1\times\mathrm{id}} \\ X_2 & \xleftarrow{d_2} & X_3 & \xrightarrow{(d_3,d_0d_0)} & X_2\times X_1 \\ {\scriptstyle (d_2,d_0)}\downarrow & (3) & \downarrow{\scriptstyle (d_2d_2,d_0)} & (4) & \downarrow{\scriptstyle (d_2,d_0)\times\mathrm{id}} \\ X_1\times X_1 & \xleftarrow[\mathrm{id}\times d_1]{} & X_1\times X_2 & \xrightarrow[\mathrm{id}\times(d_2,d_0)]{} & X_1\times X_1\times X_1 \end{array}$$

Squares (1) and (4) commute because simplicial object satisfying $d_id_j = d_{j-1}d_i$ for all $i < j$. For example, square (4) commutes meaning that $(d_2d_3x, d_0d_3x, d_0d_0x) = (d_2d_2x, d_2d_0x, d_0d_0x)$ which is equivalent to $d_2d_3 = d_2d_2$ and $d_0d_3 = d_2d_0$.

Assume now that squares (2) and (3) are homotopy pullbacks. Then consider the morphism induced between slice groupoids.

$$\begin{array}{ccccc}
\mathrm{Grpd}_{/X_1} & \xrightarrow{d_1^*} & \mathrm{Grpd}_{/X_2} & \xrightarrow{(d_2,d_0)_!} & \mathrm{Grpd}_{/X_1\times X_1} \\
\downarrow & (1)' & \downarrow & (2)' & \downarrow \\
\mathrm{Grpd}_{/X_2} & \longrightarrow & \mathrm{Grpd}_{/X_3} & \longrightarrow & \mathrm{Grpd}_{/X_2\times X_1} \\
\downarrow & (3)' & \downarrow & (4)' & \downarrow \\
\mathrm{Grpd}_{/X_1\times X_1} & \longrightarrow & \mathrm{Grpd}_{/X_1\times X_2} & \longrightarrow & \mathrm{Grpd}_{/X_1\times X_1\times X_1}
\end{array}$$

We have equivalences of paths in above diagram and this implies the original coassociativity square commutes.



The first and fourth equivalence of paths is because $(2)'$ and $(3)'$ are pullbacks, respectively. The second and third equivalence of paths is because $(1)'$ and $(4)'$ commute, respectively.

So the axioms of a decomposition space should imply that the (2) and (3) are pullbacks.

3 Categorified analogues of (A, m, u) and $\mathrm{Lin}(C, A)$

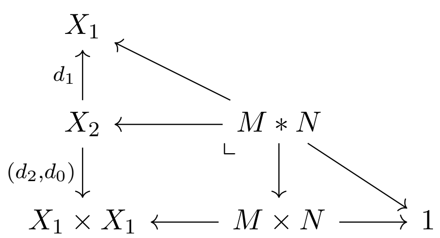
Construction 3.1. (Categorified algebra) $(\mathrm{Grpd}, \times, 1)$ is an algebra, where $X \times Y$ is the Cartesian product of groupoids. □

Construction 3.2. (Categorified convolution algebra) Let **LIN** be the category of slice groupoids whose morphisms are induced by spans.

For $F, G \in \mathbf{LIN}(\mathrm{Grpd}_{/X_1}, \mathrm{Grpd})$, suppose F is induced by the span $X_1 \leftarrow M \rightarrow 1$ and G is induced by $X_1 \leftarrow N \rightarrow 1$. Define their convolution by the span

$$X_1 \leftarrow M * N \rightarrow 1,$$

which is defined as the composition of spans:



The neutral functor for convolution is $\varepsilon : \mathrm{Grpd}_{/X_1} \rightarrow \mathrm{Grpd}$. the same ϵ function in the incidence coalgebra.

$(\mathbf{LIN}(\mathrm{Grpd}_{/X_1}, \mathrm{Grpd}), *, \epsilon)$ is called the convolution algebra. □

Fact 3.3. $\mathrm{Grpd}_{/B} \simeq \mathrm{Grpd}^B$. In particular, the convolution algebra

$$\mathbf{LIN}(\mathrm{Grpd}_{/X_1}, \mathrm{Grpd}) \simeq \mathrm{Grpd}^{X_1}.$$

□

4 Cardinality

Taking cardinality turns the objective-level theory into the classical one.

Definition 4.1. If X is a finite groupoid, define the cardinality of X be $|X| := \sum_{[x]\in\pi_0 X} \frac{1}{|\operatorname{Aut}(x)|} \in \mathbb{Q}$.

Fact 4.2. Let A, B, G be locally finite groupoids and let $A \leftarrow G \rightarrow B$ be a finite span. Then the induced functor $\mathrm{Grpd}_{/A} \rightarrow \mathrm{Grpd}_{/B}$ restricts to $\mathrm{grpd}_{/A} \rightarrow \mathrm{grpd}_{/B}$.

Construction 4.3. Let **lin** be the category of slice categories $\mathrm{grpd}_{/A}$, with A locally finite. There is a functor

$$\| \, \| : \mathbf{lin} \rightarrow \mathrm{Vect}$$

$$\begin{array}{ccccc} \mathrm{grpd}_{/A} & & \mathbb{Q}_{\pi_0 A} & & \delta_a \\ \downarrow & \longmapsto & \downarrow & & \big\downarrow \\ \mathrm{grpd}_{/B} & & \mathbb{Q}_{\pi_0 B} & & \sum_{[b]\in\pi_0 B} |B_{[b]}| \, |G_{a,b}| \, \delta_b \end{array}$$

Here

- $\mathbb{Q}_{\pi_0 A} := \mathbb{Q}[\pi_0 A]$ is the vector space with basis $\{\delta_a\}_{[a]\in\pi_0 A}$
- $B_{[b]}$ is the full subgroupoid of B on objects isomorphic to b
- $G_{a,b}$ is the fiber of $G \rightarrow A \times B$ over (a, b)

□

Construction 4.4. Dually there is a functor

$$\| \, \| : \mathrm{grpd}^B \rightarrow \mathrm{grpd}^A$$

$$\begin{array}{ccccc} \mathrm{grpd}^{B} & & \mathbb{Q}^{\pi_0 B} & & \delta^b \\ \downarrow & \longmapsto & \downarrow & & \big\downarrow \\ \mathrm{grpd}^{A} & & \mathbb{Q}^{\pi_0 A} & & \sum_{[a]\in\pi_0 A} |B_{[b]}| \, |G_{a,b}| \, \delta^a \end{array}$$

Here $\mathbb{Q}^{\pi_0 B}$ is the function space, i.e. the profinite-dimensional vector space with pro-basis given by the characteristic functions δ^b. □

Remark 4.5. One can think of

$$\mathbb{Q}^{\pi_0 B} = \varprojlim_{J\subset\pi_0 B,\ J \text{ finite}} \mathbb{Q}^J,$$

each $\mathbb{Q}^J$ finite dimensional. So a vector in $\mathbb{Q}^{\pi_0 B}$ can be represented by combination of probasis with infinite terms of nonzero coefficients. □

5 Möbius inversion at the objective level

5.1 Categorified zeta function

We want to define $\zeta : \mathrm{Grpd}_{/X_1} \to \mathrm{Grpd}$. Suppose ζ is represented by a span $X_1 \xleftarrow{f} G \to 1$. Classically, ζ is the constant function 1, we hope after taking cardinality, ζ becomes

$$\mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}, \qquad \delta_x \longmapsto 1.$$

Now let's find what ζ should be. Suppose ζ is given by $X_1 \xleftarrow{f} G \to 1$.

$$|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q},$$

$$\delta_x \longmapsto \sum_{[b]\in\pi_0 1} |1_{[b]}|\,|G_{x,b}|\,\delta_b = |G_{x,*}|\,\delta_*.$$

Remark 5.1. $1_{[*]}$ is the full subcategory of 1 consisting of objects isomorphisc to $*$. So $|1_{[*]}| = 1$ □

Hence we need to construct G and f, such that $|G_{x,*}| = 1, \qquad \text{for all } [x] \in \pi_0 X_1.$

If you thought of $G \to X_1 \times 1$ as morphism between sets and $|G_{x,*}|$ be the cardinality of the preimage set of $(x,*)$, the cheapest map to realize this is $X_1 \xleftarrow{\mathrm{id}} X_1 \to 1$. We now check $X_1 \xleftarrow{\mathrm{id}} X_1 \to 1$ will meet our requirement.

$G_{x,*}$ is the pullback

$$\begin{array}{ccc} G_{x,*} & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[(x,*)]{} & X_1 \times 1. \end{array}$$

For two objects $(*, y, \phi : x \to y), (*, y', \phi' : x \to y') \in G_{x,*}$, there is a morphism

$$(\mathrm{id}, \phi'\phi^{-1}) : (*, y, \phi) \to (*, y', \phi')$$

between them, so $\pi_0 G_{x,*}$ is a singleton.

Remark 5.2. $(\mathrm{id}, \phi'\phi^{-1})$ is a morphism, since the following diagram commutes.

$$\begin{array}{ccc} (x,*) & \xrightarrow{\mathrm{id}} & (x,*) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ (y,*) & \xrightarrow[\phi'\phi^{-1}]{} & (y',*) \end{array}$$

□

Let $(*, y, \phi) \xrightarrow{(\mathrm{id},g)} (*, y, \phi)$ be an automorphism, then the above commutative diagram forces $g = \mathrm{id}_y$. Therefore $|\operatorname{Aut}(*, y, \phi)| = 1$. Hence $|G_{x,*}| = \frac{1}{|Aut(*,y,\phi)|} = 1$, which is precisely the desired result.

So the objective-level zeta functor is represented by $X_1 \xleftarrow{\mathrm{id}} X_1 \to 1$.

5.2 Inverse of ζ at the objective level

Idea and problem: In the classical case, we construct a family of $\{\phi_n\}$ and let

$$\mu = \sum_{n \text{ even}} \phi_n - \sum_{n \text{ odd}} \phi_n.$$

Then can check $\zeta * \mu = \epsilon$. Note that in our case, ϕ_{odd} lies in $\mathrm{Grpd}^{/X_1}$, where the "$-\phi_{\text{odd}}$" does not make sense. So the right formula suggests that the Möbius inversion principle should be

$$\zeta * \sum_{n \text{ even}} \phi_n = \varepsilon + \zeta * \sum_{n \text{ odd}} \phi_n,$$

where the negative sign disappears.

Before constructing Φ_n, we first introduce completeness, which is necessary for defining incomplement of groupoids.

5.3 Completeness

Definition 5.3. A decomposition space X is *complete* if $s_0 : X_0 \to X_1$ is monomorphism, i.e., each homotopy fiber of s_0 is either empty or contractible. □

Example 5.4. Let G be a nontrivial group, and let X be the simplicial groupoid with $X_n = (BG)^n$. Then $s_0 : 1 \to BG$ is not monomorphism, so X is not complete.

Indeed, there is only one object in BG and the only fiber groupoid is

$$\begin{cases} \text{objects:} & (*, *, g \in G) \\ \text{morphisms:} & \mathrm{Hom}((*, *, g_1), (*, *, g_2)) = \begin{cases} (\mathrm{id}, \mathrm{id}) & g_1 = g_2 \\ \emptyset & \text{o/w} \end{cases} \end{cases}$$

This fiber groupoid is discrete but not connected, so it's not contractible. Hence, s_0 is not a monomorphism.

Another argument: Suppose on the contrary that s_0 is a monomorphism, then s_0 would be fully faithful, so $s_0(1)$ would be the full subgroupoid of BG. That would force $\mathrm{id} = \mathrm{Hom}_1(*, *) \simeq \mathrm{Hom}_{BG}(*, *) = G$, impossible when G is nontrivial.

□

Motivation for completeness

We want to define nondegenerate simplices for a simplicial groupoid X.

For simplicial sets, the set of nondegenerate simplices is the complement of the image of degeneracy maps. For simplicial groupoids, one would like the complement of the image *groupoid.* But in general that complement need not itself be a groupoid.

Example 5.5. Let $\mathcal{C}$ be a category with one object and only one nontrivial morphism e with $e^2 = \mathrm{id}$. Consider $1 \to \mathcal{C}$. The complement is a morphism e, does not contain any objects. So the complement is not a groupoid. □

So to make the complement of the image meaningful, we want the degeneracy map to be fully faithful, i.e. a monomorphism.

Fact 5.6. In a decomposition space, a simplex is nondegenerate if and only if all its principal edges are nondegenerate. □

Example 5.7. (Principal edges) In a 2-simplex the principal edges are x_{01} and x_{12}, not x_{02}. In a 3-simplex the principal edges are x_{01}, x_{12}, x_{23}.

Hence, it suffices to make nondegenerate 1-simplices make sense, i.e. it suffices to make the complement of $s_0 : X_0 \to X_1$ makes sense. This illustrates the condition of completeness.

Constructing Φ_n

Classical situation

$$\Phi_0 = \varepsilon, \qquad \Phi_1 = \zeta - \varepsilon, \qquad \Phi_n = \Phi_{n-1} * (\zeta - \varepsilon).$$

Then $\Phi_n = (\zeta - \varepsilon)^n = \Phi_1^n$.

Objective level

Define $\vec{X}_n \subset X_n$ to be the full subgroupoid of nondegenerate n-simplices. ϵ is induced by span Φ_0 is induced by $X_1 \xleftarrow{s_0} X_0 \to 1$, and ζ is induced by span $X_1 \xleftarrow{\mathrm{id}} X_1 \to 1$. Hence it's reasonable to define $\Phi_1 = \zeta - \epsilon$ induced by $X_1 \leftarrow \vec{X}_1 \to 1$. More generally, define Φ_n induced by span $X_1 \leftarrow \vec{X}_n \to 1$ where $\vec{X}_0 = X_0$ and $\vec{X}_n \to X_1$ is the restriction of the canonical morphism $X_n \to X_1$.

The next proposition illustrate our Φ_n have same behavior as classical case.

Remark 5.8. There is a canonical map $X_n \to X_1$ obtained by composing all arrows:

$$X_n \xrightarrow{d_{i_n}} X_{n-1} \xrightarrow{d_{i_{n-1}}} X_{n-2} \to \cdots \to X_2 \xrightarrow{d_{i_2}} X_1,$$

where each d_{i_j} is an inner face map (since we only consider composition of arrows), so $1 \leq i_j \leq j-1$.
Claim: $d_{i_2} d_{i_3} \cdots d_{i_n} = d_1 d_1 \cdots d_1$.

Indeed, one repeatedly uses

$$d_{j-1} d_i = d_i d_j$$

to move d_j leftward until its index drops to 1. This can always be achieved. There are $j-2$ maps preceding d_{i_j}. Since $1 \leq i_j \leq j-1$, the maximum value i_j can take is $j-1$. By commuting d_{i_j} to the left (moving it forward), one can ultimately shift it past $j-2$ maps to reduce the index by $j-2$; that is, $d_{j-1-(j-2)} = d_1$. For example,

$$X_5 \to X_1, \qquad d_2 d_1 d_3 d_4 = d_1 d_2 d_1 d_3 = d_1 d_1 d_2 d_1 = d_1 d_1 d_1 d_1.$$

□

Property 5.9. $\Phi_n = (\Phi_1)^n$

Proof sketch. We prove by induction. Assume $\Phi_{n-1} = (\Phi_1)^{n-1}$. Then $(\Phi_1)^n = \Phi_{n-1} * \Phi_1.$, which is given by

$$\begin{array}{ccccc} X_1 & & & & \\ \uparrow & & & & \\ X_2 & \longleftarrow & ? & & \\ \downarrow & & \downarrow & & \\ X_1 \times X_1 & \longleftarrow & \vec{X}_{n-1} \times \vec{X}_1 & \longrightarrow & 1 \end{array}$$

Using [**Lemma 3.5,** [3]] one can show $? = X_{a \cdots a} = \vec{X}_n$, which shows $(\Phi_1)^n = \Phi_n$.

6 Sign-free Möbius inversion for complete decomposition spaces

Lemma 6.1. The square

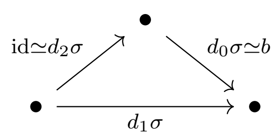
$$\begin{array}{ccc} \vec{X}_1 + \vec{X}_2 & \longrightarrow & X_2 \\ \downarrow & \lrcorner & \downarrow {\scriptstyle (d_2, d_0)} \\ X_1 \times \vec{X}_1 & \longrightarrow & X_1 \times X_1 \end{array}$$

is a pullback.

Proof. (While not strictly formal, this provides an intuitive sense of the underlying mechanism.) An object in the pullback is a tuple (a, b, σ, ϕ) with

$$(a, b) \in X_1 \times \vec{X}_1, \quad , \sigma \in X_2, \text{ and } (a, b) \xrightarrow{\phi} (d_2\sigma, d_0\sigma).$$

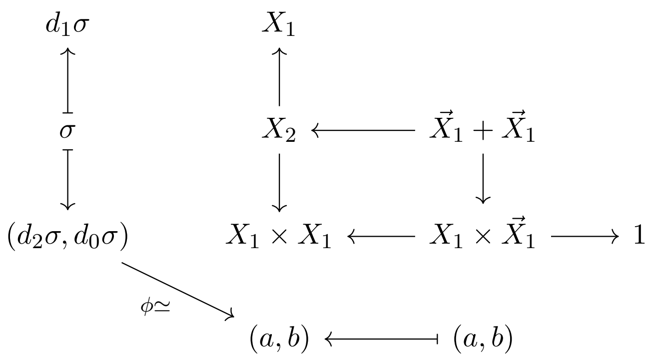
Case 1. If a is degenerate, then σ is determined by $b \in \vec{X}_1$ (up to equivalence)



in this case $d_1\sigma \simeq b$. So (a, b, σ, ϕ) can be encoded in $b \in \vec{X}_1$ (up to equivalence)

Case 2. If a is nondegenerate, then (a, b, σ, ϕ) is encoded by a nondegenerate 2-simplex $\sigma \in \vec{X}_2$, whose principal edges are $d_2\sigma \simeq a$, $d_0\sigma \simeq b$., both are nondegenerate. For a complete decomposition space, nondegeneracy of the principal edges implies nondegeneracy of the whole simplex. Hence, $\sigma \in \vec{X}_2$. And in this case, $d_1\sigma \simeq b \circ a$. □

Therefore,



The above diagram shows

Corollary 6.2. $\zeta * \Phi_1 \simeq \Phi_1 + \Phi_2$. Higher-dimensional analogues gives $\zeta * \Phi_n \simeq \Phi_n + \Phi_{n+1}$. □

Remark 6.3. For a rigorous version, there is a proof using "word notation". See **Section 2.8**, **Lemma 2.12**, **Proposition 2.13**, **Proposition 3.7** in [3]. □

Theorem. For a complete decomposition space,

$$\zeta * \Phi_{\text{even}} = \varepsilon + \zeta * \Phi_{\text{odd}}, \qquad \Phi_{\text{even}} * \zeta = \varepsilon + \Phi_{\text{odd}} * \zeta,$$

where

$$\Phi_{\text{even}} := \sum_{k \geq 0} \Phi_{2k}, \qquad \Phi_{\text{odd}} := \sum_{k \geq 0} \Phi_{2k+1}.$$

Proof.

$$\zeta * \Phi_{\text{even}} = \zeta * \sum_{k\geq 0} \Phi_{2k} = \sum_{k\geq 0} \zeta * \Phi_{2k} = \sum_{k\geq 0}(\Phi_{2k} + \Phi_{2k+1}) = \sum_{k\geq 0} \Phi_k,$$

$$\zeta * \Phi_{\text{odd}} + \varepsilon = \zeta * (\sum_{k\geq 0} \Phi_{2k+1}) + \Phi_0 = \sum_{k\geq 0}(\Phi_{2k+1} + \Phi_{2k+2}) + \Phi_0 = \sum_{k\geq 0} \Phi_k.$$

Hence, $\zeta * \Phi_{\text{even}} = \zeta * \Phi_{\text{odd}} + \varepsilon$. Similarly, $\Phi_{\text{even}} * \zeta = \Phi_{\text{odd}} * \zeta + \varepsilon$.

The Möbius inversion will turn to classical one after taking cardinality.

Fact 6.4. If X is a Möbius decomposition space, then $|\zeta| * |\mu| = |\varepsilon| = |\mu| * |\zeta|$,, where $|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|$. □

7 Example for $\mathbf{N} := N((\mathbb{N}, +))$

$(\mathbb{N}, +)$ is a monoid and $B(\mathbb{N}, +)$ is the category with one object with hom set be $(\mathbb{N}, +)$.

Let $\mathbf{N}$ be the nerve of the monoid $(\mathbb{N}, +)$.

7.1 Möbius function and its cardinality

The objective-level Φ_r. Φ_r is represented by the span $\mathbf{N}_1 \leftarrow \vec{\mathbf{N}}_r \rightarrow 1$. Here $\mathbf{N}_r = N(B\mathbb{N}) = \{(n_1, \dots, n_r) \mid n_i \in \mathbb{N}\}$, and the nondegenerate part $\vec{\mathbf{N}}_r = \{(n_1, \dots, n_r) \mid n_i \in \mathbb{N} \setminus \{0\}\} = (\mathbb{N} \setminus \{0\})^r$.

The fiber of

$$\vec{\mathbf{N}}_r \to \mathbf{N}_1 = \mathbb{N}, \qquad (n_1, \dots, n_r) \longmapsto n_1 + \cdots + n_r$$

over $n \in \mathbb{N}$ is

$$(\vec{\mathbf{N}}_r)_n = \{(n_1, \dots, n_r) \in (\mathbb{N} \setminus \{0\})^r \mid n_1 + \cdots + n_r = n\} = \{n \twoheadrightarrow r\},$$

the set of r-partitions of n.

Therefore, $\Phi_r(\ulcorner n \urcorner) = \{n \twoheadrightarrow r\}$.

Hence, $\Phi_{\text{even}}(n) = \{n \twoheadrightarrow r \mid r \text{ even}\}$, $\qquad \Phi_{\text{odd}}(n) = \{n \twoheadrightarrow r \mid r \text{ odd}\}$.

With an abusive sign notation one would like to write

$$\mu(n) = \sum_{r\geq 0}(-1)^r \{n \twoheadrightarrow r\}.$$

Remark 7.1. Note that here we use "negative sets", which does not make sense. A discussion of negative sets can be found in [6].

Cardinality computation.

$$\big|\{n \twoheadrightarrow r\}\big| = \binom{n-1}{r-1}.$$

Remark 7.2. Since each cell must contain at least one element, the problem reduces to choosing $r - 1$ positions for the bars from the $n - 1$ available gaps between elements. □

Thus $|\mu(n)| = \sum_{r\geq 0}(-1)^r \binom{n-1}{r-1}$. Writing $m := n - 1, \qquad s := r - 1$, this becomes $|\mu(n)| = \sum_{s\geq -1}(-1)^{s+1}\binom{m}{s}$.

- For $n = 1$ one gets $\mu(1) = (-1)^0\binom{0}{-1} + (-1)^1\binom{0}{0} = -1$,

- For $n=0$ one gets $|\mu(0)| = \underbrace{(-1)^0\binom{-1}{-1}}_{1} + \underbrace{\binom{-1}{0} + \binom{-1}{1} + \ldots}_{0} = 1$

- For $n>1$, $|\mu(n)| = \sum_{s=0}^{m}(-1)^{s+1}\binom{m}{s} = -(x-1)^m\big|_{x=1} = 0$.

Hence $|\mu(n)| = \begin{cases} 1, & n=0, \\ -1, & n=1, \\ 0, & \text{otherwise.} \end{cases}$

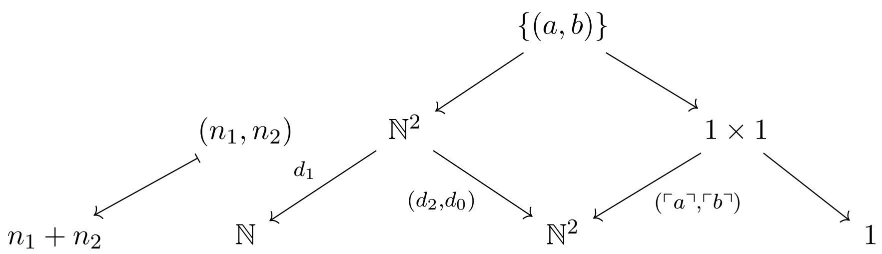
So, in $\mathbb{Q}^{\pi_0\mathbb{N}} \cong \mathbb{Q}^{\mathbb{N}}$, $\mu = \sum |\mu(n)|\delta^n = \delta^0 - \delta^1$.

Another way to compute $\mu = \delta^0 - \delta^1$

Property 7.3. The incidence algebra $(\mathrm{grpd}^{\mathbb{N}}, *, \varepsilon)$ has basis $\{h^n\}_{n\in\mathbb{N}}$ with convolution $h^a * h^b = h^{a+b}$.

Proof. **Fact:**grpd^S is the full subcategory of $Grpd^s \simeq \mathbf{LIN}(\mathrm{Grpd}_{/S}, \mathrm{Grpd})$. So an object in grpd^S is a linear functor $\mathrm{Grpd}_{/S} \to \mathrm{Grpd}$. Since functor is linear, it's induced by span $S \leftarrow G \to 1$. The basis $h^s \in \mathrm{grpd}^S$ is induced by $S \xleftarrow{\ulcorner s \urcorner} 1 \to 1$.

By the following diagram, one can show $h^a * h^b$ is given by $\mathbb{N} \leftarrow \{(a,b)\} \to 1$, $a+b \leftarrow\!\dashv (a,b)$, which is $\mathbb{N} \xleftarrow{\ulcorner a+b \urcorner} 1 \to 1$. So $h^a * h^b = h^{a+b}$.



The cardinality of the incidence algebra is $\mathbb{Q}^{\pi_0\mathbb{N}} = \mathbb{Q}^{\mathbb{N}}$.

Property 7.4.
$$\mathbb{Q}^{\mathbb{N}} \cong \mathbb{Q}[[z]], \qquad \delta^n = |h^n| \longleftrightarrow z^n.$$

Proof.
$$f : \mathbb{N} \to \mathbb{Q} \qquad \longleftrightarrow \qquad \sum_{n\geq 0} f(n) z^n.$$

Here $f : \mathbb{N} \to \mathbb{Q}$ represents a vector in $\mathbb{Q}^{\mathbb{N}}$ whose coefficients of δ^n is $f(n)$. $|h^a * h^b| = |h^{a+b}|$, i.e., $\delta^a * \delta^b = \delta^{a+b}$

corresponds to $z^a \cdot z^b = z^{a+b}$.

In $\mathbb{Q}[[z]]$, $\zeta = \sum_{n\geq 0} z^n = \frac{1}{1-z}$, since $f(n) \equiv 1$. so its inverse is $\mu = 1 - z = 1\cdot z^0 + (-1)\cdot z^1$, corresponding to

$$\mu = 1\cdot\delta^0 + (-1)\cdot\delta^1 = \delta^0 - \delta^1 \text{ in } \mathbb{Q}^{\mathbb{N}}$$

7.2 Cancellation of $\Phi_{odd}(n)$ and $\Phi_{even}(n)$ for $n \geqslant 2$

At the objective level, the vanishing for $n \geq 2$ comes from an actual cancellation of groupoids. There is a bijection between odd and even decompositions.

We denote an r-partition of n elements in $\Phi_r(n)$ as $(x_1, \ldots, x_r)$ for $x_i \in \mathbb{N}^*$ and $\sum_i x_i = n$.

For $n \geq 2$, there is a bijection

$$\Phi_{\text{odd}}(n) \longrightarrow \Phi_{\text{even}}(n)$$

by

$$(x_1, \ldots, x_r) \longmapsto \begin{cases} (1, x_1 - 1, x_2, \ldots, x_r), & \text{if } x_1 > 1, \\ (x_2 + 1, x_3, \ldots, x_r), & \text{if } x_1 = 1. \end{cases}$$

Remark 7.5. This morphism is only well-defined for $n \geq 2$, since $\Phi_{\text{even}}(1) = \varnothing$.. $\square$

Therefore, $|\Phi_{\text{even}}(n)| = |\Phi_{\text{odd}}(n)|, \qquad n \geq 2$ This would illustrate why $\mu(n) = 0$ for $n \geq 2$ from the objective level.

Check the Möbius principle

Goal: check $\zeta * \Phi_{\text{even}} = \zeta * \Phi_{\text{odd}} + \varepsilon$.

The counit ε is represented by $\mathbf{N}_1 \xleftarrow{s_0} \mathbf{N}_0 \to 1$, $0 \leftarrow\!\shortmid *$, since $\text{id}_* = 0 \in \mathbb{N}$ is the identity under composition in $B\mathbb{N}$.

This induces $\epsilon : \text{Grpd}_{/\mathbf{N}_1} \longrightarrow \text{Grpd}_{/\mathbf{N}_0} \longrightarrow \text{Grpd}_{/1} \simeq \text{Grpd}$

$$\begin{array}{ccccccccc} & & & & ? & \longrightarrow & 1 & & \\ & & & & \big\downarrow & & \big\downarrow{\scriptstyle \ulcorner n \urcorner} & & \\ 1 \xrightarrow{\ulcorner n \urcorner} \mathbb{N} & \longmapsto & & & * & \longrightarrow & \mathbb{N} & \longmapsto & ? = \begin{cases} \emptyset \text{ ,when } n \neq 0 \\ * \text{ ,when } n = 0 \end{cases} \end{array}$$

Hence, $\varepsilon(\ulcorner n \urcorner) = \begin{cases} \varnothing, & n \neq 0, \\ 1, & n = 0. \end{cases}$

For $n \geq 2$, since $\Phi_{\text{even}}(n) \simeq \Phi_{\text{odd}}(n)$ and $\varepsilon(\ulcorner n \urcorner) = \varnothing$, the identity

$$\zeta * \Phi_{\text{even}} = \zeta * \Phi_{\text{odd}} + \varepsilon$$

holds automatically. Then it remains to check the cases $n \leq 1$.

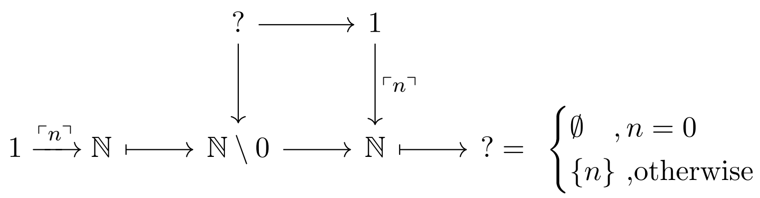
For $n \leq 1$ one has $\Phi_{\text{even}}(n) = \Phi_0(n) = \epsilon(n), \qquad \Phi_{\text{odd}}(n) = \Phi_1(n)$. Therefore it suffices to show $\zeta * \epsilon = \zeta * \Phi_1 + \varepsilon$.

Recall:

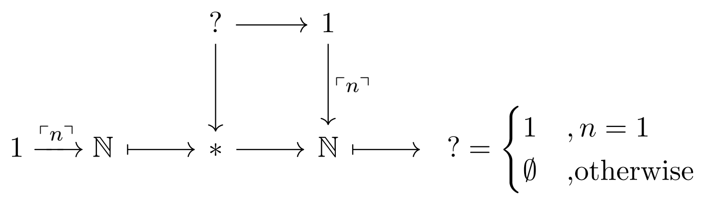
- Φ_r is given by $\mathbb{N} \leftarrow (\mathbb{N} \setminus \{0\})^r \to 1, \qquad x_1 + \cdots + x_r \leftarrow\!\shortmid (x_1, \ldots, x_r)$,
- ϵ is given by $\mathbb{N} \leftarrow * \to 1, \qquad 0 \leftarrow\!\shortmid *$,
- Φ_1 is given by $\mathbb{N} \leftarrow \mathbb{N} \setminus \{0\} \to 1, \qquad n \leftarrow\!\shortmid n$,
- ζ is given by $\mathbb{N} \xleftarrow{\text{id}} \mathbb{N} \to 1$.

Property 7.6. For $n \leq 1$, $\Phi_1(\ulcorner n \urcorner)$ coincides with $\delta^1(\ulcorner n \urcorner)$ of the element 1

Proof. $\Phi_1 : \mathrm{Grpd}_{/\mathbb{N}} \to \mathrm{Grpd}$ is computed by



$\delta^1 : \mathrm{Grpd}_{/\mathbb{N}} \to \mathrm{Grpd}$ is computed by:

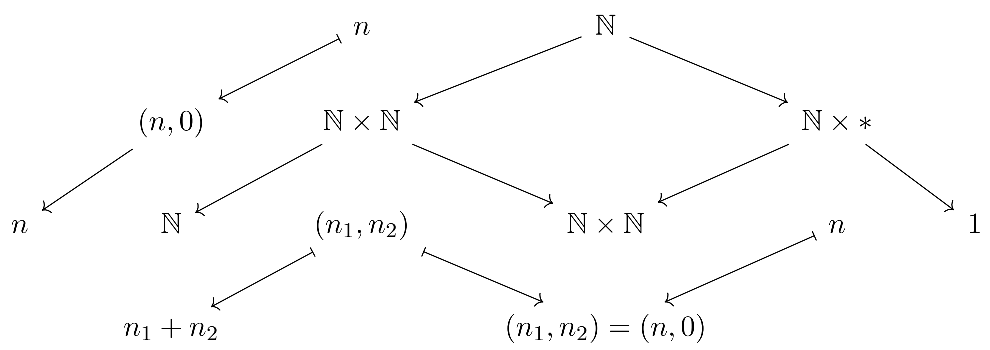


Hence, $\Phi_1(\ulcorner n \urcorner) = \delta^1(\ulcorner n \urcorner)$ for $n \leq 1$. □

So it suffices to check $\zeta * \epsilon = \zeta * \delta^1 + \epsilon$ for $n \leq 1$.

Property 7.7. $\zeta * \epsilon$ is given by $\mathbb{N} \xleftarrow{\mathrm{id}} \mathbb{N} \to 1$.

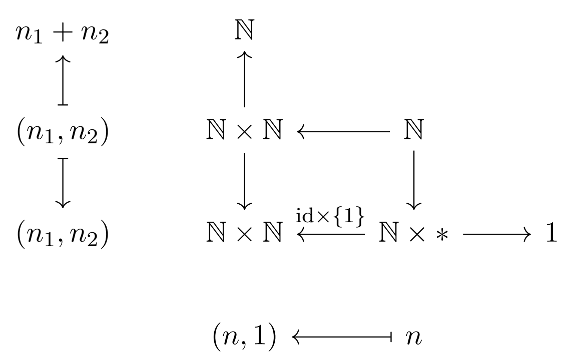
Proof.



□

Property 7.8. $\zeta * \delta^1$ is given by $\mathbb{N} \leftarrow \mathbb{N} \to 1, \qquad n+1 \leftarrow\!\shortmid n,$

Proof.



□

Corollary 7.9. $(\zeta * \delta^1)(\ulcorner n \urcorner) = \begin{cases} \{n-1\}, & n \geq 1, \\ \varnothing, & n = 0. \end{cases}$

Proof. $\zeta * \delta^1 : \mathrm{Grpd}_{/\mathbf{N}} \to \mathrm{Grpd}$ computed by

$$
\begin{array}{ccccccccc}
 & & & & ? & \longrightarrow & 1 & & \\
 & & & & \big\downarrow & & \big\downarrow{\scriptstyle \ulcorner n \urcorner} & & \\
1 & \xrightarrow{\ulcorner n \urcorner} & \mathbb{N} & \longmapsto & \mathbb{N} & \longrightarrow & \mathbb{N} & \longmapsto & \begin{cases} \{n-1\} & , n \geq 0 \\ \emptyset & , n = 0 \end{cases}
\end{array}
$$

$$x \longmapsto x + 1$$

□

Thus:

- For $n = 1:$ $\quad (\zeta * \epsilon)(\ulcorner 1 \urcorner) = 1, \quad (\zeta * \delta^1)(\ulcorner 1 \urcorner) = \{n-1\} \simeq 1, \quad \varepsilon(\ulcorner 1 \urcorner) = \varnothing$
- For $n = 0:$ $\quad (\zeta * \epsilon)(\ulcorner 0 \urcorner) = 1, \quad (\zeta * \delta^1)(\ulcorner 0 \urcorner) = \varnothing, \quad \varepsilon(\ulcorner 0 \urcorner) = 1.$

Hence, for $n \leq 1$,

$$\zeta * \Phi_0 = \zeta * \delta^1 + \varepsilon.$$

Combining the cases $n \geq 2$ and $n \leq 1$, we obtain the objective-level Möbius principle for $\mathbf{N}$.

The fact that the Möbius function vanishes for $n \geq 2$ is not a coincidence of alternating sums; rather, at the objective level, the collections of odd and even decompositions for all $n \geq 2$ are themselves equivalent, which leads to their cancellation.

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