

Analytic Geometry and Homotopy Groups of the $K(n)$ -Local Sphere

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Our Goal

Theorem(Barthel-Schlank-Stapleton-Weinstein , 2024)

There is an isomorphism of graded \mathbb{Q} -algebras

$$\mathbb{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \dots, \zeta_n),$$

where the latter is the exterior \mathbb{Q}_p -algebra with generators ζ_i in degree $1 - 2i$.





▣ Rationalization of the $K(n)$ -Local Sphere

▣ Analytic Geometry





Rationalization of the
 $K(n)$ -Local Sphere



Morava E-theories and Morava K-theories

Let G_0 be a formal group over a perfect field k with characteristic p , then a deformation of G_0 to R is a triple (G, i, Ψ) , where G is a formal group over R , $i : k \rightarrow R/m$, $\Psi : \pi^* G \cong i^* G_0$ is an isomorphism of formal groups over R/m .

Theorem (Lubin-Tate, 1966)

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \dots, v_{n-1}]]$ in the following sense: for every infinitesimal thickening A of k , there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \rightarrow \mathrm{Def}(A).$$

There is a spectrum E_n called **Morava E-theory**, whose homotopy group is

$$\pi_* E_n = W(k)[[v_1, \dots, v_{n-1}]][\beta^{\pm 1}],$$

This is an even spectrum $K(n)$ called **Morava K-theory**, whose homotopy groups is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \dots, t_{p^n-2}, t_{p^n}, \dots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$



Morava Stabilizer Groups

We let G_0 denote a formal group of height n over a perfect field $\overline{\mathbb{F}}_p/\mathbb{F}_p$

The small Morava stabilizer group $\text{Aut}_{\overline{\mathbb{F}}_p}(G_0)$ is the group of automorphism of G_0 with coefficients in $\overline{\mathbb{F}}_p$,

$$\text{Aut}(G_0) = \{f(x) \in \overline{\mathbb{F}}_p[[x]] : f(G_0(X, Y)) = G_0(f(x), f(y)), f'(0) \neq 0\}$$

Since G_0 is defined over $\overline{\mathbb{F}}_p$, the Galois group $\text{Gal} = \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ act on G_0 by acting on the coefficients. The Morava stabilizer group \mathbb{G}_n is defined by

$$\mathbb{G}_n = \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \rtimes \text{Aut}(G_0)$$

Theorem (Devinatz-Hopkins , Goerss-Hopkins-Miller)

The Morava stabilizer group acts on E_n , and it gives essential all automorphisms of E_n

$$E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$$

Stable Homotopy Groups of Sphere

Lemma

The $K(1)$ -local sphere $L_{K(1)}S$ is given by the homotopy fiber of the map $\Psi^g - 1 : \widehat{KU} \rightarrow \widehat{KU}$.

$$\begin{aligned}\pi_{2n}(\widehat{KU}^{\Psi^g-1}) &\simeq 0 \\ \pi_{2n-1}(\widehat{KU}^{\Psi^g-1}) &\simeq \mathbb{Z}^p / (g^n - 1).\end{aligned}$$

By this theorem, we can compute the homotopy group of $L_{K(1)}S$

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Q}_p / \mathbb{Z}_p & n = -2 \\ \mathbb{Z} / p^{k+1} \mathbb{Z} & n+1 = (p-1)p^k m, p \nmid m \\ 0 & \text{otherwise} \end{cases}$$



Homotopy fixed point spectral sequence

Proposition

There is a homotopy fixed point spectral sequence (descent spectral sequence)

$$E_2^{s,t} = H_{gp}^s(G; \pi_t(X)) \implies \pi_{t-s}(X^{hG})$$

similarly for X_{hG} , X^{tG} .

We have $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$, then we get

$$E_2^{s,t} \cong H_{cts}^s(\mathbb{G}, \pi_t E_n) \implies \pi_{t-s} L_{K(n)} S^0.$$



The structure of Morava stabilizer group

For f a formal group law over $\overline{\mathbb{F}}_p$.

$$\text{End}f = \{g(t) \in tR[[t]] \mid f(g(x), g(y)) = gf(x, y)\}$$

Proposition

$\text{End}(f)$ is a noncommutative local ring: The collection non-invertible elements is the left ideal generated by $\pi(t) = \nu(t^p)$, where $\nu f^p(x, y) = f(\nu(x), \nu(y))$.

Let $D = \mathbb{Q} \otimes \text{End}(f)$.

Lemma

D is a central division algebra over \mathbb{Q}_p . $\text{End}(f) = \{x \in D : \nu(x) \geq 0\}$.

Morava Stabilizer Group

$$\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times \quad \det : \mathbb{S}_n \rightarrow \mathbb{Z}_p^\times$$

Composition with $\mathbb{Z}_p^\times / \mu \cong \mathbb{Z}_p$.

$$\zeta_n : \mathbb{G}_n \rightarrow \mathbb{Z}_p.$$

Let $\mathbb{G}_n^1 = \ker \zeta_n$, we have

$$\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p, \quad \mathbb{S}_n \cong \mathbb{S}_n^1 \rtimes \mathbb{Z}_p.$$

As a consequence of $\mathbb{G}_n / \mathbb{G}_n^1 \rtimes \mathbb{Z}_p$, there is a equivalence $L_{K(n)}S^0 \simeq (E_n^{h\mathbb{G}_n^1})^{h\mathbb{Z}_p}$.

$$L_{K(n)}S^0 \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi-1} E_n^{h\mathbb{G}_n^1} \xrightarrow{\delta} \Sigma L_{K(n)}S^0.$$



The action of Morava stabilizer group

Let F_n be the universal deformation over $(E_n)_0$ of G_0 . If we have $\alpha = (f, \sigma) \in \mathbb{G}_n$. The universal property of F_n implies that there is ring isomorphism $\alpha_* : (E_n)_0 \rightarrow (E_n)_0$ and an isomorphism of formal group laws $f_\alpha : \alpha_* F_n \rightarrow F_n$. The action can extend to $(E_n)_* \cong \mathbb{W}_n[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$

1. $\alpha = (id, \sigma)$ for $\sigma \in \text{Gal}(k/\mathbb{F}_p)$. Then the action is action of Galois group on \mathbb{W}_n .
2. If $\omega \in \mathbb{S}_n$ is a primitive $(p^n - 1)$ -th root of the unity, then $\omega_*(u_i) = \omega^{p^i - 1} u_i$ and $\omega_*(u) = \omega u$.
3. $\psi \in \mathbb{Z}_p^\times \subset \mathbb{S}_n$ is the center, then $\psi_*(u_i) = u_i$ and $\psi_* u = \psi u$.

Theorem (Devnatz-Hopkins)

Let $1 \leq i \leq n - 1$ and $f = \sum_{j=0}^{n-1} f_j \zeta^j \in \mathbb{S}_n$, where $f_j \in \mathbb{W}_n$. Then modulo $(p, u_1, \dots, u_{n-1})^2$,

$$f_*(u) \equiv f_0 u + \sum_{j=1}^{n-1} f_{n-j}^{\sigma^j} u u_j \quad f_*(u u_i) \equiv \sum_{j=1}^i f_{i-j}^{\sigma^j} u u_j + \sum_{j=i+1}^n p f_{n+i-j}^{\sigma^j} u u_j$$

Local-to-global Principle

The Hasse square is a pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_p \prod_p \mathbb{Z}_p \end{array}$$

This is the special case of a local-to-global principle for any chain complex $M \in \mathcal{D}_{\mathbb{Z}}$.

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M_p^\wedge \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes M & \longrightarrow & \mathbb{Q} \otimes_p \prod_p M_p^\wedge \end{array}$$

which is a homotopy pullback square, where M_p^\wedge denote the derived p -completion (p -local and $\text{Ext}^i(\mathbb{Q}, M_p^\wedge) = 0$, for $i = 0, 1$.)



Rationalization of the $K(n)$ -Local Sphere

Theorem(Barthel-Schlank-Stapleton-Weinstein, 2024)

There is an isomorphism of graded \mathbb{Q} -algebras

$$\mathbb{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \dots, \zeta_n),$$

where the latter is the exterior \mathbb{Q}_p -algebra with generators ζ_i in degree $1 - 2i$.



Lemma

For all $t \neq 0$ and all $s \in \mathbb{Z}$, we have $H_{cts}^s(\mathbb{G}_n, \mathbb{Q} \otimes \pi_t E_n) = 0$.

Proof: There is a short exact sequence

$$1 \rightarrow \mathcal{O}_D^\times \rightarrow \mathbb{G}_n \cong \mathcal{O}_D^\times \rtimes \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \rightarrow 1$$

where \mathcal{O}_D^\times is isomorphic to the automorphism group of our chosen formal group law \mathbb{G}_n over $\overline{\mathbb{F}}_p$. The center of \mathcal{O}_D^\times is isomorphic to \mathbb{Z}_p^\times . The central subgroup $\mathbb{Z}_p \subset \mathbb{Z}_p^\times \subset \mathcal{O}_D^\times$ which can be generated by the element $1 + p \in \mathbb{Z}_p^\times$. We have the convergent Lydon-Hochschild-Serre spectral sequence

$$H^p(\mathcal{O}_D^\times/\mathbb{Z}_p, H_{cts}^q(\mathbb{Z}_p, \mathbb{Q} \otimes \pi_t E_n)) \implies H_{cts}^{p+q}(\mathcal{O}_D^\times, \mathbb{Q} \otimes \pi_t E_n)$$

The generator acts on $\mathbb{Q} \otimes \pi_t E_n$ by multiplication by $(1 + p)^t$. Consider the complex

$$\mathbb{Q} \otimes \pi_t E_n \xrightarrow{(1+p)^t - 1} \mathbb{Q} \otimes \pi_t E_n$$

Since $\mathbb{Q}_p \otimes \pi_t E_n$ is a \mathbb{Q}_p -vector space, when $t \neq 0$ the action by $(1 + p)^t - 1$ is invertible, so the complex is acyclic, $H_{cts}^q(\mathcal{O}_D^\times, \mathbb{Q} \otimes \pi_t E_n) = 0$ for $t \neq 0$.



We continue to consider the spectral sequence

$$H^p(\mathbb{G}_n/\mathcal{O}_D^\times, H_{cts}^q(\mathcal{O}_D^\times, \mathbb{Q} \otimes \pi_t E_n)) \implies H_{cts}^{p+q}(\mathbb{G}_n, \mathbb{Q} \otimes \pi_t E_n),$$

we get $H_{cts}^s(\mathbb{G}_n, \mathbb{Q} \otimes \pi_t E_n) = 0$ for all $t \neq 0$.



Cohomology of Morava Stabilizer Group

Proposition

For every integer $s \geq 0$, the natural map $W = W(\overline{\mathbb{F}}_p) \rightarrow \pi_0 E_n = W[[u_1, \dots, u_{n-1}]]$ induces a split injection

$$H_{cts}^s(\mathbb{G}_n, W) \hookrightarrow H_{cts}^s(\mathbb{G}_n, \pi_0 E_n)$$

whose complement killed by a power of p . In particular,

$$H_{cts}^s(\mathbb{G}_n, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow H_{cts}^s(\mathbb{G}_n, \pi_0 E_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

Proof:

The cohomology groups $H_{cts}^i(\mathcal{O}_D^\times, A^c)$ and $H_{cts}^i(\mathbb{G}_n, A^c)$ are p -power torsion.

$$\begin{array}{ccc} & \mathcal{X} & \\ \text{GL}_n(\mathbb{Z}_p) \swarrow & & \searrow \mathcal{O}_D^\times \\ \text{LT}_K & & \mathcal{H}_K \end{array}$$

This diagram induces an isomorphism in $D(\text{Solid})$:

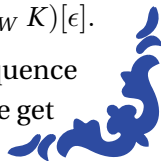
$$R\Gamma(\text{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+)^{h\mathcal{O}_D^\times} \cong R\Gamma(\mathcal{H}_{K,\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+)^{h\text{GL}_n(\mathbb{Z}_p)}$$

We have

$$H^*(R\Gamma(\mathcal{H}_{K,\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+)^{h\text{GL}_n(\mathbb{Z}_p)}) \otimes_W K \cong \Lambda_K(y_1, y_3, \dots, y_{2n-1})[\epsilon]$$

$$H^*(R\Gamma(\text{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+)^{h\mathcal{O}_D^\times}) \otimes_W K \cong \Lambda_K(x_1, x_3, \dots, x_{2n-1})[\epsilon] \oplus ((A^c)^{h\mathcal{O}_D^\times} \otimes_W K)[\epsilon].$$

We then have $H_{cts}^*(\mathcal{O}_D^\times, A^c) \otimes_W K = 0$, using the Hochschild-Serre spectral sequence combined with the fact that the cohomological dimension $\mathbb{G}_n/\mathcal{O}_D^\times \cong \widehat{\mathbb{Z}}$ is 1, we get $H_{cts}^i(\mathbb{G}_n, A^c)$ is also p -power torsion.



Galois Cohomology of Witt Rings

Lemma

Let $W = W(\overline{\mathbb{F}}_p)$ and $K = W[1/p]$.

1. $H_{cts}^i(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), W)$ is \mathbb{Z}_p if $i = 0$, and is 0 otherwise.
2. Let \mathbb{G}_n action on K through its quotient $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. There is an isomorphism of graded \mathbb{Q}_p -algebras:

$$H_{cts}^*(\mathbb{G}_n, K) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1}).$$

Proof:

1. $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \widehat{\mathbb{Z}}$, so it is enough to prove in degree 1. W is p -adically complete, this is further reducing that $H_{cts}^1(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \overline{\mathbb{F}}_p) = 0$, this is true because $x \mapsto x^p - x$ is surjective on $\overline{\mathbb{F}}_p$.
2. Consider the spectral sequence

$$H_{cts}^i(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), H_{cts}^j(\mathcal{O}_D^\times, K)) \implies H_{cts}^{i+j}(\mathbb{G}_n, K)$$

Consider the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on $H_{cts}^j(\mathcal{O}_D^\times, K) = H_{cts}^j(\mathcal{O}_D^\times, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} W$.

The action on the first factor is trivial by the following lemma, and the action on the factor has no higher cohomology by 1. Therefore

$$H_{cts}^*(\mathbb{G}_n, K) \cong H_{cts}^*(\mathcal{O}_D^\times, \mathbb{Q}_p),$$

then again apply the following lemma.

Lemma

Let G be either of the group $GL_n(\mathbb{Z}_p)$ or \mathcal{O}_D^\times . Consider the trivial action of G on \mathbb{Q}_p . There is an isomorphism of graded \mathbb{Q}_p -algebras:

$$H_{cts}^*(G, \mathbb{Q}_p) \cong H^*(\text{Lie}G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1}).$$

In the case of $G = \mathcal{O}_D^\times$, the outer morphism $\text{ad}\Pi$ (where Π is a uniformizer of D^\times) act as the identity on $H_{cts}^*(G, \mathbb{Q}_p)$.

Proof of the Main Theorem

The Devinatz-Hopkins spectral sequence

$$E_2^{s,t} \cong H_{cts}^s(\mathbb{G}, \pi_t E_n) \implies \pi_{t-s} L_{K(n)} S^0$$

converges strongly and collapses on a finite page with a horizontal vanishing line. Tensor with \mathbb{Q} , we get a convergent spectral sequence

$$\mathbb{Q} \otimes E_2^{s,t} \cong H_{cts}^s(\mathbb{G}, \mathbb{Q} \otimes \pi_t E_n) \implies \mathbb{Q} \otimes \pi_{t-s} L_{K(n)} S^0$$

By above lemmas, the E_2 page of the rationalization of the Devinatz-Hopkins spectral sequence only have one nonvanishing line, which is $t = 0$ in the (s, t) coordinate system. So we get an isomorphism

$$H_{cts}^*(\mathbb{G}, \mathbb{Q} \otimes \pi_0 E_n) \cong \mathbb{Q} \otimes \pi_* L_{K(n)} S^0$$

By the computation of the cohomology groups of Morava stabilizer groups, the left hand side equals to

$$H_{cts}^*(\mathbb{G}, \mathbb{Q} \otimes \pi_0 E_n) \cong H_{cts}^*(\mathbb{G}, \mathbb{Q} \otimes W) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \dots, x_n).$$

with x_i in cohomological degree $2i - 1$.





Analytic Geometry



The Langlands correspondence in number theory (Langlands 67) is a conjectural correspondence (a bijection subject to various conditions) between

1. n -dimensional complex linear representations of the Galois group $\text{Gal}(\bar{F}/F)$ of a given number field F
2. certain representations-called automorphic representations of the n -dimensional general linear group $GL_n(\mathbb{A}_F)$ with coefficients in the ring of adeles of F , arising within the representations given by functions on the double coset space $GL_n(F) \backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$.

moduli spaces of shtukas	Shimura varieties
moduli spaces of local shtukas	local Shimura varieties
Drinfled's upper half spaces	Lubin-Tate towers



Shtukas over Function Fields

Definition

- Let S/\mathbb{F}_p be a scheme. A shtuka of rank n with legs $x_1, \dots, x_m \in X(S)$ is a rank n vector bundle \mathcal{E} over $S \times_{\mathbb{F}_p} X$ together with an isomorphism

$$\phi_{\mathcal{E}} : (\text{Frobs})^* \mathcal{E}|_{S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}}$$

on $S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}$, where $\Gamma_{x_i} \subset S \times_{\mathbb{F}_p} X$ is the graph of x_i .

- Let \widehat{X} be the formal completion of X at one of its \mathbb{F}_p rational points, so that $\widehat{X} \cong \text{Spf} \mathbb{F}_p[[T]]$. A local shtuka of rank n over an adic space S/\mathbb{F}_p with legs $x_1, \dots, x_m \in \widehat{X}(S)$ is a rank n vector bundle \mathcal{E} over $S \times_{\mathbb{F}_p} \widehat{X}$ together with an isomorphism

$$\phi_{\mathcal{E}} : (\text{Frobs})^* \mathcal{E}|_{S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}}$$

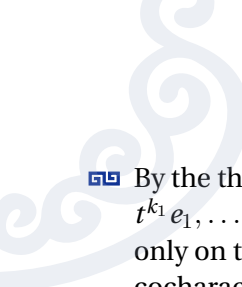
over $S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}$

Suppose that we are given a shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank n over $\text{Spec} k$, where k is an algebraically closed. Then it can be described by the following data:

1. The collection of points $x_1, \dots, x_m \in X(k)$ where $\phi_{\mathcal{E}}$ is undefined. We call these points legs of the shtuka.
2. For each $i = 1, \dots, m$ a conjugacy class μ_i of cocharacters $G_m \rightarrow GL_n$, encoding the behaviour of $\phi_{\mathcal{E}}$ near x_i .

Now we explain the second item. Let $x \in X(k)$ be a leg of shtuka, and let $t \in \mathcal{O}_{X,x}$ be a uniformizing parameter at x . We have the complete stacks $(\text{Frob}_S^* \mathcal{E})_x^{\wedge}$ and \mathcal{E}_x^{\wedge} . These two are free rank modules over $\mathcal{O}_{X,x}^{\wedge} \cong k[[t]]$, whose generic fibers are identified using $\phi_{\mathcal{E}}$. That is we have two $k[[t]]$ lattices in the same n dimensional $k((t))$ vector space.





By the theory of elementary divisors, there exists a basis e_1, \dots, e_n of \mathcal{E}_x^\wedge such that $t^{k_1}e_1, \dots, t^{k_n}e_n$ is a basis of $(\mathrm{Frob}_S^* \mathcal{E})_x^\wedge$, where k_1, \dots, k_n . These integers depend only on the shtuka. Another way to package of this data is as conjugacy class μ of cocharacters $G_m \rightarrow GL_n$ via $\mu(t) = \mathrm{diag}(t^{k_1}, \dots, t^{k_n})$.

Thus there are some discrete data attach to a shtuka: the number of legs m and the ordered collection of cocharacter classes (μ_1, \dots, μ_m) . Fixing these, we can define a moduli space $\mathrm{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}}$ whose k -points classify the following data:

1. An m -tuple of points (x_1, \dots, x_m) of $X(k)$.
2. A shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank n with legs x_1, \dots, x_m , for which the relative position of $\mathcal{E}_{x_i}^\wedge$ and $(\mathrm{Frob}_S^* \mathcal{E})_{x_i}^\wedge$ is bounded by the cocharacter μ_i for all $i = 1, \dots, m$.



It can be proved that $\text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}}$ is representable by a Deligne-Mumford stack. We have a structure map

$$f : \text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}} \rightarrow X^m$$

by sending a shtuka to its m-tuple of legs.

We can add level structures to these spaces of shtukas, parametrized by finite closed subscheme $N \subset X$. A level N-structure on $(\mathcal{E}, \phi_{\mathcal{E}})$ is then a trivialization of the pullback of \mathcal{E} to N which is compatible with $\phi_{\mathcal{E}}$. By this additional structure, we can get a family of shtukas $\text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}, N}$ and morphisms

$$f_N : \text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}, N} \rightarrow (X/N)^m.$$

The stack $\text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}, N}$ carries an action of $GL_n(\mathcal{O}_N)$, by altering the trivialization of \mathcal{E} on N. The inverse limit $\varprojlim_N \text{Sht}_{GL_n, \{\mu_1, \dots, \mu_m\}, N}$ admits an action of $GL_n(\mathbb{A}_K)$, via the Hecke correspondences. Assume the relative dimension of f is d . We consider the cohomology $R^d(f_N)_! \overline{\mathbb{Q}}_l$, this an $\overline{\mathbb{Q}}_l$ étale sheaf on X^m .

Passing to the limit over N , one gets a big representation of $GL_n(A_K) \times \text{Gal}(\bar{K}/K) \times \cdots \times \text{Gal}(\bar{K}/K)$ on $R^d(f_N)_! \bar{\mathbb{Q}}_l$. Roughly, we expect this space to decompose as follows

$$\lim_{\substack{\longrightarrow \\ N}} R^d(f_N)_! \bar{\mathbb{Q}}_l = \bigoplus_{\pi} \pi \otimes (r_1 \circ \sigma(\pi)) \otimes \cdots \otimes (r_m \circ \sigma(\pi))$$

- ▣ π run over cuspidal automorphic representations of $GL_n(K)$,
- ▣ $\sigma(\pi) : \text{Gal}(\bar{K}/K) \rightarrow GL_n(\bar{\mathbb{Q}}_l)$ is the corresponding L-parameter,
- ▣ $r_i : GL_n \rightarrow GL_{n_i}$ is an algebraic representation corresponding to μ_i .

Drinfeld (1980, $n=2$) and L. Lafforgue (general n , 2002) considered the case of $m = 2$, with μ_1 and μ_2 corresponding to the n -tuples $(1, 0, \dots, 0)$ and $(0, \dots, 0, -1)$ respectively. V. Lafforgue considered general reductive group G in place of GL_n .



Shimura Varieties

A Shimura datum is a pair (G, μ) , where G is a reductive group over \mathbb{Q} , and $\mu : C^\times \rightarrow G(\mathbb{R})$ is a morphism of real groups, such that the conjugacy class \mathcal{H}_μ of μ is a complex manifold. The tower of Shimura varieties is

$$\mathrm{Sh}(G, \mu)_K = G(\mathbb{Q}) \backslash (\mathcal{H}_\mu) \times G(\mathbb{A}_f) / K$$

where K runs over all compact open subgroups of $G(\mathbb{A}_f)$. The l -adic cohomology of the tower admits an action $G(\mathbb{A}_f) \times \mathrm{Gal}(\bar{E}/E)$. Let

$$H^i(\xi) = \lim_{\rightarrow K} H^i(\mathrm{Sh}(G, \mu)_{K, \bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_l)$$

$$H^*(\xi) = \sum_i (-1)^i H^i(\xi)$$

Conjecture

$$H^*(\xi) = \sum_{\pi} a(\pi, \xi) \pi_f \otimes (R_\mu \circ \phi_\pi)|_{\mathrm{Gal}(\bar{\mathbb{Q}}/E)}$$

Here π runs over cuspidal automorphic representations of G , $R_\mu : {}^L G \rightarrow GL_n$ is the representation of highest weight μ , and $a(\pi, \xi)$ is an integer.

Adic Spaces

Definition

- ▣ A Huber ring is a topological ring A , such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such A has the I -adic topology.
- ▣ A Huber ring A is Tate if it contains a continuous topologically nilpotent unit. Such an element is called a pseudo-uniformizer
- ▣ A subset S of a topological ring A is bounded if for all open neighborhoods U of 0 , there exists an open neighborhood V of 0 such that $V \cdot S \subset U$.
- ▣ An element $f \in A$ is power-bounded if $\{f^N\} \subset A$ is bounded. Let A° be the subset of power-bounded elements. If A is linearly topologized (for instance if A is Huber) then $A^\circ \subset A$ is a subring.
- ▣ A Huber ring A is uniform if $A^\circ \subset A$ is bounded.

Definition

- Let A be a Huber ring. A subring $A^+ \subset A$ is a ring of integral elements if it is open and integrally closed and $A^+ \subset A^\circ$. A Huber pair is a pair (A, A^+) , where A is a Huber and $A^+ \subset A$ is ring of integral elements.
- Given a Huber pair, we let $\text{Spa}(A, A^+) \subset \text{Cont}(A)$ be the subset of continuous valuations x for which $|f| \leq 1$ for all $f \in A^+$. Write $\text{Spa}A$ for $\text{Spa}(A, A^\circ)$.

Example

$A = \mathbb{Q}_p\langle T \rangle$ and $A^+ = A^\circ = \mathbb{Z}_p\langle T \rangle$, we define

$$A^{++} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in A^+ \mid |a_n| < 1 \text{ for all } n \geq 1 \right\}$$

We have $A^{++} \subset A^+$, so $\text{Spa}(A, A^+) \subset \text{Spa}(A, A^{++})$.

Topology of Adic Spaces

The topology of an adic spectrum $X = \mathrm{Spa}(A, A^+)$ is generated by *rational sets* of the form

$$U = U\left(\frac{f_1, \dots, f_r}{g}\right) = \{v \in \mathrm{Spa}(A, A^+) \mid v(f_i) \leq v(g) \neq 0, i = 1, \dots, r\}$$

For $U = U(f_i/g)$ a rational set

- ▣ $\mathcal{O}_X(U)$, the completion of $A[f_i/g]$.
- ▣ $\mathcal{O}_X^+(U)$, the completion of the integer closure of $A^+[f_i/g]$ in $A[f_i/g]$.

Definition

An adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ which is locally isomorphic to an affinoid adic space $\mathrm{Spa}(A, A^+)$.

Rigid Analytic Spaces

Definition

A rigid affinoid is an algebra A which has form T_n/I , where $T_n = K\langle z_1, \dots, z_n \rangle$ is the subring of the of all power series $K[[z_1, \dots, z_n]]$ consisting of the power series $\sum_{\alpha} c_{\alpha} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in K[[z_1, \dots, z_n]]$ satisfying $\lim c_{\alpha} = 0$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index.

One define the **Gauss norm** on T_n by

$$\|\sum_{\alpha} c_{\alpha}\| = \max |c_{\alpha}|.$$

Further $T_n^{\circ} := \{f \in T_n \mid \|f\| \leq 1\}$ and $T_n^{\circ\circ} := \{f \in T_n \mid \|f\| < 1\}$.

Rigid analytic spaces are adic spaces, locally are $\text{Spa}(T_n, T_n^{\circ})$.



Perfectoid Spaces

Definition

A ring R is perfectoid if R is a complete Tate ring R which is uniform and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ holds in R° , and such that the p -th power Frobenius map

$$\phi : R^\circ / \varpi \rightarrow R^\circ / \varpi^p$$

is an isomorphism.

Definition

A perfectoid field is a perfectoid Tate ring R which is a nonarchimedean field. That is, it is a complete non-archimedean field K of residue characteristic p , equipped with a non-discrete valuation of rank 1, such that the Frobenius map $\theta : \mathcal{O}_K / p \rightarrow \mathcal{O}_K / p$ is surjective, where $\mathcal{O}_K \subset K$ is the subring of elements of norm ≤ 1 .

Definition

- ▣ A perfectoid space is an adic space that may be covered by affinoids of the form $\mathrm{Spa}(A, A^+)$, where A is perfectoid.
- ▣ Let A be a perfectoid ring. We define its tilt to be

$$A^b := \varprojlim_{x \rightarrow x^p} A$$

- ▣ A diamond is a pro-étale sheaf \mathcal{D} on Perf such that one can write $\mathcal{D} = X/R$ as a quotient of a perfectoid space X of characteristic p by an equivalence relation $R \subset X \times X$ such that R is a perfectoid space with $s, t : R \rightarrow X$ proétale.

Definition

In Perfd , $\{f_i : X_i \rightarrow Y\}_{i \in I}$ is a cover if and only if for all quasi-compact open subsets $V \subset Y$ there is some finite subset $I_V \subset I$ and quasicompact open $U_i \subset X_i$ for $i \in I_V$ such that $V = \cup_{i \in I_V} f_i(U_i)$.

Definition

An Artin v -stack is a small v -stack X such that the diagonal map $\Delta_X : X \rightarrow X \times X$ is representable in locally spatial diamonds, and there is some surjection map $f : U \rightarrow X$ from a locally spatial diamond U such that f is separated and cohomologically smooth.

Theorem (Fargues-Scholze, 2021)

The stack Bun_G is a cohomologically smooth Artin v -stack of l -dimension 0.

Mix-Characteristic Shtukas

Theorem

Let $S \in \text{Perf}$. The following sets are naturally identified:

1. Sections of $(S \dot{\times} \text{Spa} \mathbb{Z}_p)^\diamond \rightarrow S$,
2. Morphisms $S \rightarrow \text{Spd} \mathbb{Z}_p$,
3. Untilts S^\sharp of S .

Definition

Let S be a perfectoid space in characteristic p . Let $x_1, \dots, x_m : S \rightarrow \text{Spd} \mathbb{Z}_p$ be a collection of morphism; We let $\Gamma_{x_i} : S_i^\sharp \rightarrow S \dot{\times} \text{Spa} \mathbb{Z}_p$ be the corresponding closed Cartier divisor. A mixed-characteristic shtuka of rank n over S with legs x_1, \dots, x_m is a rank n vector bundle \mathcal{E} over $S \dot{\times} \text{Spa} \mathbb{Z}_p$ together with an isomorphism

$$\phi_{\mathcal{E}} : \text{Frob}_S^* \mathcal{E}|_{S \dot{\times} \text{Spa} \mathbb{Z}_p \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \dot{\times} \text{Spa} \mathbb{Z}_p \setminus \cup_i \Gamma_{x_i}}$$

that is meromorphic along $\cup_i \Gamma_{x_i}$.

Theorem (Scholze-Weinstein, 2020)

The following categories are equivalent:

1. Shtukas over $\mathrm{Spa}C^b$ with one leg at $\phi^{-1}(x_C)$, i.e., vector bundles \mathcal{E} on $\mathcal{Y}_{[0,\infty]}$ together with an isomorphism $\phi_{\mathcal{E}} : (\phi^* \mathcal{E})|_{\mathcal{U}_{[0,\infty)} \setminus \phi^{-1}(x_C)} \cong \mathcal{E}|_{\mathcal{U}_{[0,\infty)} \setminus \phi^{-1}(x_C)}$.
2. Pairs (T, E) , where T is a finite free \mathbb{Z}_p -module, and $E \subset T \otimes_{\mathbb{Z}_p} B_{dR}$ is a B_{dR}^+ lattice.
3. Quadruples $(\mathcal{F}, \mathcal{F}', \beta, T)$, where \mathcal{F} and \mathcal{F}' are vector bundles on the Fargues-Fontaine curve X_{FF} such that \mathcal{F} is trivial, $\beta : \mathcal{F}|_{X_{FF} \setminus \{\infty\}} \cong \mathcal{F}'|_{X_{FF} \setminus \{\infty\}}$ is an isomorphism, and $T \subset H^0(X_{FF}, \mathcal{F})$ is a \mathbb{Z}_p lattice.
4. Vector bundles $\tilde{\mathcal{E}}$ on \mathcal{Y} together with an isomorphism $\phi_{\tilde{\mathcal{E}}} : (\phi^* \tilde{\mathcal{E}})|_{\mathcal{Y} \setminus \phi^{-1}(x_C)} \cong \tilde{\mathcal{E}}|_{\mathcal{Y} \setminus \phi^{-1}(x_C)}$.
5. Breuil-Kisin-Fargues modules over A_{inf} , i.e., finite free A_{inf} -modules M together with isomorphism $\phi_M : (\phi^* M)[\frac{1}{\phi(\xi)}] \cong M[\frac{1}{\phi(\xi)}]$.

Local Mix-Characteristic Shtukas

Let k be a discrete algebraically closed field, and $L = W(k)[\frac{1}{p}]$.

Let $(\mathcal{G}, b, \{\mu_i\})$ be a triple consisting of a smooth group scheme \mathcal{G} with reductive generic fiber G and connected special fiber, and element $b \in G(L)$, and a collection μ_1, \dots, μ_m of conjugacy class of cocharacters $G_m \rightarrow G_{\overline{\mathbb{Q}_p}}$. For $i = 1, \dots, m$, let E_i/\mathbb{Q}_p be the field of definition of μ_i , and let $\check{E}_i = E_i \cdot L$.

For any perfectoid space $S = \text{Spa}(R, R^+)$ over k , a shtuka associated with $(\mathcal{G}, b, \{\mu_i\})$ is a quadruples $(\mathcal{P}, \{S_i^\sharp\}, \phi_{\mathcal{P}}, \iota_r)$, where:

1. \mathcal{P} is a \mathcal{G} -torsor on $S \times \text{Spa} \mathbb{Z}_p$,
2. S_i^\sharp is a an untilt of $S \rightarrow \check{E}_i$, for $i = 1, \dots, m$,
3. $\phi_{\mathcal{P}}$ is an isomorphism

$$\phi_{\mathcal{P}} : \text{Frob}_S^* \mathcal{P}|_{S \times X \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{P}|_{S \times X \setminus \cup_i \Gamma_{x_i}},$$

4. ι_r is an isomorphism

$$\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r, \infty]}(S)} \rightarrow G \times \mathcal{Y}_{[r, \infty]}(S)$$

for large enough r , under which $\phi_{\mathfrak{P}}$ gets identified with $b \times \text{Frob}_S$.



By the definition of local shtukas, we can define a moduli functor

$$\begin{aligned} \text{Shtuka}_{\mathcal{G}, b, \mu} &: \text{Perf}_k \rightarrow \text{Set} \\ S &\rightarrow \{(\mathcal{P}, \{S_i^\sharp\}, \phi_{\mathcal{P}, \iota_r})\} \end{aligned}$$

Theorem

The moduli space $\text{Shtuka}_{\mathcal{G}, b, \mu}$ is a locally spatial diamond.

Theorem (Scholze-Weinstein, 2013)

There is a natural isomorphism

$$\mathcal{M}_{\mathbb{X}, \check{\mathbb{Q}}_p} \cong \text{Shtuka}_{(GL_n, b, \mu)}$$

as diamonds over $\text{Spf} \check{\mathbb{Q}}_p$.

Definition

A local Shimura datum is a triple (G, b, μ) consists a reductive group G over \mathbb{Q}_p , a conjugacy class μ of minuscule cocharacters $G_m \rightarrow G_{\bar{\mathbb{Q}}_p}$, and $b \in B(G, \mu^{-1})$, that is $\nu_b \leq (\mu^{-1})^\diamond$ and $\kappa(b) = -\mu^\natural$.

There is a étale map

$$\pi_{GM} : \text{Shtuka}_{G,b,\mu,K} \rightarrow \text{Gr}_{G,\text{Spd}\check{E},\leq\mu}.$$

By the construction of diamonds, there exists a unique smooth rigid space $\mathcal{M}_{G,b,\mu,K}$ over \check{E} with an étale map towards $\mathcal{F}_{G,\mu,\check{E}}$.

Definition

The local Shimura variety associated with (G, b, μ) is the tower

$$(\mathcal{M}_{G,b,\mu,K})_{K \subset G(\mathbb{Q}_p)}$$

of smooth rigid space over \check{E} , together with its étale period map to $\mathcal{F}_{G,\mu,\check{E}}$

Condensed Mathematics

Definition

1. We define $*_{\text{proét}}$ as the proétale site of a point, which is the category of profinite sets S , with finite jointly surjective families of maps as covers.

A condensed set /group/ring, ... is a functor

$$T : \{\mathbf{profinite\ sets}\}^{op} \rightarrow \{\mathbf{sets/rings/groups/ \dots}\}$$

$$S \mapsto T(S)$$

satisfies $T(\emptyset) = *$ and satisfying the following condition

1. For any profinite set S_1, S_2 , the natural map

$$T(S_1 \cup S_2) \rightarrow T(S_1) \times T(S_2)$$

is a bijection.

2. For any surjection $S' \rightarrow S$ of profinite sets with the fibre product $S' \times_S S'$ and its projection p_1, p_2 to S' , the map

$$T(S) \rightarrow \{x \in T(S) \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection.



Solid Abelian Groups

Definition

1. For a profinite set $S = \varprojlim_i S_i$, we define the condensed abelian group

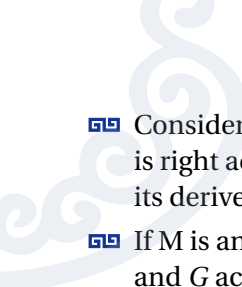
$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i].$$

There is a natural map $S = \varprojlim_i S_i \rightarrow \mathbb{Z}[S]^{\blacksquare}$, inducing a map $\mathbb{Z}[S] \rightarrow \mathbb{Z}[S]^{\blacksquare}$.

2. A solid abelian group is a condensed abelian group A such that for all profinite set S and all maps $f : S \rightarrow A$, there is a unique map $\tilde{f} : \mathbb{Z}[S]^{\blacksquare} \rightarrow A$ extending f .
3. A complex $C \in D(\text{Cond}(\text{Ab}))$ of condensed abelian groups is solid if for all profinite sets, the natural map

$$R\text{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \rightarrow R\Gamma(S, C) = R\text{Hom}(\mathbb{Z}[S], C)$$

is an isomorphism.

- 
- Consider the functor of fixed points $\text{Solid}_G \rightarrow \text{Solid}$ defined by $\mathcal{M} \rightarrow \mathcal{M}^G$, which is right adjoint to the trivial action functor $\text{Solid} \rightarrow \text{Solid}_G$. Let $\mathcal{C} \rightarrow R\Gamma(G, \mathcal{C})$ be its derived functor $D(\text{Solid}_G) \rightarrow D(\text{Solid})$.
- If M is an abelian group which is separated and complete for a linear topology, and G act continuously on M , then

$$H^i(R\Gamma(G, M)) \cong \underline{H_{cts}^i(G, M)}.$$

- Let G be a profinite group, write

$$\begin{aligned} D(\text{Solid}_G) &\rightarrow D(\text{Solid}) \\ \mathcal{C} &\rightarrow \mathcal{C}^{hG} \end{aligned}$$

for the functor $R\Gamma(G, C)$.



Pro-étale Cohomology of Rigid Analytic Spaces

Definition

Let X be a rigid-analytic space over K , the object of the pro-étale site $X_{\text{proét}}$ are formal limits $U = \varprojlim U_i$, where i runs over a filtered index set and U_i are rigid analytic spaces which are étale over X .

Let $\widehat{\mathcal{O}}^+ = \varprojlim \mathcal{O}^+ / p^n$.

Proposition

We have an isomorphism in $D(\text{Cond}(\text{Ab}))$:

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \widehat{\mathcal{O}}^+) \cong R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+).$$

Let $Y \rightarrow X$ be a pro-étale G -torsor. There is an isomorphism in $D(\text{Solid})$:

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+) \cong R\Gamma(Y_{\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+)^{hG}$$

The Proétale Cohomology of LT_K and \mathcal{H}

Theorem

- There is a morphism of differential graded solid W -algebras, which is equivariant for the action of \mathbb{G}_n :

$$A[\epsilon] \rightarrow R\Gamma(LT_{K,\text{proét}}, \mathcal{O}_{\text{cond}}^+).$$

- There is a morphism of differential graded solid \mathbb{Z}_p -algebras, which is equivariant for the action of $\text{GL}_n(\mathbb{Z}_p)$:

$$\mathbb{Z}_p[\epsilon] \rightarrow R\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}_{\text{cond}}^+).$$

Let A be the cofiber of either of the above morphism. Then $H^i(A) = 0$ for $i \leq 0$, and all $H^i(A)$ for $i \geq 1$ are annihilated by a single power of p .



Thanks for Listening!

