Kerz-Strunk-Tamme's vanishing theorem on K-theory

Yunhao Sun

Southern University of Science and Techonolgy

June 18, 2024

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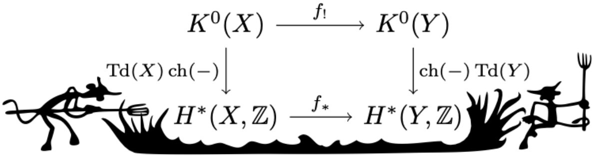
A brief introduction to algebraic K-theory

The notion of K-theory first appeared in Grothendieck's famous work on Riemann-Roch theorem:



Where $f : X \to Y$ is a proper morphism of complex manifolds. It induces a morphism $f_! : \sum(-1)^i R^i f_* : K^0(X) \to K^0(Y)$ on K-groups. The bottom line is the push forward in cohomology.

Definition

Let A be a ring.

$$K_0(A) := \mathbb{Z}[\text{iso. classes of f.g. proj } A\text{-mod}]/(0 \to L \to M \to N \to 0)$$

For a f.g. A-module M, we denote by $[M]$ its class in $K_0(A)$. If M sits in a short exact sequence

$$0 \to L \to M \to N \to 0.$$

Then $[M] = [N] + [L]$. This is the prototype of the universal property of K-theory. We assume A is commutative for a moment. Recall that the Chern class functor ch satisfies $ch(M) = ch(L) + ch(N)$. One can show that the functor ch factors through K_0. Further, we can prove Grothendieck's Riemann-Roch theorem by the universal property of K-theory.

Let A be a ring. The set $P(A)$ of isomorphism classes of finitely generated projective A-modules, together with direct sum $\oplus$ and identity 0, forms an abelian monoid. The group $K_0(A)$ can be defined as the group completion of $P(A)$ equivalently.

We list some calculations of K_0:

Example

(1) For a PID A, $K_0(A) = \mathbb{Z}$.

(2) For a Dedekind domain A, $K_0(A) = \mathbb{Z} \oplus Cl(A)$ where $Cl(A)$ is the class group of A.

(3) Let X be a 1-dimensional separated regular noetherian scheme. Then $K_0(X) = H^0_{Zar}(X; \mathbb{Z}) \oplus \mathrm{Pic}(X)$.

Theorem (théorème dévissage)

If I is a nilpotent ideal of A, then $K_0(A) \cong K_0(A/I)$.

Proof.

Every A-module M has a filtration: $M = M_0 \supset IM \supset \cdots \supset I^n M = 0$. Then $[M] = \sum_{i=0}^{n-1}[M_i/M_{i+1}] \in K_0(A)$, where $M_i = I^i M$. This follows by $[M_i] = [M_{i+1}] + [M_i/M_{i+1}]$. So $K_0(A/I) \to K_0(A)$ is onto. Given a short exact sequence $0 \to M' \to M \to M'' \to 0$. We may construct a filtration $\{M_i\}$ on M by combining filtrations on M' and M''. □

As a corollary, K_0 is nil-invariant, i.e. $K_0(A_{red}) = K_0(A)$.

Let A be a commutative ring with unit. Identifying each $n \times n$ matrix g with the larger matrix $\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$ gives an embedding of $GL_n(A)$ into $GL_{n+1}(A)$. The union of the resulting sequence

$$GL_1(A) \subset GL_2(A) \subset \cdots \subset GL_n(A) \subset GL_{n+1}(A) \subset \dots$$

is called the infinite general linear group $GL(A)$.

Definition

$K_1(A)$ is the abelian group $GL(A)/[GL(A), GL(A)]$.

Lemma

Whitehead taught us $K_1(A) = GL(A)/E(A)$.

Example

The determinant of a matrix provides a group homomorphism from $GL(A)$ to $A^\times$. We writ $SK_1(A)$ for the kernel of the induced surjection $det : K_1(A) \to A^\times$. Since the natural inclusion of the units $A^\times$ in $GL(A)$ as $GL_1(A)$ is split by the homomorphism $det : GL(A) \to A^\times$, we see that $GL(A) \cong SL(A) \rtimes A^\times$, and there is a direct sum decomposition: $K_1(A) = A^\times \oplus SK_1(A)$.
If k is a field, then $K_1(k) = k^\times$. $K_1(\mathbb{Z}) = \mathbb{Z}^\times$.

Example (Bass-Milnor-Serre)

If k is a number field and A is an integrally closed subring of k. Then $SK_1(A) = 1$, so $K_1(A) = A^\times$.

Example

$K_1(k) = k^\times \neq (k[x]/(x^2))^\times = K_1(k[x]/(x^2))$.

Definition

For $n > 0$, we inductively define $K_{-n}(A)$ to be the cokernel of the map

$$K_{-n+1}(A[t]) \oplus K_{-n+1}(A[t^{-1}]) \to K_{-n+1}(A[t, t^{-1}]).$$

Theorem

If A is a regular noetherian ring, then $K_0(A) \cong K_0(A[t]) \cong K_0(A[t, t^{-1}])$.

Corollary

If A is regular noetherian, then $K_n(A) = 0$ for all $n < 0$.

Theorem (Bass-Heller-Swan)

$K_0(A) = coker(K_1(A[t]) \oplus K_1(A[t^{-1}]) \to K_1(A[t, t^{-1}]))$.

Fix a regular ring A, and let $\Delta^n(A)$ denote the coordinate ring $A[t_0, \ldots, t_n]/(f)$, $f = t_0 \ldots t_n(1 - \sum t_i)$, of the n-dimensional tetrahedron over A. Using $I = (1 - \sum t_i)\Delta^n(A)$ and $\Delta^n(A)/I \cong A[t_1, \ldots, t_n]$ via $t_0 \mapsto 1 - (t_1 + \cdots + t_n)$, we have a pullback square of rings

$$\begin{array}{ccc} \Delta^n(A) & \longrightarrow & A_n \\ \downarrow & & \downarrow \\ A[t_1, \ldots, t_n] & \longrightarrow & \Delta^{n-1}(A) \end{array}$$

where $A_n = A[t_0, \ldots, t_n]/(t_0 \ldots t_n)$. The negative K-groups of A_n vanish and $K_i(A_n) = K_i(A)$ for $i = 0, 1$. Thus $K_0(\Delta^n(A)) \cong K_0(A) \oplus K_1(\Delta^{n-1}(A))/K_1(A)$ for $n \geq 0$, and $K_{-j}(\Delta^n(A)) \cong K_{1-j}(\Delta^{n-1}(A))$ for $j > 0$. These groups vanish for $j > n$, with $K_{-n}(\Delta^n(A)) \cong K_0(A)$. If k is a field, then $\Delta^n(k)$ is an n-dimensional noetherian ring with $K_{-n}(\Delta^n(k)) \cong \mathbb{Z}$.

Let A be a commutative noetherian ring of Krull dimension d. Charles A. Weibel conjectured that $K_{-j}(A)$ vanishes for all $j > d$ (1980).
In 2017, Kerz, Strunk and Tamme proved Weibel's conjecture and more:

Theorem

For a noetherian scheme X of dimension $d < \infty$ the following hold.
(i) For $i < -d$ we have $K_i^{TT}(X) = 0$.
(ii) For $i \leq -d$ and any integer $r \geq 0$ the map

$$K_i^{TT}(X) \to K_i^{TT}(\mathbb{A}^r_X)$$

is an isomorphism.

Here K^{TT} is Bass-Thomason-Trobaugh K-theory. In algebraic geometry, we have two K-theories: Quillen K-theory and Thomason K-theory. These two theories are agree for qcqs schemes such that every coherent sheaf is a quotient of a vector bundle.

Definition of higher algebraic K-theory

In early time, people had defined K_1, K_2, K_3 in an ad hoc way. It was difficult to define general K_n. In 1969, Quillen proposed defining the higher K-theory of a ring A to be the homotopy groups of a certain topological space, whcih he called $BGL(A)^+$.

Definition

The notation $BGL(A)^+$ will denote any CW complex X which has a distinguished map $BGL(A) \to BGL(A)^+$ such that the following are true:
(1) $\pi_1 BGL(A)^+ \cong K_1(A)$, and the natural map from $GL(A) = \pi_1 BGL(A)$ to $\pi_1 BGL(A)^+$ is onto with kernel $E(A)$.
(2) $H_*(BGL(A); M) \xrightarrow{\sim} H_*(BGL(A)^+; M)$ for every $K_1(A)$-module M.

For $n \geq 1$, $K_n(A)$ is defined to be the homotopy group $\pi_n BGL(A)^+$.

Definition (K-theory space)

Write $K(A)$ for the product $K_0(A) \times BGL(A)^+$. That is, $K(A)$ is the disjoint union of copies of the connected space $BGL(A)^+$, one for each element of $K_0(A)$.

Let Spc be the category of CW complexes, and $Spc^{hyp} \subset Spc$ be the full sub-category of hypoabelian spaces, i.e. those X for which $\pi_1(X, x)$ has no perfect subgroup except the trivial group $\{e\}$ for every base point $x \in X$.

Proposition (Kervaire, Quillen)

The inclusion $Spc^{hyp} \subset Spc$ admits a left adjoint $(-)^+ : Spc \to Spc^{hyp}$.

This is the so called plus construction. Quillen proved that $BGL(A)^+$ is an infinite loop space and extends to an Ω-spectrum $\mathbb{K}(A)$.

Corollary

One can show that $K_2(A) \cong H_2(E(A); \mathbb{Z})$ and $K_3(A) \cong H_3(St(A); \mathbb{Z})$.

Proposition

For any ring A and all $i \geq 1$, there are a canonical isomorphisms

$$K_i(A) \otimes \mathbb{Q} \cong colim_n H_i(GL_n(A); \mathbb{Q}).$$

Theorem (Borel)

If $\mathcal{O}_K$ is the ring of integers in a number field K, then the rational algebraic K-groups of $\mathcal{O}_K$ are given by

$$K_i(\mathcal{O}_K) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{if } i = 1 \text{ or } i > 0 \text{ even} \\ \mathbb{Q}^{r+s} & \text{if } i \equiv 1 \text{ mod } 4 \text{ and } i > 1 \\ \mathbb{Q}^s & \text{if } i \equiv 3 \text{ mod } 4 \end{cases}$$

where r and s are the number of real and complex embeddings of k, respectively.

Borel's calculation is pretty cray. He used the theory of arithmetic groups.

Quillen determined all K-groups of finite fields. Let $\mathbb{F}_q$ be a finite field with q some power of a prime number p.

Theorem (Quillen)

Any embedding $\rho : \overline{\mathbb{F}}_p^\times \to \mathbb{C}^\times$ as the group $\bigoplus_{\ell \neq p} \mu_{\ell^\infty}$ of all roots of unity of order coprime to p gives an equivalence

$$Br^\rho : K(\mathbb{F}_q) \xrightarrow{\sim} hofib(\psi^q - id : BU \to BU).$$

Consequently,

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/(q^n - 1) & \text{if } i = 2n - 1 \\ 0 & \text{else} \end{cases}$$

Quillen and Lichtenbaum conjectured that if k is algebraically closed and char$k \neq \ell$, $K_*(k)/\ell^v$ should be the same as ordinary topological K-theory. Their conjecture was proved by Suslin:

Theorem

(1) Let $i : k \subset l$ be an extension of algebraically closed fields. Then $i_* : K_*(k)/n \to K_*(l)/n$ is an isomorphism for all n.
(2) The natural map $K(\mathbb{C})^\wedge_\ell \to ku^\wedge_\ell$ is an equivalence.

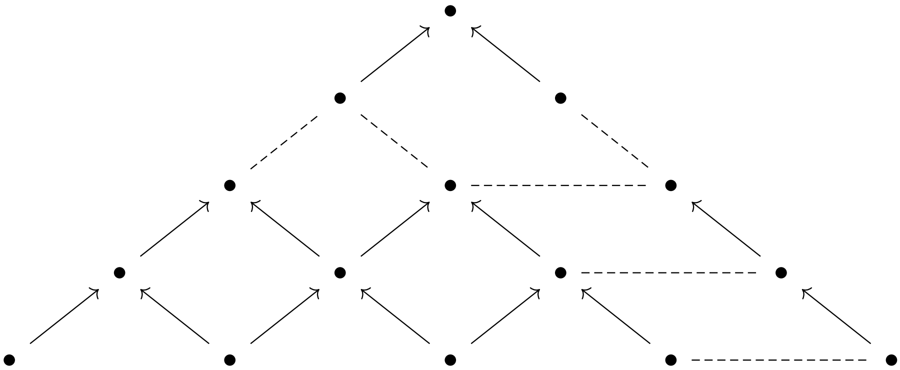
It is a difficult problem to calculate algebraic K-groups. For example, Vandiver's conjecture(still open) on class number is equivalent to $K_{4i}(\mathbb{Z}) = 0, i \geq 1$. Voevodsky-Rost theorem implies $K_{4k+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2^{(k+1)\text{mod}2}\mathbb{Z}$. The group $K_1(\mathbb{Z}[\pi_1 M])$ of a smooth compact manifold M is connected with deep problems in geometry topology.

Quillen gave another categorical construction of higher algebraic K-theory soon after his plus construction.

Definition

Let $\mathcal{C}$ be a category. The twisted arrow category $\mathrm{TwAr}(\mathcal{C})$ is defined by $\mathrm{TwAr}(\mathcal{C})_n := \mathrm{Hom}((\Delta^n)^{op} \star \Delta^n, \mathcal{C})$.

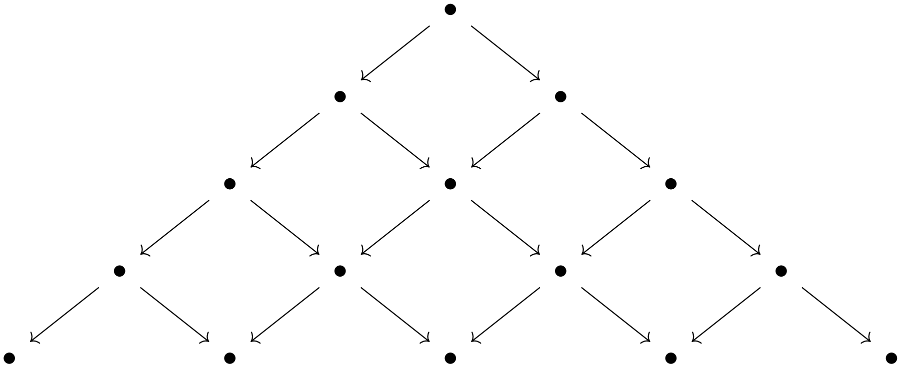
If we draw in some of the edges in $\mathrm{TwAr}(\mathcal{C})$, we obtain a picture as follows:



Definition

Let $Q_n(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{TwAr}([n])^{op}, \mathcal{C})$ be the full sub-category consisting of those functors that take all "squares" in $\mathrm{TwAr}([n])^{op}$ to pullbacks.

A functor $\mathrm{TwAr}([n])^{op} \to \mathcal{C}$ can be pictured as a diagram:



Every square is a pullback square.

We have that $\mathrm{Fun}(\mathrm{TwAr}([-])^{op}, \mathcal{C}) : \Delta^{op} \to \mathrm{Cat}$ is a functor. All the boundary and degeneracy maps preserve the full subcategories $Q_n(\mathcal{C})$. Thus $Q(\mathcal{C})$ assembles into a simplicial category.

Definition

Let $\mathcal{C}$ be a category with pullbacks. The category $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$ is defined by $Q(\mathcal{C})^{\simeq}$. I.e. we ignore all non-invertible arrows in $Q(\mathcal{C})$.

Definition

We define its algebraic K-space as $\mathcal{K}(\mathcal{C}) := \Omega|\mathrm{Span}(\mathcal{C})|$, where the loop space is formed with base point $0 \in \mathcal{C}^{\simeq} = (\mathrm{Span}(\mathcal{C}))^{\simeq}$. The connective K-theory spectrum is $K(\mathcal{C}) := \Sigma^{\infty}\mathcal{K}(\mathcal{C})$.

There is another equivalent construction due to Waldhausen which using arrow category rather than twisted arrow category.

There is also a non-connective K-theory spectrum $\mathbb{K}($ of any stable quasi-category) due to Thomason. But its construction is very delicate. We omit it for convenient.

Definition

Let X be a scheme. The Quillen K-theory $K^Q(X)$ of X is given by the Q-construction of the category of vector bundles $\mathrm{Vect}(X)$. The Bass-Thomason-Trobaugh K-theory $\mathbb{K}(X)$ of X is given by the Q-construction of the derived category of perfect complexes $D_{perf}(X)$.

If X is a noetherian scheme, we get the algebraic G-theory if we replace the category of vector bundles by the category of coherent sheaves. Of course, Thomason had gave a more delicate definition of G-theory.

Kerz, Strunk and Tamme's proof

Weibel's conjecture has been known under some assumptions:

Theorem (Cortiñas-Haesemeyer-Schlichting-Weibel(Annals paper))

Weibel's conjecture is true for a variety X/k over a field k with $ch(k) = 0$.

Theorem (Geisser-Hesselholt, Krishna)

Weibel's conjecture is true for X/k a variety such that a strong from of resolution of singularities holds over the field k.

Their method is based on cdh-descent of infinitesimal K-theory and cyclic homology. Kelly proved the following result in his thesis based on Gabber's refined alterations and cdh-descent:

Theorem

Let X be a quasi-excellent noetherian scheme and p a prime that is nilpotent on X. Then $K_n(X) \otimes \mathbb{Z}[1/p] = 0$ for $n < -\dim X$ where K_n is the K-theory of Bass-Thomason-Trobaugh.

For any morphism $f : X \to Y$, the inverse image functor $f^* : D_{perf}(Y) \to D_{perf}(X)$ gives rise to a map $f^* : \mathbb{K}(Y) \to \mathbb{K}(X)$. Thus algebraic K-theory gives rise to a functor $\mathbb{K} : (Sch/\mathbb{Z})^{op} \to Sp$. It's a natural question to ask whether $\mathbb{K}$ is a τ-sheaf for a certain topology τ.

Theorem (Brown-Gersten)

If X is a separated noetherian regular scheme then K-theory presheaf satisfies Zariski descent.

Does K-theory satisfies more useful étale descent? No! This is a folklore but hard to write it down. A related question is Quillen-Lichtenbaum conjecture(now a theorem of Voevodsky-Rost).
In order to study the K-theory presheaf further, we need the so called descent spectral sequence.

Theorem (Thomason)

Let F be a sheaf of spectra on a site C. Suppose the topos of C has enough points. There is a spectral sequence

$$E_2^{p,q} = H^p(C; \tilde{\pi}_p F) \Rightarrow \pi_{q-p} F(C), p \geq 0, \infty > q > -\infty.$$

Differentials d_r have bi-degree $(r, r-1)$.
The E_2 term is the sheaf cohomology of the topos C with coefficients in the sheafification of the presheaf $\pi_q F$.
The spectral sequence converges strongly if either there is an N such that $\tilde{\pi}_q F = 0$ for $q > N$ or if C has bounded cohomological dimension for the sheaves $\tilde{\pi}_* F$.

Descent spectral sequence will be used in Kerz, Strunk and Tamme's proof.

Definition

Let X be a noetherian scheme. A cartesian diagram of schemes

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array}$$

(1) is called an abstract blow-up square if i is a closed immersion of finite presentation, and p is aproper morphism inducing an isomorphism $(\tilde{X} - E) \cong (Y - Y)$.
(2) is called a Nisnevich square if i is an open immersion, and p is an étale morphism inducing an isomorphism $(\tilde{X} - E)_{red} \cong (X - Y)_{red}$.

Definition (cdh topology)

The cdh topology on Sch/X is the topology generated by abstract blow-up squares and Nisnevich squares.

Example

(1) If $Z \to X$ is a closed immersion with quasi-coherent ideal sheaf I. Then the blow-up along Z produces an abstract blow-up square:

$$\begin{array}{ccc} \mathrm{Proj} \bigoplus_{d\geq 0} I^d/I^{d+1} & \longrightarrow & \mathrm{Bl}_Z X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

(2) Let $A \to B$ be a finite extension of finite type k-algebras. We have the the conductor ideal $\mathrm{Ann}_A(B) = \{a \in A : aB \subset A\}$. The cartesian square in schemes

$$\begin{array}{ccc} \mathrm{Spec} B/B\mathrm{Ann}_A(B) & \longrightarrow & \mathrm{Spec} B \\ \downarrow & & \downarrow p \\ \mathrm{Spec} A/\mathrm{Ann}_A(B) & \xrightarrow{i} & \mathrm{Spec} A \end{array}$$

is an abstract blow-up square which is not an actual blow-up.

Theorem (Khan)

A presheaf F of stable quasi-category on Sch/X satisfies cdh descent if and only if os satisfies the following conditions:
(1) It sends empty scheme to a zero object. (2) It sends Nisnevich(abstract blow-up) squares to homotopy cartesian squares. (3) For every $Y \in Sch/X$ and every regular closed immersion $Z \to Y$, it sends the blow-up square of $Z \to Y$ to a homotopy cartesian square.

We refer to SGA6 for the definition of regular closed immersion which is different from the definition in EGA4.

Example

Let $X_{red} \to X$ be the closed immersion given by the reduced locus of X; then

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow p \\ X_{red} & \xrightarrow{i} & X \end{array}$$

is an abstract blow-up square.

This square show that a cdh sheaf cannot detect nilpotent elements. We can think cdh sheaf sheafification as something like resolution of singularities because it killed all nilpotent. We have see that K_1 is not nil-invariant. Thus algebraic K-theory does not satisfy cdh descent. Instead, Kerz, Strunk and Tamme proved that algebraic K-theory satisfies pro-cdh descent. The ideal is using infinitesimal thickening. Consequently, Weibel's vanishing conjecture follows.

Given a category $\mathcal{C}$, and any small cofiltered category I.

Definition

A pro-system in $\mathcal{C}$ is a functor $X : I \to \mathcal{C}$. Following Deligne, we denote by $\lim_n X_n$.
If the pro-systems $X = \lim_n X_n$ and $Y = \lim_n Y_n$ are indexed by the same category I, then by a level map $X \to Y$ we mean a natural transformation between these I-shaped diagrams.

Definition

A morphism of pro-spectra $f : X \to Y$ is an equivalence if

$$f_* : \lim_n \pi_i(X_n) \to \lim_n \pi_i(Y_n)$$

is an isomorphism of pro-groups for all $i \in \mathbb{Z}$.

We say that a commutative square of pro-spectra

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is homotopy cartesian if X' is equivalent to a level-wise homotopy limit of the other part of the square. It's equivalent to the existence of a natural long exact sequence of pro-groups

$$\dots \pi_i(X') \to \pi_i(Y') \oplus \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(X') \to \dots.$$

Theorem (Kerz-Strunk-Tamme)

Let X be a noetherian scheme. For any abstract blow-up square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

The diagram of pro-spectra of non-connective algebraic K-theory

$$\begin{array}{ccc} K(X) & \longrightarrow & \lim_n K(Y_n) \\ \downarrow & & \downarrow \\ K(\tilde{X}) & \longrightarrow & \lim_n K(E_n) \end{array}$$

is homotopy cartesian. Where Y_n respectively E_n the n-th infinitesimal thickening of Y in X respectively E in $\tilde{X}$.

The proof of the theorem is technical. Their proof used derived algebraic geometry. Now, we formulate their proof of Weibel's conjecture.

Proposition

Let E be a Zariski sheaf of spectra on the category of noetherian schemes. Let X be a noetherian scheme of finite Krull dimension d. If for every $x \in X$ the stalk $\pi_i(E)_x$ vanishes for all $i < -\dim(\mathcal{O}_{X,x})$, then $\pi_i(E(X))$ vanishes for all $i < -d$.

This proposition shows that it is enough to show vanishing result of K-theory on affine schemes.

The proof is arguing inductively on the dimension d of the noetherian scheme X. We may assume X is an affine scheme by the previous proposition. Recall that

$$K_{-1}(A) := \operatorname{coker}(K_0(A[t]) \oplus K_0(A[t^{-1}]) \to K_0(A[t, t^{-1}]))$$

and K_0 is nil-invariant. Inductively, negative algebraic K-theory of affine schemes is nil-invariant. Hence we may also assume X is reduced. A first general vanishing result of negative K-theory is due to Bass:

Proposition

Let A be a noetherian ring. If $\dim A = 0$, then $K_*(A) = 0$ for $* < 0$.

Let $d > 0$ and assume that it is true for all noetherian schemes of dimension less than d.

Kerz and Strunk showed the following proposition by using platification par éclatement de Raynaud et Gruson:

Proposition

Let X be a reduced quasi-projective scheme over a noetherian scheme. Let $f : Y \to X$ a smooth and quasi-projective morphism. Let $k > 0$ be a integer and let $\xi \in K_{-k}(Y)$. There exists a birational projective morphism $p : X' \to X$ such that $q^*(\xi) = 0 \in K_{-k}(Y')$ where $q : Y' \to Y$ is the pull-back of p along f.

Let $i < -d$ and consider an element $\xi \in K_i(X)$. Applying the above proposition to the identity $f = id_X$ we find a projective birational morphism $p : X' \to X$ such that $p^*(\xi) = 0$.

We choose a nowhere dense closed sub-scheme $Y \to X$ such that p is an isomorphism outside Y and obtain an abstract blow-up square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow p \\ Y & \hookrightarrow & X \end{array}$$

By pro cdh-descent, we get a long exact sequence

$$\cdots \to \lim_n K_{i+1}(Y_n') \to K_i(X) \to \lim_n K_i(Y_n) \oplus K_i(X') \to \dots$$

of pro-groups. Since $\dim Y' < d$ and $\dim Y < d$, the pro-groups $K_{i+1}(Y_n')$ and $K_i(Y_n)$ vanish by the induction hypothesis. Hence $p^* : K_i(X) \to K_i(X')$ is injective. Thus $\xi = 0$. **Q.E.D.**

Thank for listening!