An introduction to Jacobi forms

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Notation

Let $e(x)$ denote $e^{2\pi i x}$ for $x \in \mathbb{C}$. Let $q = e(\tau)$ and $\zeta = e(z)$ where $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$.

Jacobi forms are meant to be a natural generalization of Jacobi theta series.

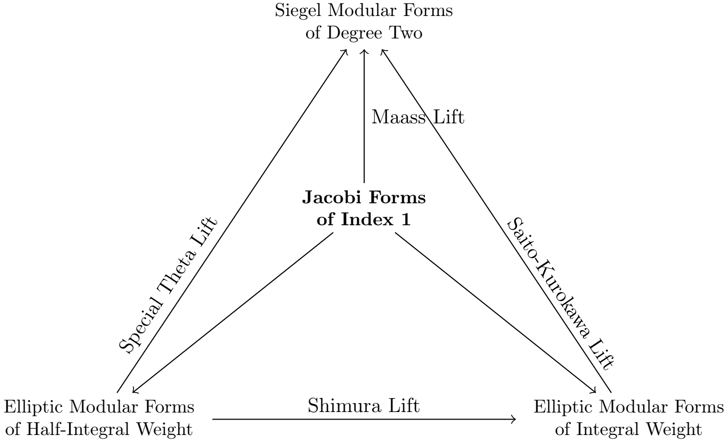
Definition 1

Let L be a lattice of rank $2k$ with a positive-definite quadratic form $Q(x)$ and bilinear form $B(x,y) = Q(x+y) - Q(x) - Q(y)$. Given a vector $y \in L$ we define the Jacobi theta series $\Theta_y(\tau, z)$ by

$$\Theta_y(\tau, z) = \sum_{x \in L} e\big((Q(x)\tau + B(x,y)z)\big),$$

where $e(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$ (as defined in the notation).

One of the interesting objects is the relations with other types of modular forms



Jacobi forms are complex functions on $\mathcal{H} \times \mathbb{C}$ which are invariant under an action of the Jacobi group.

Difinition 2

The *Jacobi group* is $\mathrm{SL}_2(\mathbb{Z})^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ where

$$[M, X][M', X'] = [MM', XM' + X'].$$

For a congruence subgroup Γ let $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$.

Definitions of Jacobi groups and group actions

Notation

Given integers k and m, the slash operator is

$$(\phi|_{k,m}\gamma)(\tau, z) = (c\tau + d)^{-k} e\left(m\left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)\right) \cdot \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)$$

for $\gamma = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right] \in \mathrm{SL}_2(\mathbb{Z})^J$. This defines an action of the Jacobi group on complex functions of $\mathcal{H} \times \mathbb{C}$.

Definition of the Jacobi form

A Jacobi form of weight k and index m for a congruence subgroup Γ is a function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ which

1. is holomorphic on $\mathcal{H} \times \mathbb{C}$,
2. satisfies $\phi|_{k,m}\gamma = \phi$ for all $\gamma \in \Gamma^J$, and
3. is holomorphic at each cusp $Mi\infty$ where $M \in \mathrm{SL}_2(\mathbb{Z})^J$, that is,

$$\phi|_{k,m}M = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn\geq r^2h}} c_M\left(\frac{n}{h}, r\right) q^{n/h}\zeta^r,$$

   where h is the width of the cusp $Mi\infty$ of Γ.

Furthermore, we say ϕ is a *Jacobi cusp form* if in addition to the conditions above ϕ vanishes at each cusp $Mi\infty$ for $M \in \mathrm{SL}_2(\mathbb{Z})^J$, that is, if

$$\phi|_{k,m}M = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>r^2h}} c_M\left(\frac{n}{h}, r\right) q^{n/h}\zeta^r.$$

Structure Theorem

Notation

Let $J_{k,m}(\Gamma)$ denote the vector space of all Jacobi forms with weight k and index m on a congruence subgroup Γ. Let the subspace of cusp forms be denoted by $J_{k,m}^{\text{cusp}}(\Gamma)$. Furthermore let $M_k(\Gamma)$ denote the space of elliptic modular forms of weight k on the congruence subgroup Γ.

Theorem 1

Given a congruence subgroup Γ structurally $\bigoplus_{k,m} J_{k,m}(\Gamma)$ forms a bigraded ring with each $J_{k,m}(\Gamma)$ finite dimensional. Moreover $J_{*,*}(\Gamma)$ is a module over $M_*(\Gamma)$.

Eisenstein series

$$E_{k,m}(\tau, z) := \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma_1^J} 1|_{k,m}\gamma,$$

where

$$\Gamma_\infty^J = \{\gamma \in \Gamma_1^J \mid 1|\gamma = 1\} = \left\{ \left[\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right] \mid n, \mu \in \mathbb{Z} \right\}.$$

Explicitly, this is

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^{2\pi i m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right)},$$

where a, b are chosen so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$.

Eisenstein series

$$E_{4,1} = 1 + \left(\zeta^2 + 56\zeta + 126 + 56\zeta^{-1} + \zeta^{-2}\right) q$$
$$+ \left(126\zeta^2 + 576\zeta + 756 + 576\zeta^{-1} + 126\zeta^{-2}\right) q^2$$
$$+ \left(56\zeta^3 + 756\zeta^2 + 1512\zeta + 2072 + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3}\right) q^3 +$$

$$E_{6,1} = 1 + \left(\zeta^2 - 88\zeta - 330 - 88\zeta^{-1} + \zeta^{-2}\right) q$$
$$+ \left(-330\zeta^2 - 4224\zeta - 7524 - 4224\zeta^{-1} - 330\zeta^{-2}\right) q^2 + \cdots,$$

$$E_{8,1} = 1 + \left(\zeta^2 + 56\zeta + 366 + 56\zeta^{-1} + \zeta^{-2}\right) q^2 + \cdots.$$

Petersson scalar product

Peterason scalar product of ϕ and ψ defined by

$$(\phi, \psi) := \int_{\Gamma\backslash\mathcal{H}\times\mathbb{C}} v^k e^{-4\pi m y^2/v} \phi(\tau, z)\overline{\psi(\tau, z)} dV$$

where

$$dV := v^{-3} dx dy du dv \quad (\tau = u + iv, z = x + iy \ (v > 0))$$

The form $v^{-2} du dv$ is the usual $SL_2(\mathbb{R})$-invariant volume form on $\mathcal{H}$; the form $v^{-1} dx dy$ is the translation-invariant volume form on $\mathbb{C}$, normalized so that the fibre $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ has volume 1.

Theorem 2

The scalar product is well defined and finite for $\phi, \psi \in J_{k,m}$, and at least one of ϕ and ψ a cusp form. It is positive-definite on $J_{k,m}^{cusp}$, and the orthogonal complement of $J_{k,m}^{cusp}$ with respect to $(,)$ is $J_{k,m}^{Eis}$.

Coefficients of Jacobi Forms

The Fourier series of Jacobi forms depend on determinant of Jacobi forms.

Theorem 3

Let ϕ be a Jacobi form of index m with Fourier development $\sum c(n,r)q^n\zeta^r$. Then $c(n,r)$ depends only on $4nm - r^2$ and on $r \pmod{2m}$. If k is even and $m = 1$ or m is prime, then $c(n,r)$ depends only on $4nm - r^2$. If $m = 1$ and k is odd, then ϕ is identically zero.

$$\begin{aligned}\sum c(n,r)q^n\zeta^r = \phi(\tau,z) &= e^{2\pi i m(\lambda^2\tau + 2\lambda z)}\phi(\tau, z + \lambda\tau + \mu) \\ &= q^{m\lambda^2}\zeta^{2m\lambda}\sum c(n,r)q^{n+r\lambda}\zeta^r \\ &= \sum c(n,r)q^{n+r\lambda+m\lambda^2}\zeta^{r+2m\lambda}\end{aligned}$$

and hence $c(n,r) = c(n + r\lambda + m\lambda^2, r + 2m\lambda)$,
i.e., $c(n,r) = c(n',r')$ whenever $r' \equiv r \pmod{2m}$ and $4n'm - r'^2 = 4nm - r^2$.

Coefficients of Jacobi Forms

If k is even, $c(n,-r) = c(n,r)$ (because applying the transformation law of Jacobi forms to $-I_2 \in \Gamma_1$ gives $\phi(\tau,-z) = (-1)^k \phi(\tau,z)$), so if m is 1 or a prime, then

$$4nm - r^2 = 4nm - r'^2 \implies r' \equiv \pm r \pmod{2m} \implies c(n,r) = c(n,r').$$

Finally, if $m = 1$ and k is odd then $\phi = 0$ because $c(n,-r) = -c(n,r)$ but $4nm - (-r)^2 = 4nm - r^2$ and $-r \equiv r \pmod{2m}$ in this case.

Some Notations

Notation

We introduce the functions

$$\phi_{10,1} = \frac{1}{144}\left(E_6 E_{4,1} - E_4 E_{6,1}\right),$$
$$\phi_{12,1} = \frac{1}{144}\left(E_4^2 E_{4,1} - E_6 E_{6,1}\right).$$

Theorem 4

The quotient

$$\frac{\phi_{12,1}(\tau,z)}{\phi_{10,1}(\tau,z)} = \frac{\zeta + 10 + \zeta^{-1}}{\zeta - 2 + \zeta^{-1}} + 12(\zeta - 2 + \zeta^{-1})^{-1} q + \dots$$

is $-3/\pi^2$ *times the Weierstrass* $\wp$*-function* $\wp(\tau, z)$.

Hecke operators

$$(1)\quad (\phi|_{k,m}U_\ell)(\tau,z)=\phi(\tau,\ell z),$$

$$(2)\quad (\phi|_{k,m}V_\ell)(\tau,z)=\ell^{k-1}\sum_{\substack{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1\backslash M_2(\mathbb{Z})\\ ad-bc=\ell}}(c\tau+d)^{-k}e^{\pi i m\frac{-cz^2}{c\tau+d}}\times$$

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{\ell z}{c\tau+d}\right),$$

$$(3)\quad (\phi|_{k,m}T_\ell)(\tau,z)=\ell^{k-4}\sum_{\substack{M\in\Gamma_1\backslash M_2(\mathbb{Z})\\ \det M=\ell^2\\ \mathrm{g.c.d}(M)=\square}}\sum_{X\in\mathbb{Z}^2/\ell\mathbb{Z}^2}(\phi|_{k,m}M)|_m X.$$

g.c.d$(M)=\square$ means that the greatest common divisor of the entries of M is a square.

V_ℓ, U_ℓ, T_ℓ map $J_{k,m}$ to J_{k,l^2m}, $J_{k,lm}$ and $J_{k,m}$,respectively.

Modular forms of integral weight

We first develop $\phi(\tau, z)$ in a Taylor expansion around $z = 0$.

$$\phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau) z^\nu$$

by the transformation equation,we get
$\chi_\nu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+\nu}\left(\chi_\nu(\tau) + \frac{2\pi imc}{c\tau+d}\chi_{\nu-2}(\tau) + \frac{1}{2!}\left(\frac{2\pi imc}{c\tau+d}\right)^2\chi_{\nu-4}(\tau) + \dots\right)$
We define ξ_ν for each ν

$$\xi_\nu(\tau) := \sum_{0\le\mu\le\frac{\nu}{2}} \frac{(-2\pi im)^\mu (k+\nu-2\mu-2)!}{(k+\nu-2)!\,\mu!}\chi^{(\mu)}_{\nu-2\mu}(\tau).$$

Modular forms of integral weight

If ϕ has a Fourier development $\sum_{n,r} c(n,r)q^n\zeta^r$, then $\chi_\nu = \frac{1}{\nu!}\sum_n \sum_r (2\pi i r)^\nu c(n,r)q^n$ and hence

$$\xi_\nu(\tau) = (2\pi i)^\nu \sum_{n\geq 0}\left(\sum_r \left(\sum_{0\leq\mu\leq\frac{v}{2}} \frac{(k+\nu-\mu-2)!}{(k+\nu-2)!}\frac{(-mn)^\mu r^{\nu-2\mu}}{\mu!(\nu-2\mu)!}\right) c(n,r)\right)$$

Let

$$D_{2\nu}\phi(\tau) = (2\pi i)^{-2\nu}\frac{(k+2\nu-2)!(2\nu)!}{(k+\nu-2)!}\xi_{2\nu}(\tau).$$

To get $D_1\phi$,$D_3\phi, \cdots$ of weight $k+1$, $k+3, \cdots$, we take $\nu \in 1/2 + \mathbf{N}_0$ and replace $(k+\nu-2)!$ by $(k+\nu-3/2)!$.

Theorem 5

the function ξ_ν and $D_\nu\phi(\tau)$ are modular forms of weight $k+\nu$ on Γ, and if $\nu > 0$, they are cusp forms.

We give some explicit examples

$$D_0\phi = \sum_n \left(\sum_r c(n,r) \right) q^n,$$

$$D_2\phi = \sum_n \left(\sum_r (nr^2 - 2nm) c(n,r) \right) q^n,$$

$$D_4\phi = \sum_n \left(\sum_r \left((k+1)(k+2)r^4 - 12(k+1)r^2nm + 12n^2m^2 \right) c(n,r) \right) q^n$$

Modular forms of integral weight

Theorem 6

$$D = \bigoplus_{\nu=0}^{2m} D_\nu \colon J_{k,m}(\Gamma) \to M_k(\Gamma) \oplus S_{k+1}(\Gamma) \oplus \cdots \oplus S_{k+2m}(\Gamma)$$

is injective. Note that half of the spaces $M_{k+\nu}(\Gamma)$ *are* 0 *if* $-I_2 \in \Gamma$*; in particular, for* $\Gamma = \Gamma_1$ *we have*

$$\dim J_{k,m} \leq \begin{cases} \dim M_k + \dim S_{k+2} + \cdots + \dim S_{k+2m} & (k \ \textit{even}), \\ \dim S_{k+1} + \dim S_{k+3} + \cdots + \dim S_{k+2m-1} & (k \ \textit{odd}). \end{cases}$$

By the theorem 3, $J_{k,1} = 0$ ($\forall k$ odd),
and $\dim J_{k,1} \leq \dim M_k + \dim S_{k+2}$ ($\forall k$ even).

Since

$$\frac{E_{6,1}(\tau,z)}{E_{4,1}(\tau,z)} = 1 - (144\zeta + 456 + 144\zeta^{-1})q + \dots$$

depends on z and hence is not a quotient of two modular forms, so the map

$$\begin{aligned} M_{k-4} \oplus M_{k-6} &\to J_{k,1} \\ (f,g) &\mapsto f(\tau)E_{4,1}(\tau,z) + g(\tau)E_{6,1}(\tau,z) \end{aligned}$$

is injective. Since $\dim M_{k-4} + \dim M_{k-6} = \dim M_k + \dim S_{k+2}$ for all k.
So $\dim J_{k,1} \geq \dim M_k + \dim S_{k+2}$.
We deduce

Modular forms of integral weight

Theorem 7

$J_{*,1}$ is a free module of rank 2 over M_*, with generators $E_{4,1}$ and $E_{6,1}$. And the map

$$D_0 + D_2 \colon J_{k,1} \to M_k + S_{k+2}$$

is an isomorphism.

similarly,

Theorem 8

$J_{*,1}^{cusp}$ is a free module of rank 2 over M_*, with generators $\phi_{10,1}$ and $E_{12,1}$. And the map

$$\begin{aligned} &M_{k-10} \oplus M_{k-12} \xrightarrow{\sim} J_{k,1}^{cusp} \\ &(f, g) \longmapsto f(\tau)\phi_{10,1}(\tau, z) + g(\tau)\phi_{12,1}(\tau, z) \end{aligned}$$

is an isomorphism.

Modular forms of half-integral weight

By theorem 3, the coefficients $c(n,r)$ of a Jacobi form of index m depend only on the "discriminant"$r^2 - 4nm$ and on the value of $r \pmod{2m}$, i.e.

$$c(n,r) \;=\; c_r\big(4nm - r^2\big), \quad c_{r'}(N) \;=\; c_r(N) \quad \text{for } r' \equiv r \pmod{2m}.$$

$c_\mu(N)$ for all $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and all integers $N \geq 0$ satisfying $N \equiv -\mu^2 \pmod{4m}$ (notice that μ^2 is well-defined modulo $4m$ if μ is given modulo $2m$), namely

$$c_\mu(N) := c\left(\frac{N + r^2}{4m}, r\right)$$

(any $r \in \mathbb{Z}$, $r \equiv \mu \pmod{2m}$)

Modular forms of half-integral weight

$$\phi(\tau, z) = \sum_{\mu \pmod{2m}} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu\,(2m)}} \sum_{n \geq \frac{r^2}{4m}} c_\mu\left(4nm - r^2\right) q^n \zeta^r$$

$$= \sum_{\mu \pmod{2m}} \sum_{r \equiv \mu\,(2m)} \sum_{N \geq 0} c_\mu(N) q^{\frac{N+r^2}{4m}} \zeta^r$$

$$= \sum_{\mu \pmod{2m}} h_\mu(\tau)\, \theta_{m,\mu}(\tau, z)$$

where

$$h_\mu(\tau) := \sum_{N=0}^{\infty} c_\mu(N)\, q^{N/4m} \quad (\mu \in \mathbb{Z}/2m\mathbb{Z})$$

and

$$\theta_{m,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{\frac{r^2}{4m}} \zeta^r.$$

Modular forms of half-integral weight

Theorem 9

This gives an isomorphism between $J_{k,m}$ and the space of vector-valued modular forms $(h_\mu)_{\mu\,(mod\ 2m)}$ on $SL_2(\mathbb{Z})$ satisfying the transformation laws

$$h_\mu(\tau+1) = e^{-4\pi i\,\frac{(\mu-y)^2}{4m}}\,h_\mu(\tau)$$

and

$$h_\mu\left(-\frac{1}{\tau}\right) = \frac{\tau^k}{\sqrt{2m\tau^i}}\sum_{\nu\,(mod\ 2m)} e^{\frac{2\pi i(\mu\nu)}{2m}}\,h_\nu(\tau)$$

and bounded as $Im(\tau)\to\infty$. The second equality is obtained by the transformation law ϕ under $(t,z)\mapsto\left(\frac{1}{t},\frac{z}{t}\right)$.

Modular forms of half-integral weight

The Petersson scalar product of ϕ and ψ as defined is equal (up to a constant) to the Petersson product in the usual sense of the vector-valued modular forms (h_μ), (g_μ) of weight $k - \frac{1}{2}$.

Theorem 10

Let

$$\phi = \sum_\mu h_\mu \theta_{m,\mu}, \quad \psi = \sum_\mu g_\mu \theta_{m,\mu}$$

be two Jacobi forms in $J_{k,m}$. Then

$$(\phi, \psi) = \frac{1}{\sqrt{2m}} \int_{\Gamma \backslash \mathcal{H}} \sum_{\mu \pmod{2m}} h_\mu(\tau) \overline{g_\mu(\tau)} v^{-k/2} \, du \, dv.$$

Modular forms of half-integral weight

W.Kohen proved (Kohen's plus subspace)

$$M^+_{k-\frac{1}{2}}(4) = \left\{ h \in M_{k-\frac{1}{2}}\big(\Gamma_0(4)\big) \;\Big|\; h = \sum_{\substack{N=0 \\ (-1)^{k-1}N \equiv 0,1 \pmod 4}}^{\infty} c(N)q^N \right\}$$

is isomorphic to $M_{2k-2}(1)$ as modules over the ring of Hecke operators.

Theorem 11

$$\sum_{\substack{N \geq 0 \\ -N \equiv 0,1 \pmod 4}} c(N)q^N \longmapsto \sum_{\substack{n,r \in \mathbb{Z} \\ 4n \geq r^2}} c(4n - r^2)q^n \zeta^r$$

gives an isomorphism between $M^+_{k-\frac{1}{2}}(4)$ and $J_{k,1}$ (k even). This isomorphism is compatible with the Petersson scalar products, with the actions of Hecke operators, and with the structures of $M^+_{2*-\frac{1}{2}}(4)$ and $J_{*,1}$ as modules over M_*. ($M^+_{*-\frac{1}{2}}(4) = \oplus_k M^+_{k-\frac{1}{2}}(4)$ is a module over M_* by $h(\tau) \to f(4\tau)h(\tau)$ ($h \in M^+_{*-\frac{1}{2}}(4)$ $f \in M_*$)

Siegel modular forms

The Siegel upper half-space of degree n is defined as the set $\mathcal{H}_n$ of complex symmetric $n \times n$ matrices Z with positive-definite imaginary part.

$Sp_{2n}(\mathbb{R}) = \left\{ M \in M_{2n}(\mathbb{R}) \mid MJ_{2n}M^t = J_{2n} \right\}, \quad J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ acts on $\mathcal{H}_n$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z = \frac{Az + B}{Cz + D} \tag{1}$$

A Siegel modular form of degree n and weight k with respect to the full Siegel modular group $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ is a holomorphic function $F : \mathcal{H}_n \to \mathbb{C}$ satisfying

$$F(M \cdot Z) = \det(CZ + D)^k F(Z)$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ with A, B, C, D being $n \times n$ matrices, and $M \cdot Z = (AZ + B)(CZ + D)^{-1}$ for $Z \in \mathcal{H}_n$.

Siegel modular forms

If $n > 1$, Siegel possess a Fourier expansion of the form

$$F(Z) = \sum_{T \geq 0} A(T)\, e(\mathrm{tr}\, TZ),$$

where the summation is over positive semidefinite semi-integral (i.e., $2T_{ij}, T_{ii} \in \mathbb{Z}$) $n \times n$ matrices T;

If $n = 2$, $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathcal{H}$, $z \in \mathbb{C}$, $\mathrm{Im}(z) < \mathrm{Im}(\tau)\,\mathrm{Im}(\tau')$.

similarly $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ with $n, r, m \in \mathbb{Z}$, $nm \geqq 0$, $r^2 \leqq 4nm$, and the Fourier expansion is

$$F(Z) = F(\tau, z, \tau') = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m)\, e^{2\pi i (n\tau + rz + m\tau')}$$

The relation to Jacobi forms is given by the following results.

Theorem 12

Let F be a Siegel modular form of weight k and degree 2 and write the Fourier development of F in the form

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z)\, e(m\tau'),$$

Then $\phi_m(\tau, z)$ is a Jacobi form of weight k and index m.

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belong to Γ_2 and act on $\mathcal{H}_2$ by

$$(\tau, z, \tau') \longmapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \tau' - \frac{cz^2}{c\tau + d} \right),$$

$$(\tau, z, \tau') \longmapsto \left(\tau, z + \lambda\tau + \mu, \tau' + 2\lambda z + \lambda^2 \tau \right),$$

Theorem 13

Let ϕ be a Jacobi form of weight k and index 1. Then the functions $\phi|V_m$ $(m \geq 0)$ are the Fourier-Jacobi coefficients of a Siegel modular form $\mathscr{V}\phi$ of weight k and degree 2, where $\mathscr{V}\phi(\tau, z, \tau') := \sum_{m\geq 0}(\phi|V_m)(\tau, z)e(m\tau')$.

Theorem 12 gives the injective map

$$\mathscr{J} : M_k(\Gamma_2) \longrightarrow J_{k,0} \times J_{k,1} \times J_{k,2} \times \cdots$$

And theorem 13 give the map

$$\mathscr{V} : J_{k,1} \longrightarrow M_k(\Gamma_2)$$

such that the composite

$$J_{k,1} \xrightarrow{\mathscr{V}} M_k(\Gamma_2) \xrightarrow{\mathscr{J}} \prod_{m\geq 0} J_{k,m} \xrightarrow{\text{pr}} J_{k,1}$$

is the identity. Thus $\mathscr{V}$ is injective and its image is the set of F with

$F = \mathscr{V}(\mathscr{J}(F))$, i.e. of Siegel modular forms whose Fourier-Jacobi expansion has the property $\phi_m = \phi_1 | V_m (\forall m)$.
Since, $\mathscr{V}\phi(\tau, z, \tau') = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m)\, e^{2\pi i(n\tau + rz + m\tau')}$ formally,
then $\phi | V_m(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ n,4nm-r^2 \geq 0}} A(n,r,m) e^{2\pi i(n\tau + rz)}$. and by the property of V_m,

$$A(n,r,m) = \sum_{\substack{d|(n,r,m) \\ (n,r,m) \neq (0,0,0)}} d^{k-1} c\left(\frac{4nm - r^2}{d^2}\right)$$

obviously, $c\left(\frac{4nm-r^2}{d^2}\right) = A(\frac{nm}{d^2}, \frac{r}{d}, 1)$.
Thus

$$A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1} A\left(\frac{n}{d^2}, \frac{r}{d}, 1\right) \quad (\forall\, n, r, m)$$

Siegel modular forms

Let $M_k^*(\Gamma_2)$ a subspace of $M_k(\Gamma_2)$ consisting of the Siegel modular forms whose coefficients satisfy this condition.
The space was studied by Maass. He called it the "Spezialschar". We established inverse isomorphisms

$$M_k^*(\Gamma_2) \rightleftarrows J_{k,1}.$$

Combining these with the isomorphism $J_{k,1} \xrightarrow{\sim} M_{k-1/2}^*(4)$, we obtain

Siegel modular forms

Theorem 14

If $h(\tau) = \sum_{\substack{N \geq 0 \\ -N \equiv 0,1 \pmod 4}} c(N) q^N$ *is a modular form in Kohnen's "plus-space"* $M^+_{k-1/2}(4)$*, then the numbers* $A(n, r, m)$ *defined are the coefficients of a modular form* F *in Maass' "Specialschar"* $M^*_k(\Gamma_2)$*. The map* $h \mapsto F$ *is an isomorphism from* $M^+_{k-1/2}(4)$ *to* $M^*_k(\Gamma_2)$*, the inverse map being given*

$$c(N) = \begin{cases} A(n,0,1) & \text{if } N = 4n \\ A(n,1,1) & \text{if } N = 4n-1 \end{cases}$$

Finally, $\mathscr{V}$ is compatible with the action of Hecke operators in $J_{k,1}$ and $M_k(\Gamma_2)$, i.e. that there is an algebra map $\iota : \mathbb{T}_S \to \mathbb{T}_J$ from the Hecke algebra for Siegel modular forms of weight k and degree 2 to the Hecke algebra for Jacobi forms of weight k and index l such that

$$\mathscr{V}(\phi) \mid T = \mathscr{V}(\phi \mid \iota(T)) \quad \forall T \in \mathbb{T}_S.$$

And Andrianov proved that $\mathbb{T}_S$ is generated by $T_S(p)$ and $T_S(p^2)$ with p prime. We choose the generators are $T_S(p)$ and $T'_S(p) = T_S(p)^2 - T_S(p^2)$.

Theorem 15

The map $\mathscr{V} : J_{k,1} \to M_k(\Gamma_2)$ is Hecke-equivariant with respect to the homomorphism of Hecke algebras $\iota : \mathbb{T}_S \to \mathbb{T}_J$ defined on generators by

$$\iota(T_S(p)) = T_J(p) + p^{k-1} + p^{-2},$$

$$\iota(T'_S(p)) = (p^{k-1} + p^{k-2})T_J(p) + 2p^{2k-3} + p^{2k-4}.$$

Saito-kurokawa conjecture

Andrianov associates to any Hecke eigenform $F \in S_k(\Gamma_2)$ the Euler product

$$Z_p(s) = \prod_p \left(1 - \gamma_p p^{-s} + (\gamma'_p - p^{-2k-4})p^{-2s} - \gamma_p p^{2k-3-3s}) + p^{4k-6-4s}\right)^{-1}$$

where $F \mid T_S(p) = \gamma_p F$, $F \mid T'_S(p) = \gamma'_p F$.

Saito-Kurokawa Conjecture(theorem)

The space $S^*_k(\Gamma_2) \subset \mathrm{M}^*_k(\Gamma_2)$ is spanned by Hecke eigenforms. These are in 1-1 correspondence with normalized Hecke eigenforms $f \in S_{2k-2}$, the correspondence being such that

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s).$$

Saito-Kurokawa lifting

The Saito-Kurokawa lifting can be constructed as the composition of two linear maps

$$M_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) \xrightarrow{\sim} J_{k,1} \to M_k(\Gamma_2).$$