Introduction to Differential Cohomology:

from Chern-Simons invariants to invertible field theories

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Degree map and Poincare-Hopf theorem

Theorem (Poincaré-Hopf)

Let M^n be a compact oriented manifold. Let v be any vector field on M with isolated zeroes. Then we have

$$\sum_i \mathrm{ind}_{m_i}(v) = \chi(M).$$

Here the index is defined as the degree of $u : \partial\mathbb{D} \to \mathbb{S}^{n-1}$ given by $u(z) = v(z)/\|v(z)\|$.

Gauss-Bonnet theorem

Theorem

Let M be a closed Riemannian 2-manifold with curvature F. Then we have

$$\int_M \frac{1}{2\pi} F = \chi(M),$$

which is independent of F.

Proof.

From v one can construct 1-form A with curvature $F_A = \mathrm{d}A$ locally.

$$\begin{aligned} \int_M \tfrac{1}{2\pi} F = \lim_{t\to 0} \int_{M-\{\mathbb{D}^t_{m_i}\}} \tfrac{1}{2\pi} F \quad &= \lim_{t\to 0} \int_{M-\{\mathbb{D}^t_{m_i}\}} \tfrac{1}{2\pi} \mathrm{d}A \\ &= \lim_{t\to 0} \int_{\{\mathbb{S}^t_{m_i}\}} \tfrac{1}{2\pi} A \\ &= \textstyle\sum_i \mathrm{ind}_{m_i}(v) \end{aligned}$$

□

Gauss-Bonnet-Chern theorem

Theorem

Let M^{2n} be a closed Riemannian manifold with curvature F. Then we have

$$\int_M p_\chi(F, \ldots, F) = \chi(M).$$

Proof.

Similarly we have A with curvature F_A from unit vector field v. Moreover we have the associated sphere bundle $\pi : S(M) \to M$ with fibers $\mathbb{S}^{2n-1}$. The trangression formula is given by a $(2n-1)$-form $Tp_\chi(F, A)$ on $S(M)$ such that

- $\pi^* p_\chi(F, \ldots, F) = \mathrm{d}Tp_\chi(F, A)$,
- $\int_{\mathbb{S}^{2n-1}} Tp_\chi(F, A) = 1$.

$$\begin{aligned} \int_M p_\chi(F) = \lim_{t\to 0} \int_{M-\{\mathbb{D}^t_{m_i}\}} p_\chi(F) &= \lim_{t\to 0} \int_{v^*(M-\{\mathbb{D}^t_{m_i}\})} \pi^* p_\chi(F) \\ &= \lim_{t\to 0} \int_{v^*(M-\{\mathbb{D}^t_{m_i}\})} \mathrm{d}Tp_\chi(F, A) \\ &= \lim_{t\to 0} \int_{\sum \deg(m_i)\mathbb{S}^t_{m_i}} Tp_\chi(F, A) \\ &= \sum_i \mathrm{ind}_{m_i}(v) \end{aligned}$$

Trangression and Chern-Weil theory

Consider a principal G-bundle $\pi : P \to M$ with A and $F_A \in \Omega^2_M(\mathfrak{g}),\ p \in \mathrm{Sym}^k(\mathfrak{g}^\vee)^G$.
Chern-Weil form is $p((\pi^* F_A)^{\wedge k}) \in \Omega^{2k}_P(\mathbb{R})$ which descends $p(F_A) \in \Omega^{2k}_M(\mathbb{R})$ such that:

- $p(F_A)$ is closed,
- $[p(F_A)] \in H^{2k}(M;\mathbb{R})$ is independent of A,
- cw: $\mathrm{Sym}^k(\mathfrak{g}^\vee)^G \to H^{2k}(M;\mathbb{R})$ is a functorial ring homomorphism.

Trangression is obtained when we consider tautological pullback

$$\pi^*(p(F_A)) = \mathrm{d}Tp(F, A).$$

The closed $2k-1$-form on P is given by

$$Tp(A) = \int_{[0,1]} p(A, \varphi_t^{k-1})\mathrm{d}t,$$

where $\varphi_t = tF_A + (t^2 - t)A \wedge A$.

Chern-Simons invariant

Taking $G = \mathrm{SO}(3)$, M^3 oriented, and $p = p_1$. Then $p_1(F_A) = 0$, i.e. $\mathrm{d}Tp_1(A) = 0$. Consider the frame bundle $F(M) \to M$ which is trivializable, take a global section s, the integral

$$\mathrm{CS}(M) := \int_M s^* Tp_1(A) \mod \mathbb{Z}$$

is a conformal invariant of the manifold called the *Chern-Simons* invariant.

Theorem (Chern-Simons)

The necessary condition for M^3 being able to be conformally immersed into $\mathbb{R}^4$ is

$$\mathrm{CS}(M) = 0.$$

What is it?

- Roughly, a differential cohomology theory $\hat{E}^n$ is a geometric refinement of a cohomology theory E^n.
- $\hat{E}^n$ detects refined geometric information which cannot be captured by a topological cohomology theory.
- Every differential cohomology theory comes equipped with a ”forgetful” map

$$I : \hat{E}^n \longrightarrow E^n,$$

  which forgets the geometric information and recovers the underlying topological information.

Secondary invariants

- Holonomy for $U(1)$-connection:
  Hermitian line bundle $(L, \nabla) \to X$ with $U(1)$-connection A. For $f : \mathbb{S}^1 \to X$, we get holonomy $\mathrm{Hol}(L, \nabla)(f) \in \mathbb{R}/\mathbb{Z}$. If L is trivialized, we have

$$\mathrm{Hol}(L, \nabla)(f) = \int_{\mathbb{S}^1} f^* \frac{A}{2\pi\sqrt{-1}} \mod \mathbb{Z}.$$

- Chern-Simons invariants:
  Hermitian vector bundle $(E, \nabla) \to X$ with $U(1)$-connection A. For $f : M^3 \to X$ with M closed oriented, we get its Chern-Simons invariant $\mathrm{CS}(E, \nabla)(f) \in \mathbb{R}/\mathbb{Z}$. If E is trivialized, we have

$$\mathrm{CS}(E, \nabla)(f) = \int_M f^* \mathrm{Tr}(\mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A) \mod \mathbb{Z}.$$

Problem

Is there an object $\hat{x}$ giving these secondary invariants by "$\int_{M^{n-1}} f^* \hat{x}$" for general X? Where does it live?

Differential properties

1. They depend on the geoemtry. The variation under bordism is measured by

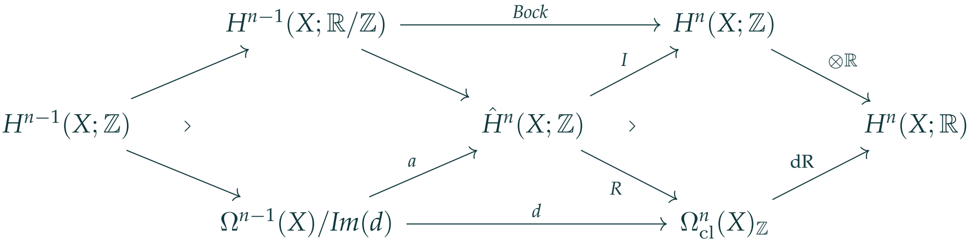
$$\int_{W^n} f^* R(\hat{x}) \quad \mathrm{mod}\, \mathbb{Z},$$

   for some $R(\hat{x}) \in \Omega^n_{\mathrm{cl}}(X)_{\mathbb{Z}}$. For example,

$$\begin{aligned} \mathrm{Hol}(L, \nabla)(f|_{M_1}) - \mathrm{Hol}(L, \nabla)(f|_{M_2}) &= \textstyle\int_{\mathbb{S}^1} f^* \frac{F_\nabla}{2\pi\sqrt{-1}}, \\ \mathrm{CS}(E, \nabla)(f|_{M_1}) - \mathrm{CS}(E, \nabla)(f|_{M_2}) &= \textstyle\int_{W^4} f^* \,\mathrm{ch}_2(F_\nabla). \end{aligned}$$

   Moreover, when the topology is trivial, we have $R(\alpha) = \mathrm{d}\alpha$.
2. They are more refined, we can recover the topology completely, including torsions.
   - $\mathrm{Hol}(L, \nabla)(f)$ for all f recovers $c_1(L) \in H^2(X; \mathbb{Z})$.
   - $\mathrm{CS}(E, \nabla)(f)$ for all f recovers $\mathrm{ch}_2(E) \in H^4(X; \mathbb{Z})$.

THE ANSWER: DIFFERENTIAL COHOMOLOGY



which is commutative and diagonal sequences are exact.

- Trivial topology case corresponds to a.
- 1. corresponds to R.
- 2. corresponds to I.

Geometric model: line bundles with connections

Definition

$\hat{H}^2_{\text{geom}}(X;\mathbb{Z}) := \{\text{Hermitian line bundle } (L,\nabla) \to X \text{ with } U(1)\text{-connection}\}/\sim$

We define the structure maps

$$
\begin{aligned}
I &: \hat{H}^2_{\text{geom}}(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}), \quad [L,\nabla] \mapsto [L]. \\
R &: \hat{H}^2_{\text{geom}}(X;\mathbb{Z}) \to \Omega^2_{\text{cl}}(X), \quad [L,\nabla] \mapsto F_\nabla/(2\pi\sqrt{-1}). \\
a &: \Omega^1(X) \to \hat{H}^2_{\text{geom}}(X;\mathbb{Z}), \quad \alpha \mapsto [X \times \mathbb{C}, d + 2\pi\sqrt{-1}\alpha].
\end{aligned}
$$

Problem

- *Hard to analyze directly.*
- *Difficult to generalize to* $\hat{H}^n(X;\mathbb{Z})$.

Cheeger-Simons' model

Definition (Cheeger-Simons's model of Differential Characters)

For $n \geqslant 0$, the group of *differential characters* $\hat{H}^n_{\mathrm{CS}}(X;\mathbb{Z})$ on X is the abelian group consisting of pairs (ω, k), where

- A closed differential form $\omega \in \Omega^n_{\mathrm{clo}}(X)$,
- A group homomorphism $\varphi : Z_{\infty,n-1}(X;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,

such that for any $c \in C_{\infty,n}(X;\mathbb{Z})$ we have

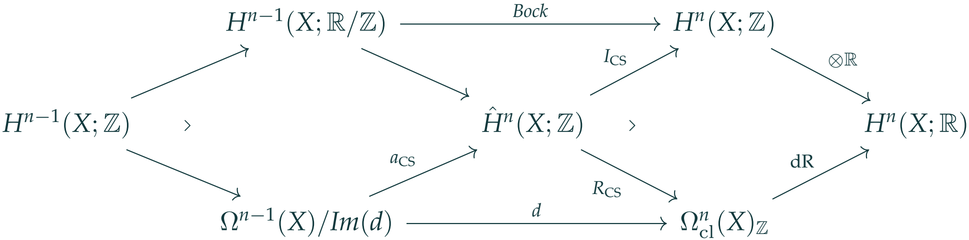
$$\varphi(\partial c) = \int_c \omega \mod \mathbb{Z}$$

Here $C_{\infty,n}(X;\mathbb{Z})$ and $Z_{\infty,n-1}(X;\mathbb{Z})$ denote the smooth singular chains and cycles with integer coefficients and automatically implies $\omega \in \Omega^n_{\mathrm{cl}}(X)_{\mathbb{Z}}$.

Theorem

We have an isomorphism $\hat{H}^2_{\mathrm{geom}}(X;\mathbb{Z}) \cong \hat{H}^2_{\mathrm{CS}}(X;\mathbb{Z})$ *given by* $[L,\nabla] \mapsto (c_1(F_\nabla), \mathrm{Hol}(L,\nabla))$.

Structure maps



$$R_{\mathrm{CS}} : \hat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) \to \Omega^n_{\mathrm{cl}}(X), \quad (\omega,\varphi) \mapsto \omega,$$
$$a_{\mathrm{CS}} : \Omega^{n-1}(X)/\mathrm{Im}(d) \to \hat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}), \quad \alpha \mapsto (d\alpha, \textstyle\int \alpha \mod \mathbb{Z}),$$
$$I_{\mathrm{CS}} : \hat{H}^n_{\mathrm{CS}}(X;\mathbb{Z}) \to \hat{H}^n_{\mathrm{CS}}(X;\mathbb{Z})/\mathrm{Im}(a_{\mathrm{CS}}) \simeq H^n(X;\mathbb{Z}), \quad (\omega,\varphi) \mapsto [\omega - \varphi_{\mathbb{R}} \circ \partial].$$

Exercise

Check: the well-defineness of I_{CS}.
$\hat{H}^0(X;\mathbb{Z}) \cong \mathbb{Z}$, $\hat{H}^1(X;\mathbb{Z}) \cong C^\infty(X;\mathbb{R}/\mathbb{Z})$, $\hat{H}^1(\bullet;\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, $\hat{H}^n(\bullet;\mathbb{Z}) \cong 0, n \geq 2$.

Chern-Simons invariant

An example of differential character is constructed from Chern-Simons invariants. The basic setting is let $(E, \nabla) \to X$ be a hermitian vector bundle with connection. For $f : M^3 \to X$ with M closed oriented, set

$$\begin{aligned} \mathrm{CS}(E, \nabla)(f : M \to X) := &\quad \mathrm{CS}(f^*E, f^*\nabla) \\ = &\quad \int_M f^* \operatorname{Tr}(\mathrm{d}A \wedge A + \tfrac{2}{3} A \wedge A \wedge A) (\operatorname{mod} Z). \end{aligned}$$

The second Chern character form is

$$\mathrm{ch}_2(F_A) = \operatorname{Tr}((\mathrm{d}\, A \wedge A + A \wedge A)^2) \in \Omega^4_{\mathrm{cl}}(X).$$

We get

$$(\mathrm{ch}_2(F_A), \mathrm{CS}(E, A)) \in \hat{H}^4(X; \mathbb{Z}).$$

Actually, the definition of the Chern-Simons invariants uses $\widehat{H\mathbb{Z}}^*$s

Differential integration

M^{n-1} closed oriented $(n-1)$-manifold. We have differential extension integration maps,

$$\int_M : \hat{H}^n(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \cong \hat{H}^1(M;\mathbb{Z}), \quad (\omega, \varphi) \mapsto \varphi(1_M)$$

Generally, for fiber bundle $p : N \to X$ with oriented fibers, we have

$$\int_{N/X} : \hat{H}^n(N;\mathbb{Z}) \to \hat{H}^{n-r}(X;\mathbb{Z}),$$

where $r = \dim N - \dim X$. Differential integration is a refinement of integrations in $H\mathbb{Z}^*$ and Ω^*.

Bordism formula

Suppose $(W^{2n}, \partial W)$ is a compact oriented manifold, for any $\hat{x} \in \hat{H}^n(W;\mathbb{Z})$, we have

$$\int_{\partial W} \hat{x}\mid_{\partial W} \equiv \int_W R(\hat{x}) \mod \mathbb{Z}$$

Digression: Chern-Simons form

The Chern character class is given by

$$\mathrm{Ch}(F_A) := [\mathrm{Tr}(e^{F_A/2\pi i})] \in H^{2\mathbb{Z}}(X),$$

which defines the topological Chern character:

$$\mathrm{Ch} : K^n(X) \to H^{2\mathbb{Z}+n}(X;\mathbb{R}) \cong H^n(X; K^*(\bullet) \otimes \mathbb{R})$$

If we have two connections A_0 and A_1, we must have $\mathrm{Ch}(F_{A_0}) - \mathrm{Ch}(F_{A_1}) \in \mathrm{im}(d)$.

Definition

The Chern-Simons form $\mathrm{CS}(A_0, A_1)$ for A_0 and A_1 as

$$\mathrm{CS}(F_{A_{0,1}}) := \int_{[0,1]} \mathrm{Ch}(F_{A_{0,1}}) \in \Omega^{2\mathbb{Z}-1}(X),$$

where $A_{0,1}$ is a homotopy between A_0 and A_1. The Chern-Simons form is

$$\mathrm{CS}(A_0, A_1) := [\mathrm{CS}(F_{A_{0,1}})] \in \Omega^{2\mathbb{Z}-1}(X)/\,\mathrm{im}(d)$$

we have the transgression formula

$$\mathrm{Ch}(F_{A_0}) - \mathrm{Ch}(F_{A_1}) = \mathrm{dCS}(A_0, A_1).$$

Differential K-theory

Roughly speaking, $\hat{K}^0(X)$ s a group of hermitian vector bundles with connections $[E, A] \in \hat{K}^0(X)$. We define structure maps

$$\begin{aligned} I &: \hat{K}^0(X) \to K^0(X), \quad [E, A] \mapsto [E] \\ R &: \hat{K}^0(X) \to \Omega^{2\mathbb{Z}}_{\mathrm{cl}}(X), \quad [E, A] \mapsto \mathrm{Ch}(F_A) \\ a &: \Omega^{2\mathbb{Z}-1}(X)/\mathrm{im}(\mathrm{d}) \to \hat{K}^0(X), \quad \mathrm{CS}(A_0, A_1) \mapsto [E, A_1] - [E, A_2]. \end{aligned}$$

$d = R \circ a$ follows by the transgression formula

$$\mathrm{Ch}(F_{A_0}) - \mathrm{Ch}(F_{A_1}) = \mathrm{dCS}(A_0, A_1).$$

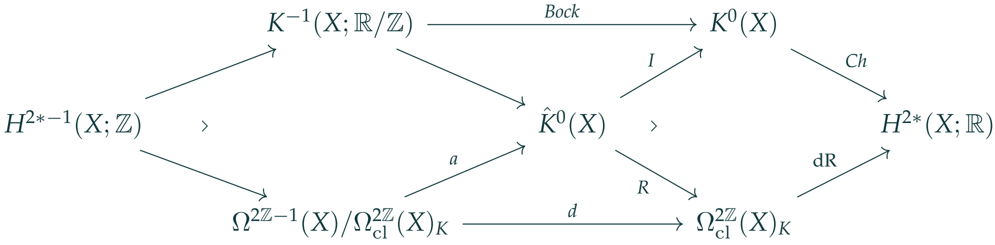
Definition (Freed-Lott)

Define $\widehat{\mathrm{Vect}}(X)$ to be the set of isomorphism classes of triples (E, A, α) where $\alpha \in \Omega^{2\mathbb{Z}-1}(X)/\,\mathrm{im}(d)$. Abelain monoid structure:

$$[E, A, \alpha] + [E', A', \alpha'] := [E \oplus E', A + A', \alpha + \alpha']$$

with relation $\sim$ on $\widehat{\mathrm{Vect}}(X)$: $[E, A, \alpha] \sim [E, A', \mathrm{CS}(A, A') + \alpha]$.
Define $\hat{K}^0(X)$ to be the associated Grothendieck group.

The hexagon for $\hat{K}^0(X)$



We define structure maps

$$
\begin{aligned}
R \quad &: \hat{K}^0(X) \to \Omega^{2\mathbb{Z}}_{\mathrm{cl}}(X), \quad [E, A, \alpha] \mapsto \mathrm{Ch}(F_A) + \mathrm{d}\alpha \\
a \quad &: \Omega^{2\mathbb{Z}-1}(X)/\mathrm{im}(\mathrm{d}) \to \hat{K}^0(X), \quad \alpha \mapsto [0, 0, \alpha].
\end{aligned}
$$

Similarly, one can define $\hat{K}^1(X)$ represented by $[E, A, U, \alpha]$, where U is a unitary automorphism on E. Set $\hat{K}^{2n} := \hat{K}^0$ and $\hat{K}^{2n-1} := \hat{K}^1$

Differential K-theory and index theory

For fiber bundles $p : N \to X$ whose fibers are $\hat{K}$-oriented, we have the differential integration map

$$p_* : \hat{K}^n(N) \to \hat{K}^{n-r}(X),$$

where $r = \dim N - \dim X$. In particular, when M^{2n-1} is a closed $\hat{K}$-oriented manifold, the differential integration along $M \to \bullet$ gives $p_* : \hat{K}^0(M) \to \hat{K}^{-2n+1}(\bullet) \cong \hat{K}^1(\bullet) \cong \mathbb{R}/\mathbb{Z}$.

Theorem

The differential integration map is given by

$$p_*[E, A, \alpha] = \bar{\eta}(D_{E,A}) + \int_M \alpha \wedge \mathrm{td}(M) \mod \mathbb{Z},$$

where the reduced eta invariant $\bar{\eta}(D_{E,A})$ is given by

$$\bar{\eta}(D_{E,A}) := \frac{1}{2}(\eta(D_{E,A}) + \dim\ker D_{E,A}) \in \mathbb{R}.$$

The bordism formula and the APS index theorem

Theorem (Atiyah-Patodi-Singer)

Suppose $(W^{2n}, \partial W)$ *a compact* $\hat{K}$*-oriented manifold.* E *Hermitian vector bundle with connection* A *on* W*. We have*

$$\operatorname{ind}(D_{E,A}) = \int_W \operatorname{Ch}(F_A) \wedge \operatorname{td}(W) - \bar{\eta}(D_{E,A} \mid_{\partial W})$$

Proposition (Bordism formula)

For any $\hat{x} \in \hat{K}^0(W)$, we have

$$(p_{\partial W})_* \hat{x} \mid_{\partial W} \equiv \int_W R(\hat{x}) \wedge \operatorname{td}(W).$$

Proof.

First assume $\hat{x} = [E, A, 0]$, then use Stokes' theorem. □

Thank you!