On the geometric, topological and combinatorial structures of surfaces

Qingrui Qu

Southern University of Science and Technology

March 11, 2026

Table of Contents

Table of Contents

$[m] = \{1, 2, ..., m\}$ and define a category $\mathcal{C}([m])$ whose objects are pairs $(\underline{X}, \underline{A})$ where

$$(\underline{X}, \underline{A}) := \{(X_1, A_1), (X_2, A_2), \ldots (X_m, A_m)\}$$

is a family of CW-pairs with base points.

Let K be a simplicial complex whose vertex set is $[m]$ and we consider K to be a category where the objects are the simplices of K and the morphisms are the inclusions relation in K i.e. $\sigma \to \Sigma$ iff $\sigma \subset \Sigma$.

Polyhedral product functors

A *polyhedral product* is a functor $Z(K; -) : \mathcal{C}([m]) \to \mathsf{Top}$ satisfying

$$Z(K; (\underline{X}, \underline{A})) := \mathsf{colim}\ D \subset \prod_{i=1}^{m} X_i$$

where D is a diagram $D : K \to CW_*$, defined by the following rules:

$$D(\sigma) := \prod_{i=1}^{m} W_i \quad \text{where} \quad W_i := \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] \setminus \mathsf{Vertex}(\sigma) \end{cases}$$

for every simplices $\sigma \subset K$.
In other words, the colimit is a union given by

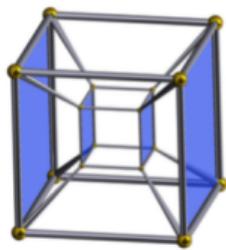$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma).$$

Example: Torus

A torus is shown as the polyhedral product $Z(K; (D^1, S^0)) \subset (D^1)^4$, where $K = C_4$ is the boundary complex of a square on the vertices $\{1, 2, 3, 4\}$. The empty simplex of K contributes $(S^0)^4$ which is identified with the 16 vertices of $(D^1)^4$, and the four vertices of K contribute the one-skeleton consisting of 32 edges. The maximal simplices of K are: $\sigma_1 = \{12\}, \sigma_2 = \{14\}, \sigma_3 = \{23\}, \sigma_4 = \{34\}$.
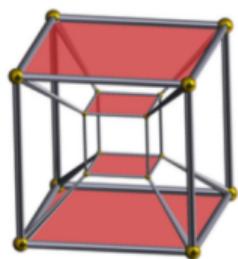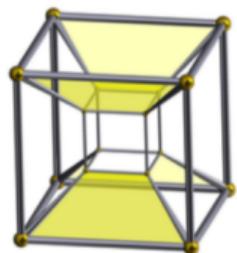
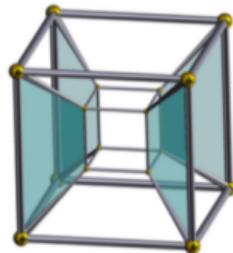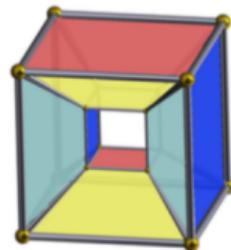$D(\sigma_1)$ $\qquad$ $D(\sigma_2)$ $\qquad$ $D(\sigma_3)$ $\qquad$ $D(\sigma_4)$ $\qquad$ the torus

Figure: Torus as the polyhedral product $Z(C_4, (D^1, S^0))$

Surface as real moment-angle complex

We study the geometry and topology of a class of real moment-angle complexes $X = Z(C_n, (D^1, S^0))$ given by polyhedral products, which is a surface, and its dual surface $X^* = Z(C_n, (D^1, S^0))^*$. The edges of the dual surface exactly form a set of shortest closed geodesics, and we investigate the homological properties of this set.

Theorem

The real moment-angle manifold $Z(C_n; (D^1, S^0))$ corresponding to an n-cycle graph C_n, for $n \geq 3$, is an oriented surface of genus $g = 1 + (n-4) \cdot 2^{n-3}$.

Theorem

The polyhedral product $Z(N_n; (D^1, S^0))$ corresponding to an n point null graph N_n is isomorphic to the hypercube graph Q_n. By the functoriality of the inclusion $N_n \hookrightarrow C_n$, we can obtain a regular embedding $Q_n \hookrightarrow X_n$.

Theorem (Q.)

For $n \geq 5$, the surface $X_n = Z(C_n, (D^1, S^0))$ can be smoothly embedded into the 3-dimensional Euclidean space $\mathbb{R}^3$. The isometry group of this embedded surface in $\mathbb{R}^3$ is isomorphic to the Klein four-group V_4.

The orientation-preserving rigid symmetry group of this configuration consists of four rotations:

- the identity;
- a $180°$ rotation about the z-axis;
- a $180°$ rotation about the axis $(1, -1, 0)$;
- and a $180°$ rotation about the axis $(1, 1, 0)$.

$n = 5, \ g = 5$
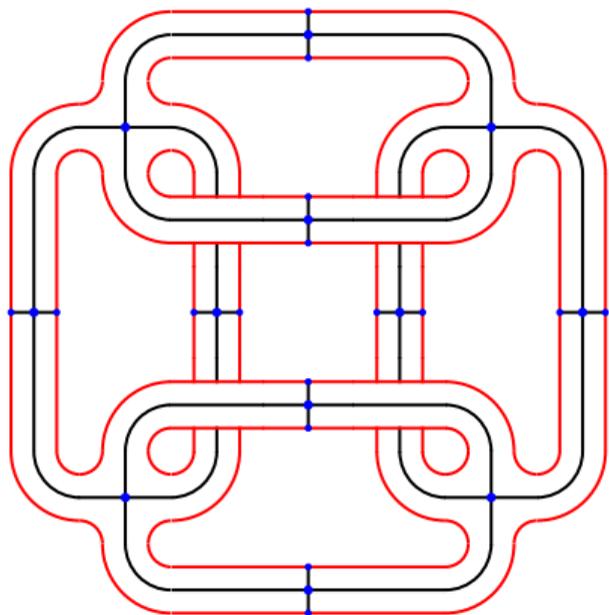
$n = 6, \ g = 17$
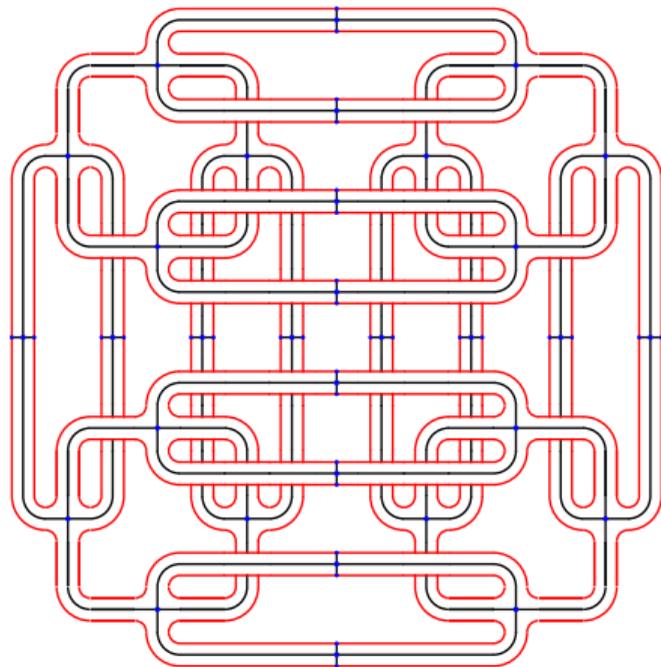
$n = 7, \ g = 49$
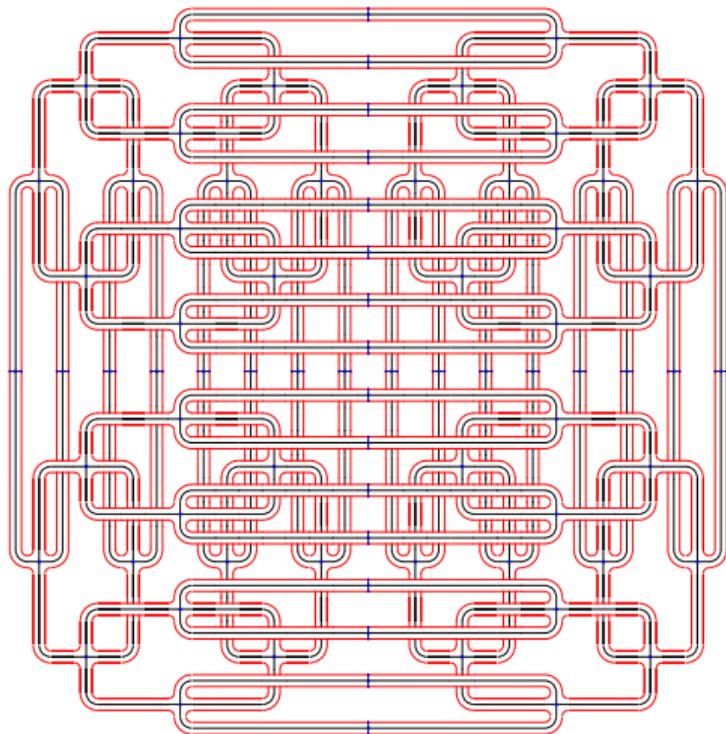
$n = 8, \ g = 129$
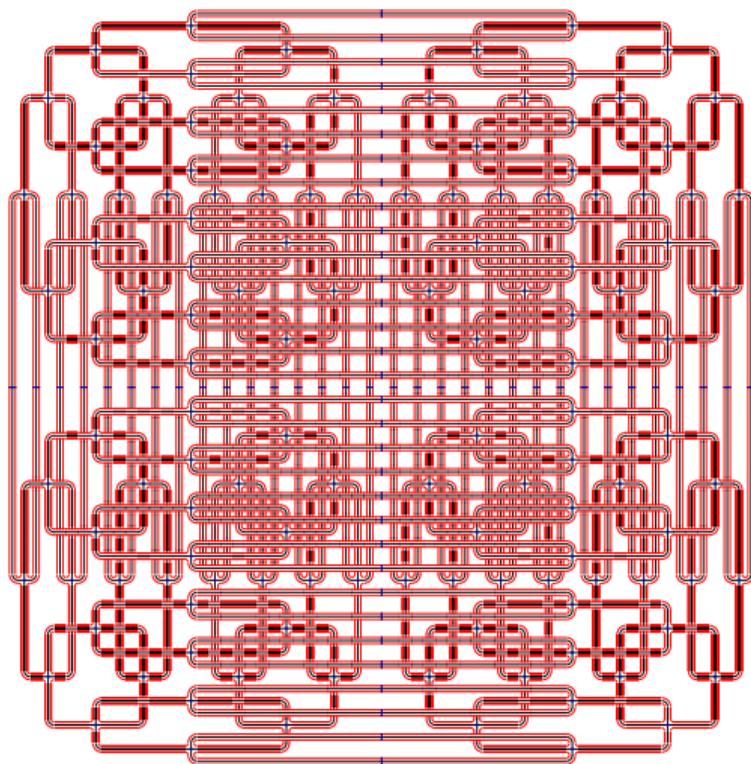
Consider the Hurewicz map of the real moment-angle complex

$$h : \pi_1(X) \to H_1(X).$$

Let S denote the systoles (the set of shortest non-contractible closed geodesics) of X. There exists a natural inclusion $\iota : S \hookrightarrow \mathrm{Conj}(\pi_1(X))$, which maps each systole to its corresponding conjugacy class. Due to $H_1(X)$ being abelian and through the universal property of free abelian groups, this morphism is unique to a homomorphism of the free abelian group $\tilde{S} := \mathbb{Z}^S$ generated by S. We can study the morphism

$$\tilde{h} : \tilde{S} \to H_1(X).$$

Basis of the kernel

Theorem (Q.)

A standard basis for the kernel of the Hurewicz map can be given, where each vector in this basis corresponds to an element of the set of nonempty independent sets of the cycle graph C_n.

The number of independent sets of a cycle graph can be obtained by standard combinatorial methods, we know this number is $L(n)$, i.e., the **Lucas numbers**, which follow the same recurrence $L(n) = L(n-1) + L(n-2)$ as the Fibonacci type numbers but with initial conditions $L(1) = 1$, $L(2) = 3$.

Corollary (Q.)

By the enumeration formula, we can determine the ranks of the free abelian groups:

$$\text{rank Ker } \tilde{h} = L(n) - 1,$$

$$\text{rank Im } \tilde{h} = n \cdot 2^{n-3} - L(n) + 1.$$

Combinatorial identity

Al-Raisi, using techniques of stable decomposition in polyhedral product theory, studied the action of the cyclic group $\mathbb{Z}/n$ on the cohomology groups of surfaces, thereby obtaining the following identity.

Theorem (Al-Raisi, 2014)

For $n \geq 4$, there is an identity

$$\frac{1}{n} \left(\sum_{d \mid n} \mu(d) 2^{\frac{n}{d}} \right) - (n-1) + \sum_{\substack{d \mid n \\ 1 < d < n}} \left(\frac{1}{d} \sum_{d_1 \mid d} \mu(d_1) 2^{\frac{d}{d_1}} \right) = \frac{1}{n} \left(\sum_{d \mid n} \varphi(d) 2^{\frac{n}{d}} \right) - (n+1)$$

relating the Möbius inversion function and the Euler totient function by equating the two different orbit counting methods.

Combinatorial identity

Similar to the previous theorem, by counting the number of orbits of the kernel under the action of a cyclic group using different methods, we obtain another identity relating the Möbius function and the Euler totient function.

$$0 \to \operatorname{Ker} \tilde{h} \to \tilde{S} \to H_1(X_n)$$

Theorem (Q.)

For $n \geq 4$, there is an identity

$$\frac{1}{n}\left(\sum_{d|n}\mu(d)L\left(\frac{n}{d}\right)\right) - (n-1) + \sum_{\substack{d|n \\ 1<d<n}}\left(\frac{1}{d}\sum_{d_1|d}\mu(d_1)L\left(\frac{d}{d_1}\right)\right) = \frac{1}{n}\left(\sum_{d|n}\varphi(d)L\left(\frac{n}{d}\right)\right) - n$$

relating the Möbius inversion function, the Euler totient function and the Lucas numbers by equating the two different orbit counting methods of the systoles.

Table of Contents

Regular embedding of hypercube graphs

- For every $n \geq 3$, Nedela and Škoviera (1997) constructed a covering map family of $\varphi(n)$ non-isomorphic orientably regular embeddings of Q_n. In that classification, each solution e of the congruence $e^2 \equiv 1 \pmod{n}$ yields a distinct regular map, and different solutions correspond to non-isomorphic embeddings. They conjectured that this lifting construction exhausts all orientably regular embeddings of Q_n.
- Du, Kwak, and Nedela (2007) proved that for odd n, this is indeed the case.
- Kwon (2004) discovered new families of orientably regular embeddings for all even $n \geq 6$ by applying a switch operator.
- Xu (2007) established that for $n \equiv 2 \pmod{4}$, Kwon's embeddings are the only additional ones beyond those from the covering construction.
- Further new embeddings were later constructed by Catalano and Nedela (2010), who provided a characterization of all orientably-regular embeddings of Q_n and introduced a construction for new regular embeddings of Q_n for all n divisible by 16, not covered by the family of embeddings previously found by Kwon.
- Catalano, Conder, Du, Kwon, Nedela, and Wilson (2011) provided a complete classification of orientably regular embeddings of the n-dimensional binary hypercubes Q_n.

Dart set:

Fix an integer $n \geq 2$. Define the Dart set by

$$D := \mathbb{F}_2^n \times \mathbb{Z}/n\mathbb{Z}.$$

An element of D is written (v, i) with $v \in \mathbb{F}_2^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{F}_2^n$. Indices are always understood modulo n and then reduced to $\{1, 2, \ldots, n\}$. Write the Hamming weight and its parity as

$$\|v\| := \sum_{j=1}^{n} v_j \in \{0, 1, \ldots, n\} \qquad \text{and} \qquad p(v) := \|v\| \bmod 2 \in \{0, 1\}.$$

Fix $e \in (\mathbb{Z}/n\mathbb{Z})^{\times}$ such that $e^2 \equiv 1 \pmod{n}$. Define maps $L, R : D \to D$ by

$$L(v, i) := (v + \mathbf{e}_i, \ i), \qquad R(v, i) := \big(v, \ i + e^{\|v\|}\big).$$

Each face has edge number

$$s(n, e) = \begin{cases} 2n, & e \equiv 1 \pmod{n}, \\ 4, & e \equiv -1 \pmod{n}, \\ \dfrac{4n}{\gcd(e+1, n)}, & 1 < e < n - 1. \end{cases}$$

Whenever $s(n, e)$ is even (it is, in all cases above), define

$$T := L \, (LR)^{\frac{s(n,e)}{2}}.$$

The *period* of T means the order of T in the permutation group of D, i.e.

$$\mathrm{ord}(T) = \min\{k \geq 1 : T^k = \mathrm{id}_D\}.$$

The edge number of each closed geodesic is the period of the operator T. It can be proved to be either 2 or 4. In most cases, it is 4, and it equals 2 if and only if $\gcd(e+1, n) = 2$.

Proposition (Q.)

The map given by $Z(C_n; (D^1, S^0))$ is $M(n, -1)$ in the NS-construction. The following table presents the computed results of the homological rank and kernel dimension of the regular embeddings $M(n, -1)$ of hypercube graphs Q_n for even values of n.

k	$n = 2k$	face dim	dim H_1	Geo num	Ker dim	rank in H_1
2	4	15	2	8	6	2
3	6	95	34	48	24	24
4	8	511	258	256	96	160
5	10	2559	1538	1280	400	880
6	12	12287	8194	6144	1732	4412
7	14	57343	40962	28672	7644	21028
8	16	262143	196610	131072	33920	97152
9	18	1179647	917506	589824	150240	439584

The ratio pattern of geodesics in the regular embedding $M(n, -1)$

k	$n = 2k$	dim H_1	Geo num	Ker dim	$\frac{\text{Ker dim}}{\text{Geo num}}$	ratio pattern
2	4	2	8	6	0.75	$\frac{3}{4} = \frac{1}{4} + \frac{1}{2}$
3	6	34	48	24	0.5	$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$
4	8	258	256	96	0.375	$\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$
5	10	1538	1280	400	0.3125	$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$
6	12	8194	6144	1732	0.281901042	$\frac{433}{1536} = \frac{1}{4} + \frac{1}{32} \cdot \frac{49}{48}$
7	14	40962	28672	7644	0.266601563	$\frac{273}{1024} = \frac{1}{4} + \frac{1}{64} \cdot \frac{17}{16}$
8	16	196610	131072	33920	0.258789063	$\frac{265}{1024} = \frac{1}{4} + \frac{1}{128} \cdot \frac{9}{8}$
9	18	917506	589824	150240	0.254720052	$\frac{1565}{6144} = \frac{1}{4} + \frac{1}{256} \cdot \frac{29}{24}$

Conjecture (Q.)

Based on this empirical evidence, we propose the following closed-form expression for the kernel dimension for all even n:

$$\dim \operatorname{Ker}_n = n \cdot 2^{n-3} \left(\frac{1}{4} + \frac{1}{2^{\frac{n}{2}-1}} \left(1 + \frac{\theta_n}{48} \right) \right),$$

where

$$\theta_n = \begin{cases} 0, & n < 8 \\ \frac{1}{2} \left(\frac{n}{2} - 4 \right) \left(\frac{n}{2} - 5 \right), & n \geq 8. \end{cases}$$

Table of Contents

In topological Morse theory, the **index** of a critical point is the dimension of its descending space. Akrout (2003) showed that the systole function on a surface X is a topological Morse function, with the index at a critical point computable as:

$$\text{index}(X) = \dim \text{span}\{\nabla \ell_\gamma(X) : \gamma \in S(X)\}.$$

Work by Schmutz Schaller (1999) shows that surfaces with high symmetries are precisely the critical points.

Work by An, Ihringer and Irmer (2025), through the search for minimal filling sets that decompose the surface, computed cases for n from 5 to 8 and conjectured that the index is $n \cdot 2^{n-3} - (n+3)$. Our work uses a different method to obtain the hyperbolic structures; this construction is consistent and holds for all n. Using this method, we computed cases for n from 5 to 13, and the results agree with the conjecture.

Calculation result of index of systole function

Proposition (Q.)

Endowing each right-angled regular polygon of the surface $X = Z(C_n, (D^1, S^0))^*$ with the standard hyperbolic metric, we computed $\text{index}(X)$ for n in the range from 5 to 13.

n	row(V)	row(S)	shape(A)	rank(A)	nullity	nullity$-n$
5	96	20	116×120	108	8	3
6	192	48	240×288	231	9	3
7	384	112	496×672	486	10	3
8	768	256	1024×1536	1013	11	3
9	1536	576	2112×3456	2100	12	3
10	3072	1280	4352×7680	4339	13	3
11	6144	2816	8960×16896	8946	14	3
12	12288	6144	18432×36864	18417	15	3
13	24576	13312	37888×79872	37872	16	3

Table of Contents

Ruled Surface

Definition

A surface formed by the movement of a straight line along a curve in 3-dimensional Euclidean space is called a **ruled surface**. A parametric vector representation of a ruled surface can be written in the following form:

$$\boldsymbol{r}(u,v) = \boldsymbol{a}(u) + v\boldsymbol{l}(u)$$

Straight lines belonging to a ruled surface are called rectilinear generatrixes or rectilinear generators or rulings. A curve intersecting all rulings of the surface is called a directrix curve or a director curve or a directing curve.

Tangent developable surface

Definition (Tangent developable surface)

A **tangent developable surface** is a particular kind of developable surface obtained from a curve $\mathbf{c}$ in Euclidean space $\mathbb{R}^3$ and swept out by its tangent lines $\mathbf{c}'(u)$ as $\mathbf{r}(u)$. That means any tangent developable surface is described by a parametric representation:

$$\mathbf{x}(u,v) = \mathbf{c}(u) + \mathbf{c}'(u) \cdot v$$

An example is shown in following figure.

Theorem (Euler 1772)

Any tangent developable surface is a developable surface. And that every surface with zero Gaussian curvature is of one of generalized cones and the cylinders and tangent developable surface these three types.
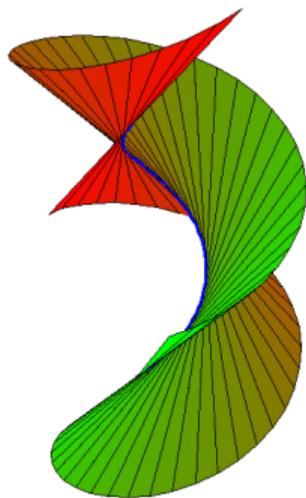
Figure: The tangent developable of a helix

Pseudo-Hermitian quantum mechanics

Standard quantum mechanics requires the Hamiltonian to be Hermitian ($H = H^\dagger$) to ensure real energy eigenvalues and unitary time evolution. **Pseudo-Hermitian quantum mechanics** generalizes this framework to study non-Hermitian systems that can still possess a real energy spectrum. The core condition is the existence of an invertible, Hermitian metric operator η such that:

$$\eta H \eta^{-1} = H^\dagger$$

This η-pseudo-Hermiticity implies that H is Hermitian with respect to a modified inner product $\langle\psi|\phi\rangle_\eta := \langle\psi|\eta|\phi\rangle$. This formalism is particularly powerful for describing open systems with balanced gain and loss, often linked to $\mathcal{PT}$-symmetry, where the system's Hamiltonian is invariant under combined parity and time-reversal operations.

Let's consider the physical implications of your chosen metric $\eta = \text{diag}(-1, 1, 1)$. This metric defines a Hilbert space with an indefinite metric signature $(-, +, +)$.

The characteristic polynomial of the pseudo-Hermitian matrix

$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$

is $g(\lambda) = \lambda^3 + \lambda^2 + (f_1^2 + f_2^2 - f_3^2 - 1)\lambda + (f_1^2 + f_2^2 + f_3^2 + 2f_1 f_2 f_3 - 1)$.

The discriminant of $g(\lambda) = 0$ is $\Delta = 4f_1^4 - 4f_1^6 + 8f_1^2 f_2^2 - 12f_1^4 f_2^2 + 4f_2^4 - 12f_1^2 f_2^4 - 4f_2^6 + 64f_1 f_2 f_3 - 72f_1^3 f_2 f_3 - 72f_1 f_2^3 f_3 + 64f_3^2 - 80f_1^2 f_3^2 + 12f_1^4 f_3^2 - 80f_2^2 f_3^2 - 84f_1^2 f_2^2 f_3^2 + 12f_2^4 f_3^2 - 144f_1 f_2 f_3^3 - 32f_3^4 - 12f_1^2 f_3^4 - 12f_2^2 f_3^4 + 4f_3^6$.
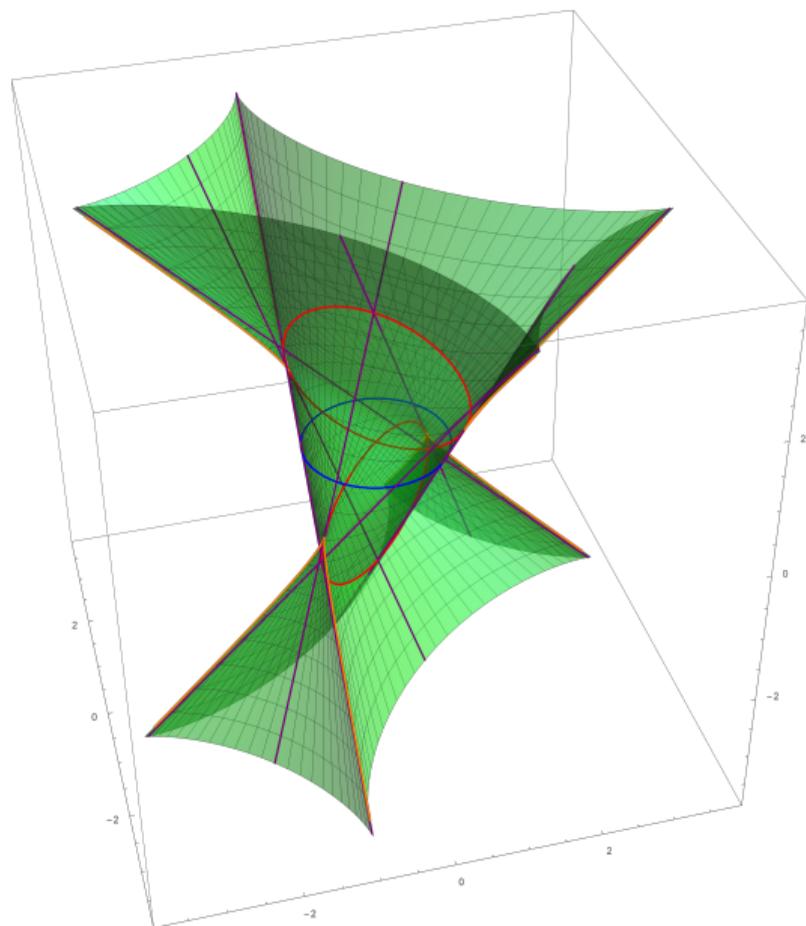
Theorem (Q.–Z.Yu)

The generic part of discriminant surface $\Delta = 0$ over real number field can be parameterized in the form

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos \varphi \\ -\sin \varphi \\ 0 \end{bmatrix} + f_3 \begin{bmatrix} \sin^3 \varphi \\ -\cos^3 \varphi \\ 1 \end{bmatrix} . \tag{1}$$

Therefore, it is a ruled surface.

Theorem (Q.–Z.Yu)

The generic part of discriminant surface of characteristic polynomial of matrix

$$H = \begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

is ruled, where g_1, g_2, g_3 are linear functions of parameters f_1, f_2, f_3, i.e.
$g_i = a_{i,1}f_1 + a_{i,2}f_2 + a_{i,3}f_3 + b_i$. Furthermore, the discriminant surface of H is tangent developable and the associated directing curve is the curve of three multiplicity.

Thanks for Your Attention!