Lightlike developable surfaces in Minkowski space

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Overview

Motivations

Why study lightlike developable surfaces?

- Developable surfaces: locally isometric to a plane, a classical topic in differential geometry.
- Lightlike surfaces: fundamental in Lorentzian geometry and relativity.
- Lightlike developable surfaces: intersection of these two classes, rich geometric structure with physical interpretations.

Developable surfaces in Euclidean space

Definition

Developable surface

The **developable surface** is a ruled surface in 3-dimensional Euclidean space with zero Gaussian curvature. It can be described by a parametric representation:

$$F(s,t) = \gamma(s) + tw(s),$$

where $\gamma(s), w(s)$ are smooth curves.

Tangent developable surface

A **tangent developable surface** is a particular kind of developable surface that can be described by a parametric representation:

$$F(s,t) = \gamma(s) + t\gamma'(s).$$

Examples

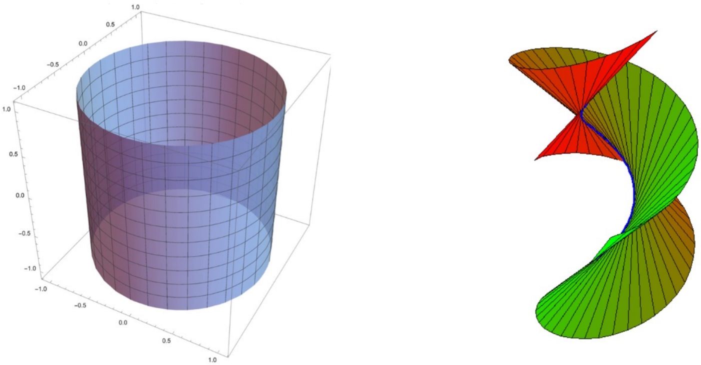


Figure: The cylinder is a developable surface; the helix (from Wikipedia) is a tangent developable surface.

Classification of developable surfaces

Theorem

A developable surface is one of the following:

- A part of a cylindrical surface.
- A part of a conical surface.
- A part of a tangent developable surface.
- A glue of the above three surfaces.

The types of curves

Finite type

Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve and denote by $\gamma(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. Consider the curve-germ γ at t_0, we say that γ at t_0 is a **finite type** if there exist natural numbers $a_1, a_2, \cdots, a_n$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ such that $x_i(t) = t^{a_i} + o(t^{a_i})$. We call that $\mathbb{A} = (a_1, a_2, \cdots, a_n)$ is **the type of** γ at t_0 and denote it as $\mathbb{A}(\gamma_{t_0})$.

Example

Let $\gamma : I \to \mathbb{R}^3$ be the twisted cubic curve $\gamma(t) = (t, t^2, t^3)$, then $\mathbb{A} = (1, 2, 3)$ is the type of γ at 0.

$\mathcal{A}$-equivalent

$\mathcal{A}$-equivalent

Two map germs $f : (M, x) \to (N, y), g : (M', x') \to (N', y')$ are $\mathcal{A}$-**equivalent** if there exist diffeomorphism germs $\phi : (M', x') \to (M, X), \psi : (N', y') \to (N, y)$ such that $f \circ \phi = \psi \circ g$.

Proposition

Let $\gamma : I \to \mathbb{R}^n, \tilde{\gamma} : I \to \mathbb{R}^n$ be smooth curves. If the type of γ at t_0 is the same as $\tilde{\gamma}$ at $\tilde{t}_0$, then the map germs $F_{(\gamma,\gamma')}(t_0, 0)$ and $F_{(\tilde{\gamma},\tilde{\gamma}')}(\tilde{t}_0, 0)$ are $\mathcal{A}$-equivalent, where $F_{(\gamma,\gamma')} : I \times \mathbb{R} \to \mathbb{R}^n, (s, t) \mapsto \gamma(s) + t\gamma'(s), F_{(\tilde{\gamma},\tilde{\gamma}')} : I \times \mathbb{R} \to \mathbb{R}^n, (s, t) \mapsto \tilde{\gamma}(s) + t\tilde{\gamma}'(s)$.

Classification of singularities

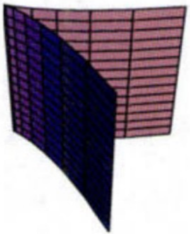
Theorem

Let $\gamma : I \to \mathbb{R}^3$ be a curve with type $\mathbb{A}$ at t_0. Then we have the following:

- If $\mathbb{A} = (1, 2, 3)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the cuspidal edge,
- If $\mathbb{A} = (1, 3, 5)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the Scherbak surface,
- If $\mathbb{A} = (2, 3, 4)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the swallowtail.

The three surfaces are parametrized as follows:
Cuspidal edge: $\{(x_1, x_2, x_3) \mid x_1 = u, x_2 = t^2 + ut, x_3 = \frac{2}{3}t^3 + \frac{1}{2}ut^2\} \times \mathbb{R}$,
Scherbak surface: $\{(x_1, x_2, x_3) \mid x_1 = u, x_2 = t^3 + ut^2, x_3 = 12t^5 + 10ut^4\}$,
Swallowtail: $\{(x_1, x_2, x_3, x_4) \mid x_1 = u, x_2 = t^3 + ut^2, x_3 = \frac{3}{4}t^4 + \frac{1}{2}ut^2, x_4 = \frac{3}{5}t^5 + \frac{1}{3}ut^3\}$.

Classification of singularities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Scherbak surface

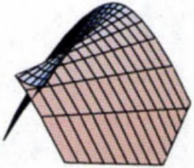
swallowtail

Figure: From “Lightlike developables in Minkowski 3-Space”, Chino, Izumiya, 2010

Lightlike developable surfaces in Minkowski space

Minkowski space

Minkowski $n+1$-space

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, i = 0, 1, \cdots, n\}$ be an $n+1$-dimensional Cartesian space. For any $x = (x_0, x_1, \cdots, x_n), y = (y_0, y_1, \cdots, y_n) \in \mathbb{R}^{n+1}$, the **pseudo-scalar product** of x and y is defined by

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i .$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ **Minkowski** $n+1$**-space**, usually written by $\mathbb{R}_1^{n+1}$ for simplicity. The norm of the vector $x \in \mathbb{R}_1^{n+1}$ is defined to be $\|x\| = \sqrt{|\langle x, x \rangle|}$

A non-zero vector $x \in \mathbb{R}_1^{n+1}$ is

- **spacelike** if $\langle x, x \rangle > 0$,
- **lightlike** if $\langle x, x \rangle = 0$,
- **timelike** if $\langle x, x \rangle < 0$.

Hypersurfaces in Minkowski $n+1$-space

Hyperplane

For a vector $v \in \mathbb{R}^{n+1}_1, c \in \mathbb{R}$, we define a **hyperplane with pseudo-normal** v by

$$HP(v,c) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, v\rangle = c\}.$$

We now define

- **Hyperbolic n-space** by $\mathbb{H}^n_{-1} = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = -1\}$,
- **De Sitter n-space** by $\mathbb{S}^n_1 = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = 1\}$,
- **Lightcone at the origin** by $LC^* = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}_1 \mid x_0 \neq 0, \langle x, x\rangle = 0\}$.

Lightlike curve and lightlike plane

Lightlike curves

A curve is a **lightlike curve** if γ' is lightlike.

Lightlike plane

A **lightlike plane** is a plane whose normal vector is lightilke.

Proposition

Let $\gamma : S^1 \to \mathbb{R}^3_1$ be a lightlike curve. Then there exists a point $t_0 \in S^1$ such that $\gamma'(t_0) = \mathbf{0}$.

Proposition

For a unit speed spacelike curve $\gamma : I \to \mathbb{R}^3_1$, if $\gamma''(s)$ is lightlike for any $s \in I$. then $\gamma(I)$ is a curve in a lightlike plane.

Lightlike developable surfaces in Minkowski 3-space

Lightlike developable surfaces

A **lightlike developable surface** in Minkowski space is a ruled surface with Gaussian curvature 0 that can be described by a parametric representation:

$$F(s,t) = \gamma(s) + tw(s).$$

such that γ is a lightlike curve.

Tangent surfaces of a curve

Let γ be a smooth curve. The **tangent surface of** γ is defined by $F(s,t) = \gamma(s) + t\gamma'(s)$. If γ is a lightlike curve, then $F(s,t)$ is called a **lightlike tangent surface**.

Classification of lightlike developable surfaces in Minkowski 3-space

The lightlike developable surface is a developable surface, so we can apply the classification theorem.

Theorem

A lightlike developable surface is one of the follwoing surfaces:

- A part of a lightlike plane,
- A part of the lightcone,
- A part of the tangent surface of a curve in a lightlike plane,
- A part of the tangent surface of a lightlike curve,
- A glue of the above four surfaces.

Lightlike tangent surface

Let $\gamma : I \to \mathbb{R}^3_1$ be a lightlike curve. If there exists an interval $J \subset I$ such that $\gamma'(t) = 0$, $\forall t \in J$, then γ is constant on J. Thus, we assume that γ has only isolated singular points. In this case, we denote that $\gamma(t) = (x_0(t), x_1(t), x_2(t))$ and $\gamma'(t) = (x_0'(t), x_1'(t), x_2'(t))$. Since $\gamma'(t)$ is lightlike, $x_0'(t) = 0$ if and only if $\gamma'(t) = 0$.
If $x_0'(t_0) = 0$, by Taylor's theorem, we have

$$x_0'(t) = a_0(t - t_0)^{r_0} + o(t^{r_0}), x_1'(t) = a_1(t - t_0)^{r_1} + o(t^{r_1}), x_2'(t) = a_2(t - t_0)^{r_2} + o(t^{r_2})$$

at any $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for sufficiently small $\epsilon > 0$ where $a_0, a_1, a_2 \neq 0$.
Since $x_0'(t)^2 = x_1'(t)^2 + x_2'(t)^2$, we have

$$a_0^2(t - t_0)^{2r_0} + o(t^{2r_0}) = a_1^2(t - t_0)^{2r_1} + o(t^{2r_1}) + a_2^2(t - t_0)^{2r_2} + o(t^{2r_2}),$$

so we have $r_0 = \min\{r_1, r_2\}$ and $\frac{x_1'(t)}{x_0'(t)}, \frac{x_2'(t)}{x_0'(t)}$ are smooth functions at t_0. This means that there exist smooth functions $r(t)$ and $\theta(t)$ such that

$$\gamma'(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)).$$

Lightlike tangent surface

If $x_0'(t) \neq 0$, then there exists a smooth function $\theta(t)$ such that

$$\frac{x_1'(t)}{x_0'(t)} = \cos\theta(t), \frac{x_2'(t)}{x_0'(t)} = \sin\theta(t).$$

Therefore, we can always assume that there exist smooth functions $r(t)$ and $\theta(t)$ such that

$$\gamma'(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)).$$

We now consider the following set of curves in $\mathbb{R}^3_1$:
$L(I, \mathbb{R}^3_1) = \{\sigma(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)) \mid t \in I, r(t), \theta(t)$ are smooth functions such that $r(t)$ has isolated zero points$\}$.
For any $\gamma : I \to \mathbb{R}^3_1$ with $\gamma' \in L(I, \mathbb{R}^3_1)$, we have $\gamma'(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t))$.
In this case $\tilde{\gamma}'(t_0) = (1, \cos\theta(t_0), \sin\theta(t_0))$ determines the tangent surface of γ at t_0.
Therefore, we can define the tangent surface of γ by

$$F_{(\gamma,\tilde{\gamma}')}(t, u) = \gamma(t) + u\tilde{\gamma}'(t).$$

Classification of lightlike tangent surfaces in Minkowski 3-space

Theorem

Let $\gamma : I \to \mathbb{R}^3_1$ be s smooth curve such that $\gamma' \in L(I, \mathbb{R}^3_1)$. Then we have the followings:

- The tangent surface germ $F_{(\gamma,\tilde{\gamma}')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the cuspidal edge if $r(t_0) \neq 0$ and $\theta'(t_0) \neq 0$.
- The tangent surface germ $F_{(\gamma,\tilde{\gamma}')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the Scherbak surface if $r(t_0) \neq 0, \theta'(t_0) = 0, \theta''(t_0) \neq 0$.
- The tangent surface germ $F_{(\gamma,\tilde{\gamma}')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the swallowtail if $r(t_0) \neq 0, r'(t_0) \neq 0$ and $\theta'(t_0) \neq 0$.

Lightlike curves in de Sitter 3-space

Recall that de Sitter 3-space is defined by $\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = 1\}$.
A natural question is: why study lightlike curves in de Sitter 3-space rather than in hyperbolic 3-space or the lightcone?
In fact, for the hyperbolic space, there is no lightlike curve. For the lightcone, there is no lightlike curve except the straight lightlike curve. Research on the lightlike curve in the lightcone or in hyperbolic space will lose its significance. Therefore, we only study the lightlike curves and singularities for the tangent developable surface of a lightlike curve in de Sitter 3-space.

Proposition

Let $\gamma : I \to \mathbb{S}^3_1$ be a lightlike curve in de Sitter 3-space, then the lightlike tangent developable surface of $\gamma(t)$ is also in de Sitter 3-space.

Classification of lightlike tangent surfaces in de Sitter 3-space

Theorem

Let $\gamma : I \to \mathbb{S}^3_1$ be a lightlike curve with type $\mathbb{A}$ at t_0. Then we have the following:

- If $\mathbb{A} = (1,2,3,4)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the cuspidal edge,
- If $\mathbb{A} = (1,3,4,5)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the Mond surface,
- If $\mathbb{A} = (2,3,4,5)$, then the germ of the tangent surface $F_{(\gamma,\gamma')}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the swallowtail.

The parametrizations of three surfaces are as followed:
Cuspidal edge: $\{(x_1, x_2, x_3) \mid x_1 = u, x_2 = t^2 + ut, x_3 = \frac{2}{3}t^3 + \frac{1}{2}ut^2\} \times \mathbb{R}$,
Mond surface: $\{(x_1, x_2, x_3, x_4) \mid x_1 = u, x_2 = t^3 + ut^2, x_3 = \frac{3}{4}t^4 + \frac{2}{3}ut^3, x_4 = \frac{3}{5}t^5 + \frac{1}{2}ut^4\}$,
Swallowtail: $\{(x_1, x_2, x_3, x_4) \mid x_1 = u, x_2 = t^3 + ut^2, x_3 = \frac{3}{4}t^4 + \frac{1}{2}ut^2, x_4 = \frac{3}{5}t^5 + \frac{1}{3}ut^3\}$.

Classification of lightlike tangent surfaces in de Sitter 3-space

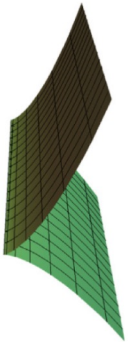

Fig. 1. Cuspidal edge.

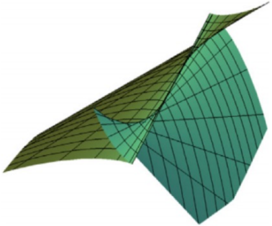

Fig. 2. Mond surface.

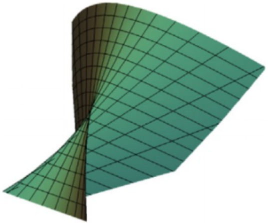

Fig. 3. Swallowtail.

Figure: From "Lightlike tangent developables in de Sitter 3-space", Li, Wang, 2021

Applications

Applications

- Event Horizons: The boundaries of black holes are essentially lightlike surfaces. They act as one-way membranes where lightlike geodesics fail to escape to infinity.
- Null Shells: They model the history of a flash of light or a shockwave of "null dust" traveling through space, helping physicists understand how gravity interacts with high-energy radiation.
- Characteristic Surfaces: In the theory of Partial Differential Equations (PDEs), they represent the characteristic surfaces of the wave equation. They are the paths along which physical information and "shocks" propagate.

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