You Could've Invented Cotangent Complexes

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Kähler Differential Recollection

Differential Geometry	Algebraic Geometry
Euclidean spaces	affine Schemes
manifolds/orbifolds/$\cdots$,etc	schemes/stacks/$\cdots$,etc
vector bundles	quasi-coherent sheaves
$\cdots$	$\cdots$

Kähler Differential Recollection

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differential forms	Kähler differentials

Kähler Differential Recollection

Definition

Let A be a commutative ring and let M be an A-module. A **derivation** from A into M is a map $d : A \to M$ satisfying conditions

$$d(x + y) = d(x) + d(y) \quad d(xy) = xdy + ydx.$$

The collection of derivations of A into M forms an abelian group, which we will denote by $Der(A, M)$.

If A is fixed, then the functor $M \mapsto Der(A, M)$ is corepresented by the A-module Ω_A, called the A-module of **absolute Kähler differentials**. Note that there exists a universal A-linear derivation from A into Ω_A.

Kähler Differential Recollection

Let $\phi : \tilde{R} \to R$ be a **square-zero extension** of a commutative ring, that is, a surjective ring homomorphism such that $ker(\phi)^2 = 0$ as an ideal of $\tilde{R}$. In this case, the kernel $M = ker(\phi)$ inherits the structure on R-module. Then there exists a ring homomorphism

$$(R \oplus M) \times_R \tilde{R} \to \tilde{R},$$

given by the formula

$$(r, m, \tilde{r}) \mapsto \tilde{r} + m.$$

Consequently, in some sense square-zero extensions of R by M can be viewed as **torsors** for the **trivial square-zero extension** $R \oplus M$.

Kähler Differential Recollection

In general, if $\phi : \tilde{R} \to R$ is a square-zero extension of R by $M \simeq ker(\phi)$, we say that $\tilde{R}$ is trivial if ϕ admits a section. We have a bijection

$$\{\text{sections of } \phi\} \xrightarrow{\sim} \{\text{isomorphisms } \tilde{R} \xrightarrow{\sim} R \oplus M\}.$$

Warning: Can be empty!

Kähler Differential Recollection

If there were sections, then any two sections of ϕ differs by a derivation from R into M, which is classified by an R-linear map $\Omega_R \to M$. Consequently, we have an isomorphism

$$\mathrm{Aut}(\phi) \xrightarrow{\sim} \mathrm{Ext}^0_R(\Omega_R, M).$$

Furthermore, we have

$$\{\text{sections of } \phi\}/\text{isomorphism} \xrightarrow{\sim} \mathrm{Ext}^1_R(\Omega_R, M).$$

Indeed, given an element $\eta \in \mathrm{Ext}^1_R(\Omega_R, M)$, one can construct a square-zero extension $\tilde{R} \to R$ (by M) as follows:

Kähler Differential Recollection

Construction

Unwinding definitions, the element $\eta \in \mathrm{Ext}^1_R(\Omega_R, M)$ *determines a short exact sequence of* R*-modules*

$$0 \to M \to \tilde{M} \to \Omega_R \to 0.$$

Now pulling back along the universal R*-linear derivation gives us the short exact sequence*

$$0 \to M \to \tilde{R} \to R \to 0$$

of abelian groups, where $\tilde{R} = \tilde{M} \times_{\Omega_R} R$*.*

Exercise

Define a natural multiplication on $\tilde{R}$ such that $\tilde{R}$ is a square-zero extension of R by M.

Kähler Differential Recollection

We can also consider the **relative Kähler differentials** $\Omega_{B/A}$ associated with a ring map $A \to B$. Moreover, given a sequence of commutative ring homomorphisms $A \to B \to C$, there exists an associated short exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B}.$$

Modules as Tangent Vectors

The module of Kähler differentials has a higher analogue, which we call the **cotangent complex**. But before we dive in, let's first look at a prototypical example.

Definition

Let CRing be the 1-category of commutative rings. A **Beck Module** over a commutative ring R is an abelian group object in the slice category $\mathrm{CRing}_{/R}$. Beck modules over R form an abelian category.

Modules as Tangent Vectors

One may think of CRing as a "space" and morphisms in CRing as paths. Then taking a Beck module over a ring is analogous to taking a tangent vector of a path at the target.

Remark (For Geometers)

Would it make you feel better if we look in the opposite direction–that is, **affine schemes**?

Modules as Tangent Vectors

Proposition

For any commutative ring R there exists a canonical equivalence of abelian categories

$$\mathrm{Mod}_R \xrightarrow{\sim} \mathrm{Ab}(\mathrm{CRing}_{/R}).$$

Moreover, these are assembled into a category T_{CRing} ***fibered over*** CRing, *which we call the* ***tangent bundle*** *over* CRing.

In higher category theory, we replace abelian groups by **spectra**. Thus, the tangent bundle over an ∞-category can be defined as a family of **stabilizations**.

Modules as Tangent Vectors

Definition

Let $\mathcal{C}$ be a presentable ∞-category. A **tangent bundle** over $\mathcal{C}$ is an ∞-category $T_{\mathcal{C}}$ equipped with a map $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ such that

- the composite $p : T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a **biCartesian fibration**, where the latter map is given by evaluating at $\{1\}$, and
- for each object $X \in \mathcal{C}$, the induced functor $\Omega^\infty : T_{\mathcal{C}} \times_{\mathcal{C}} \{X\} \to \mathcal{C}_{/X}$ in Pr^R exhibits the source as a **stabilization** of the target.

In this case, the tangent bundle $T_{\mathcal{C}}$ exists and is unique up to equivalence over $\mathcal{C}$.

Modules as Tangent Vectors

One important fact is that, the tangent bundle respects monoidal structures on the underlying category $\mathcal{C}$.

Proposition (HA 7.3.1.15)

Let $\mathcal{C}$ be a presentable ∞-category equipped with an $\mathbb{E}_k$-monoidal structure, such that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves sequential colimits. Then the tangent bundle $T_{\mathcal{C}}$ inherits an $\mathbb{E}_k$-monoidal structure.

Modules as Tangent Vectors

As in ordinary algebra, we may again identify modules with tangent vectors.

Theorem

Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a stable $\mathcal{O}$-monoidal ∞-category, and let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ be an $\mathcal{O}$-algebra object of $\mathcal{C}$. Then there is a canonical equivalence of ∞-categories

$$\mathrm{Sp}\left(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}\right) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{O}}\left(\mathcal{O}, \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})\right)$$

Corollary

Let A be an $\mathbb{E}_\infty$-ring. There exists a canonical equivalence of ∞-categories

$$\mathrm{Sp}\left(\mathrm{CAlg}_{/A}\right) \xrightarrow{\sim} \mathrm{Mod}_A$$

Cotangent Complex

Now we are ready to give the definition of the **cotangent complex** construction.

Definition

Let $\mathcal{C}$ be a presentable ∞-category. Then the tangent bundle $p : T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to T_{\mathcal{C}}$. For any object $A \in \mathcal{C}$, we denote the image under L in $\mathrm{Sp}(\mathcal{C}_{/A})$ by L_A, and call it the **absolute cotangent complex** associated with A.

Cotangent Complex

we can also define the relative version of cotangent complex.

Definition

In the above situation, let $A \to B$ be a map in $\mathcal{C}$. Then, there exists a canonical coCartesian square

$$\begin{array}{ccc} L_A & \longrightarrow & L_B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & L_{B/A} \end{array}$$

in $T_{\mathcal{C}}$ such that the objects on each vertical arrow project on to A and B, respectively. We say $L_{B/A}$ is the **relative cotangent complex** associated with the map $A \to B$, regarded as an object in $\mathrm{Sp}(\mathcal{C}_{/B})$.

Cotangent Complex

Remark

Suppose $A \simeq \emptyset$ is an initial object. Then $L_A \simeq 0 \in T_{\mathcal{C}}$ and the canonical map $L_B \to L_{B/A}$ in $\mathrm{Sp}(\mathcal{C}_{/B})$ is an equivalence.

Proposition

Given a sequence of maps $A \to B \xrightarrow{f} C$ in $\mathcal{C}$, there exists an associated fiber sequence

$$f^* L_{B/A} \to L_{C/A} \to L_{C/B}.$$

Here, $f^* : \mathrm{Sp}(\mathcal{C}_{/B}) \to \mathrm{Sp}(\mathcal{C}_{/C})$ is the functor induced by post-composing with f.

Digression: Operadic Modules

Let $\mathcal{C}$ be a presentably stable $\mathcal{O}$-monoidal ∞-category. According to the last section, we can think of the cotangent complex L_A associated with $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ as an object in $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$.

If $\mathcal{O} \simeq \mathbb{E}_\infty$, $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ is simply the stable symmetric monoidal ∞-category $\mathrm{Mod}_A(\mathcal{C})$. However, for general $\mathcal{O}$, the ∞-operad $\mathrm{Mod}_{\mathcal{O}}^A(\mathcal{C})$ is mysterious.

Digression: Operadic Modules

Fortunately, we can still say something concrete for $\mathcal{O} \simeq \mathbb{E}_{\mathsf{k}}$.

Proposition (HA 5.1.3.2)

Let $\mathcal{C}$ be a $\mathbb{E}_{\mathsf{k}}$-monoidal ∞-category where geometric realizations exist and are preserved by tensor products. Let A be an $\mathbb{E}_{\mathsf{k}}$-algebra in $\mathcal{C}$. Then $\mathrm{Mod}_A^{\mathbb{E}_{\mathsf{k}}}(\mathcal{C})$ is an $\mathbb{E}_{\mathsf{k}}$-monoidal ∞-category.

Digression: Operadic Modules

We can think of an object in $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ as an object in $\mathcal{C}$ equipped with a family of actions of A parametrized by rays in $\mathbb{R}^k$ startting from the origin, hence a single left action by the **factorization homolgy** $\int_{S^{k-1}} A$, which is the "free $\mathbb{E}_k$-A-module" on the unit $\mathbf{1} \in \mathcal{C}$.

Example

- When $k = 1$, the monoidal ∞-category $\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})$ is equivalent to the category of A-A-bimodules ${}_A\mathrm{BMod}_A(\mathcal{C})$, which can also be identified with $\mathrm{LMod}_{A \otimes A^{\mathrm{op}}}(\mathcal{C})$.
- When $k = 2$, $\mathrm{Mod}_A^{\mathbb{E}_2}(\mathcal{C})$ is a braided monoidal category and is equivalent to $\mathrm{LMod}_{\mathrm{HH}(A/\mathcal{C})}(\mathcal{C})$. Note that when $\mathcal{C} \simeq \mathrm{Sp}$, we usually write $\mathrm{HH}(-/\mathcal{C})$ as $\mathrm{THH}(-)$ and call it the **topological Hochschild homology**.

Digression: Operadic Modules

Let R be an commutative k-algebra over some field k. Denote the kernel of the multiplication $R \otimes R \to R$ by I. Then there exists a canonical isomorphism of R-modules $I/I^2 \xrightarrow{\sim} \Omega_{R/k}$. The following result can be viewed as a generalization of the above statement.

Theorem (HA 7.3.5.1)

Let $\mathcal{C}^{\otimes}$ be a presentably stable symmetric monoidal ∞-category and let $k \geq 0$. For every $\mathbb{E}_k$-algebra object $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, there is a canonical fiber sequence

$$\int_{S^{k-1}} A \to A \to \Sigma^k L_A$$

in the stable ∞-category $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$.

Derivations

Let R be a commutative ring. Recall that we assign to each extension class $\eta \in \mathrm{Ext}^1_R(\Omega_R, M)$ a square-zero extension $\tilde{R} \to R$ of R by M.
As homotopy theorists, we'd love to write

$$\mathrm{Ext}^1_R(\Omega_R, M) \simeq \pi_0(\mathrm{Map}_{\mathrm{Mod}_R}(\Omega_R, \Sigma M)).$$

Under this interpretation, an extension class is represented by a map $\Omega_R \to \Sigma M$ in Mod_R. Note that the zero map corresponds to the trivial square-zero extension $R \oplus M$.

Derivations

The goal of this section:

- define the notion of derivations for higher algebra, as well as square-zero extensions
- describe the relationship between them

In fact, we can do this for arbitrary presentable $\mathcal{C}$(which we still refer to as algebras) with $T_{\mathcal{C}}$(which we call modules).

Derivations

The idea is simple, and is the same as the example we gave for ordinary algebra. First, we define a derivation to be a module map of the form $\eta : L_A \to M$. Given such a derivation, we may construct a algebra map as follows:

- find the extension of modules classified by η,
- pullback along the universal derivation and
- write down the multiplicative structure on the module obtained in this way

Derivations

The idea is simple, and is the same as the example we gave for ordinary algebra. First, we define a derivation to be a module map of the form $\eta : L_A \to M$. Given such a derivation, we may construct a algebra map as follows:

- find the extension of modules classified by η,
- pullback along the universal derivation and
- write down the multiplicative structure on the module obtained in this way (impossible!!!)

In higher algebra, we cannot write down all data for the ring structure as they are infinite.

Derivations

One solution: replace the cotangent complex functor L with the inclusion into the "mapping cylinder".

Definition

Let $\mathcal{C}$ be a presentabe ∞-category. Then there exists a coCartesian fibration

$$q : \mathcal{M} \to \Delta^1$$

with $\mathcal{M} \times_{\Delta^1} \{0\} \simeq \mathcal{C}$ and $\mathcal{M} \times_{\Delta^1} \{1\} \simeq T_{\mathcal{C}}$ such that the corresponding functor can be identified with the cotangent complex $L : \mathcal{C} \to T_{\mathcal{C}}$. We will refer to $\mathcal{M}^T(\mathcal{C}) := \mathcal{M}$ as a **tangent correspondence** to $\mathcal{C}$.

Derivations

Definition

A **derivation** from A into M is a map $\eta : A \to M$ in the tangent correspondence $\mathcal{M}^T(\mathcal{C})$, where $A \in \mathcal{C} \subseteq \mathcal{M}^T(\mathcal{C})$ and $M \in T_{\mathcal{C}} \times_{\mathcal{C}} \{A\} \simeq \mathrm{Sp}(\mathcal{C}_{/A})$. We denote the ∞-category of derivations in $\mathcal{C}$ by $\mathrm{Der}(\mathcal{C})$.

Remark

Let $L : \mathcal{C} \to T_{\mathcal{C}}$ be the cotangent complex functor. A derivation $\eta : A \to M$ can be identified with a map $d : L_A \to M$ in $\mathrm{Sp}(\mathcal{C}_{/A})$. We will abuse of terminology by identifying η with d, and call d a derivation from A into M.

Square-zero Extensions

We may assign to each derivation $\eta : A \to M$ a map $f : A^\eta \to A$ in $\mathcal{C}$, such that f fits into a Cartesian diagram in $\mathcal{M}^T(\mathcal{C})$:

$$\begin{array}{ccc} A^\eta & \xrightarrow{f} & A \\ \downarrow & \lrcorner & \downarrow \eta \\ 0_A & \longrightarrow & M \end{array} .$$

Here 0_A is a zero object in the fiber $\mathrm{Sp}(\mathcal{C}_{/A})$. Moreover, this assignment assembles to a functor $\Phi : \mathrm{Der}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. In this case, the map f being a map of "algebras" is by construction!!!

Square-Zero Extensions

Definition

We say a map $f : \tilde{A} \to A$ in $\mathcal{C}$ a **square-zero extension** if there exists a derivation $\eta : A \to M$ together with an equivalence $\tilde{A} \xrightarrow{\sim} A^{\eta}$ over A. In this case, we also say that $\tilde{A} \to A$ is a square-zero extension by $\Sigma^{-1}M$.

Exercise:explain to yourself why there is a degree shifting.

Square-zero Extensions

The name "square-zero extension" seems abusive, as we simply define it as given by a derivation. However, the name will justfied by the rest of the talk. We first show that these square-zero extensions are really "square-zero":

Proposition (HA 7.4.1.14)

Let $\mathcal{C}$ be a presentably stable monoidal ∞-category. Let $f : A^\eta \to A$ be a square-zero extension in $\mathrm{Alg}(\mathcal{C})$, and let I denote the fiber of f. Then the multiplication map

$$\theta : I \otimes_{A^\eta} I \to I$$

is nullhomotopic (as a map of A^η-bimodules).

Square-zero Extensions

Remark

Let $\eta : L_A \to M$ be a derivation. Let $A \oplus M$ denote the image of M under $\Omega^\infty : \mathrm{Sp}(\mathcal{C}_{/A}) \to \mathcal{C}$. There exists a Cartesian square in $\mathcal{C}$ as follows:

$$\begin{array}{ccc} A^\eta & \longrightarrow & A \\ \big\downarrow & \lrcorner & \big\downarrow d_\eta \\ A & \xrightarrow{d_0} & A \oplus M \end{array} .$$

Square-zero Extensions

Remark (continue)

$$\begin{array}{ccc} A^{\eta} & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow d_{\eta} \\ A & \xrightarrow{d_0} & A \oplus M \end{array} .$$

Here we identify d_0 with the map associated with the zero derivation $L_A \to M$. Given any map $\phi : B \to A$ in $\mathcal{C}$, it determines a map $\eta' : \phi^* L_B \to L_A \to M$ in $\mathrm{Sp}(\mathcal{C}_{/A})$. It follows that the anima

$$\mathrm{Map}_{\mathcal{C}}(B, A^{\eta}) \times_{\mathrm{Map}_{\mathcal{C}}(B,A)} \{\phi\}$$

of lifts of ϕ to A^{η} is equivalent to the anima of homotopies from 0 to η' in $\mathrm{Sp}(\mathcal{C}_{/A})$.

Small Extensions

In this section, we fix a presentably stable $\mathbb{E}_k$-monoidal ∞-category $\mathcal{C}$ with a t-structure compatible with the monoidal structure. Our goal is to study the square-zero extensions of $\mathbb{E}_k$-algebras in this category.

Notation:Let $\mathcal{D} \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ denote the category of $\mathbb{E}_k$-algebras. Remember, we have made an identification $\mathrm{Sp}(\mathcal{D}_{/A}) \simeq \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$, and we can talk about t-structure homotopy groups as we always do in Sp!

Small Extensions

Recall that we have a functor $\Phi : \mathrm{Der}(\mathcal{D}) \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ sending an derivation $\eta : A \to M$ to a map $f : A^\eta \to A$ of $\mathbb{E}_k$-algebras in $\mathcal{C}$.

Warning: The functor Φ is neither full nor faithful!!! Given a square-zero extension

$$f : \tilde{A} \to A,$$

there exists a derivation $\eta : A \to M$ and an equivalence $\tilde{A} \xrightarrow{\sim} A^\eta$. However, neither η nor M are determined uniquely by f.

However, we can define subcategories on both sides, where the restrictions on each side induce equivalences. Moreover, each subcategorie has a relatively simple characterization.

Small Extensions

Definition

Let $f\colon A \to B$ be a map of $\mathbb{E}_k$-algebras in $\mathcal{C}$ and let $n \geq 0$. We say that f is an *n*-**small extension** if:

- $A \in \mathcal{C}_{\geq 0}$,
- $\mathrm{fib}(f) \in \mathcal{C}_{\geq n, \leq 2n}$ and moreover,
- the multiplication map $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to \mathrm{fib}(f)$ is nullhomotopic.

We denote by $\mathrm{Fun}_{n-\mathrm{sm}}(\Delta^1, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{D})$ spanned by the *n*-small extensions.

Small Extensions

Remark

Given a map f such that all but the last conditions are satisfied. Then

$$\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to \mathrm{fib}(f)$$

being nullhomotopic is equivalent to the vanishing of

$$\pi_n(\mathrm{fib}(f)) \otimes \pi_n(\mathrm{fib}(f)) \to \pi_{2n}(\mathrm{fib}(f)).$$

It follows that all conditions listed above can be determined once the homotopy groups are known.

Small Extensions

Also, we consider subcategories on the right-hand side.

Definition

We denote $\mathrm{Der}_{n-\mathrm{sm}}(\mathcal{D})$ the full subcategory of $\mathrm{Der}(\mathcal{D})$ spanned by those pairs $(A, \eta\colon L_A \to \Sigma M)$ such that:

- $A \in \mathcal{C}_{\geq 0}$ and
- $M \in \mathcal{C}_{\geq n, \leq 2n}$.

Small Extensions

Now we are ready to state our main story:

Theorem (HA 7.4.1.26)

For each $n \geq 0$, the functor $\Phi : \mathrm{Der}(\mathcal{D}) \to \mathsf{Fun}(\Delta^1, \mathcal{D})$ induces an equivalence of ∞-categories

$$\Phi^{n-\mathrm{sm}} \colon \mathrm{Der}^{n-\mathrm{sm}}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{n-\mathrm{sm}}(\Delta^1, \mathcal{D}).$$

It follows that, if we only focus on an extension given by a "small" object (for example, concentrated in a single degree), then it is necessarily uniquely given by a derivation!

Postnikov Tower

Let us describe an application of the previous result, which is extremely useful in higher algebra.

Corollary

Let $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$ *be a connective* $\mathbb{E}_k$*-algebra. Then each arrow in the Postnikov tower*

$$\cdots \to \tau_{\leq 3}A \to \tau_{\leq 2}A \to \tau_{\leq 1}A \to \tau_{\leq 0}A$$

is a square-zero extension, that is, by definition, classified by a derivation.

Postnikov Tower

As the corollary suggests, for an $\mathbb{E}_k$-algebra B there exists a Cartesian square

$$\begin{array}{ccc} B_{\tau \leq n} & \longrightarrow & B_{\tau \leq n-1} \\ \downarrow & \lrcorner & \downarrow \\ B_{\tau \leq n-1} & \longrightarrow & B_{\tau \leq n-1} \oplus \Sigma^{n+1}\pi_n(B) \end{array}$$

It follows that studying $\mathbb{E}_k$-algebra maps $\mathrm{Map}(A, B)$ can be reduced to studying the set $\mathrm{Hom}(\pi_0(A), \pi_0(B))$ and the groups $\mathrm{Ext}^{n+1}(L_A, \pi_n B)$ representing "linear problems" for all $n \geq 0$.

Étale Morphism

In the last section, we briefly state an application of this theory. In HA 7.5, Lurie generalized the notion of étale morphism to higher algebra.

Definition

Let $f : A \to B$ be a map of $\mathbb{E}_k$-ring spectra, $k \geq 2$. We say f is étale if the following conditions hold:

- the induced map $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ in $\mathrm{CAlg}^\heartsuit$ is étale in the ordinary sense
- the canonical maps $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$ in $\mathrm{Ab} \simeq \mathrm{Sp}^\heartsuit$ are isomorphisms for all $n \geq 0$

Étale Morphism

Theorem (HA 7.5.0.6)

Let $2 \leq k \leq \infty$, let R be a $\mathbb{E}_{k+1}$-ring, and let A be a $\mathbb{E}_k$-algebra over R. Let $(\mathrm{Alg}_R^{(k)})^{\acute{e}t}_{A/}$ denote the full subcategory of $(\mathrm{Alg}_R^{(k)})_{A/}$ spanned by the étale morphisms. Then the construction $B \mapsto \pi_0(B)$ induces an equivalence of ∞-categories:

$$(\mathrm{Alg}_R^{(k)})^{\acute{e}t}_{A/} \xrightarrow{\sim} \mathrm{CAlg}^{\heartsuit,\acute{e}t}_{\pi_0(A)}.$$

Here, the right-hand side denotes (the nerve of) the category of ordinary étale $\pi_0(A)$-algebras.