Algebraic K-theory of finite fields

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Introduction I

This is an exposition of Quillen's paper

On the cohomology and K-theory of the general linear groups over a finite field (1972).

Among other things, it determines $\mathrm{K}_*(\mathbf{F}_{p^n})$ for all p and n. This is one of the few instances where we completely understand $\mathrm{K}_*(R)$; it also serves as input for various other computations in algebraic K-theory.

Introduction II

Notations:

- p is a prime and $q = p^n$ for some n.
- ℓ is a prime different from p.
- k is the finite field $\mathbf{F}_q$.
- r is the least integer ≥ 1 such that $q^r \equiv 1 \bmod \ell$.

Introduction III

For R a comutative ring, Quillen defines

$$\Omega^\infty \mathrm{K}(R) := \mathrm{K}_0(R) \times \mathrm{BGL}(R)^+.$$

where X^+ denotes the **$+$-construction** of the space X.

A major result: computation of $\mathrm{K}_i(k)$ (recall $k = \mathbf{F}_q$):

$$\mathrm{K}_i(k) := \pi_i(\mathbf{Z} \times \mathrm{BGL}(k)^+) \cong \begin{cases} \mathbf{Z}, & i = 0, \\ \mathbf{Z}/(q^j - 1), & i = 2j - 1, \\ 0, & i = 2j,\ j > 0. \end{cases}$$

Introduction IV

In fact, Quillen determines the homotopy type

$$\mathrm{BGL}(k)^+ \simeq \mathrm{BU}^{\psi^q}$$

where $\psi^q : \mathrm{BU} \to \mathrm{BU}$ represents the unstable[1] Adams operation, and

$$\mathrm{BU}^{\psi^q} := \mathrm{hofib}(\mathrm{BU} \xrightarrow{\psi^q - 1} \mathrm{BU}).$$

From this, the computation of $\mathrm{K}_i(k)$ easily follows.

[1] The map ψ^q does not preserve the $\mathbb{E}_\infty$-structure on BU and hence does not lift to ku. To see this, take $L_{K(1)}\mathrm{ku} \simeq \mathrm{KU}_p$ and apply to $\beta^{\pm 1}$.

Outline I

1. Identify $\Omega^\infty \mathrm{K}(k) \simeq \mathrm{K}_0(k) \times \mathrm{BGL}(k)^+$.
2. Using **Brauer lifting**, find a map $\mathrm{Br} : \mathrm{BGL}(k) \to \mathrm{BU}^{\psi^q}$, which lifts to

$$\begin{array}{ccc} \mathrm{BGL}(k) & \xrightarrow{\mathrm{Br}} & \mathrm{BU}^{\psi^q} \\ \downarrow & \nearrow & \\ \mathrm{BGL}(k)^+ & & \end{array}$$

by the universal property of the $+$-construction (since $\pi_1(\mathrm{BU}^{\psi^q})$ is abelian and contains no perfect subgroups).

Outline II

3. Show that

$$f : \mathrm{BGL}(k) \to \mathrm{BU}^{\psi^q}$$

induces isomorphism on cohomology groups

$$\mathrm{H}^*(\mathrm{BGL}(k); \kappa) \cong \mathrm{H}^*(\mathrm{BU}^{\psi^q}; \kappa),$$

for $\kappa = \mathbf{Q}, \mathbf{F}_\ell$ and $\mathbf{F}_p$, hence for $\kappa = \mathbf{Z}$. Furthermore, by construction,

$$\mathrm{H}^*(\mathrm{BGL}(k)^+; \mathbf{Z}) \cong \mathrm{H}^*(\mathrm{BGL}(k); \mathbf{Z}).$$

Since both spaces are **simple**[2] and have the same cohomology,

$$\mathrm{BGL}(k)^+ \simeq \mathrm{BU}^{\psi^q}$$

Outline III

4. Compute $\mathrm{K}_i(k) = \pi_i(\mathrm{BGL}(k)^+) \cong \pi_i(\mathrm{BU}^{\psi^q})$ for $i > 1$ using the fiber sequence
$$\mathrm{BU}^{\psi^q} \longrightarrow \mathrm{BU} \xrightarrow{\psi^q - 1} \mathrm{BU}.$$
Recall that
$$\pi_{2j}(\mathrm{BU}) \cong \mathbf{Z}\{\beta^j\}, \quad \psi^q \beta^j = q^j \beta^j.$$
Taking the associated long exact sequence gives
$$0 \to \pi_{2j}(\mathrm{BU}^{\psi^q}) \to \pi_{2j}(\mathrm{BU}) \xrightarrow{\cdot(q^j - 1)} \pi_{2j}(\mathrm{BU}) \to \pi_{2j-1}(\mathrm{BU}^{\psi^q}) \to 0.$$
5. Conclusion:
$$\mathrm{K}_{2j}(k) = \pi_{2j}(\mathrm{BU}^{\psi^q}) = 0, \quad \mathrm{K}_{2j-1}(k) = \pi_{2j-1}(\mathrm{BU}^{\psi^q}) \cong \mathbf{Z}/(q^j - 1).$$

[2] X is simple if $\pi_1(X)$ is abelian and acts trivially on higher $\pi_j(X)$. All H-spaces are simple.

The $+$-construction I

The $+$-construction is an operation on topological spaces. When applied to $\mathrm{BGL}(R)$ it magically yields a model for $(\Omega^\infty \mathrm{K}(R))_{\geq 1}$.

Definition

A group P is **perfect** if

$$P^{\mathrm{ab}} := P/[P, P] = 0.$$

There exists a maximal perfect subgroup $P \triangleleft G$ for any G since $P_1, P_2 \triangleleft G$ perfect implies that $P_1 P_2$ is perfect.

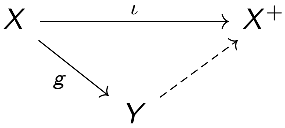
The +-construction II

Let X be a pointed CW-complex and $P \triangleleft \pi_1(X)$ the maximal perfect normal subgroup. There exists a space X^+ and a map $\iota : X \to X^+$, called the **+-construction** of X, such that

1. ι induces an isomorphism on (co)homology.
2. ι induces a surjection on π_1 with kernel P.
3. Given any $g : X \to Y$ such that

$$P < \ker(\pi_1(X) \xrightarrow{g_*} \pi_1(Y)),$$

there exists a dotted arrow unique up to pointed homotopy



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The $+$-construction proceeds by attaching 2-cells to kill P, and then attaching 3-cells to recover the homology group. It can be made functorial.

Example ([6], Proposition 1)

For $X = \mathrm{BGL}(R)$ the maximal perfect subgroup of $\pi_1(X) \cong \mathrm{GL}(R)$ is

$$E(R) := [\mathrm{GL}(R), \mathrm{GL}(R)],$$

the union of all elementary matrices.

Definition (Quillen)

Define the **higher K-groups** of a ring R to be

$$\mathrm{K}_i(R) := \pi_i(\mathrm{BGL}(R)^+), \ \forall i \geq 1.$$

Higher K-theory for rings II

Why does this definition work? We illustrate its relationship to an alternative construction of $\mathrm{K}(R)$, due to Segal in [7].
Denote

$$M := \mathrm{Proj}_{\mathsf{f.g.}}(R)^{\simeq}.$$

Grothendieck defined

$$\mathrm{K}_0(R) := \pi_0(M)^{\mathrm{gp}},$$

the group completion of the **commutative monoid** $\pi_0(M)$ under $\oplus$.
On the other hand, since $\oplus$ makes $\mathrm{Proj}_{\mathrm{f.g.}}(R)$ into a symmetric monoidal category, M is an $\mathbb{E}_\infty$**-monoid**. Instead of group completing $\pi_0(M)$ we may group complete M itself. This leads to

$$\Omega^\infty \mathrm{K}(R) := M^{\mathrm{gp}} \simeq \Omega \mathrm{B} M.$$

Higher K-theory for rings III

A naive alternative approach to "group complete" M is by "inverting" f.g. projective R-modules P. Namely, we take the telescope

$$M[P^{-1}] := \mathrm{tel}_P(M) \simeq \mathrm{hocolim}(M \xrightarrow{\oplus P} M \xrightarrow{\oplus P} \cdots)$$

for all P.
Note that it suffices to take $M[R^{-1}]$: since P is projective,

$$P \oplus Q \cong R^{\oplus n}$$

for some n and some projective Q, so

$$[-P] = [Q] - [R^{\oplus n}].$$

Higher K-theory for rings IV

If $R = k$ is a field, the telescope $\mathrm{tel}_k(M) \simeq M[k^{-1}]$ is a colimit

$$
\begin{array}{cccccccccc}
 & & & & \mathrm{BGL}_0(k) & \coprod & \mathrm{BGL}_1(k) & \coprod & \mathrm{BGL}_2(k) & \cdots \\
 & & & & & & \big\downarrow & & \big\downarrow & \\
 & & \mathrm{BGL}_0(k) & \coprod & \mathrm{BGL}_1(k) & \coprod & \mathrm{BGL}_2(k) & & \cdots & \\
 & & & & \big\downarrow & & \big\downarrow & & & \\
\mathrm{BGL}_0(k) & \coprod & \mathrm{BGL}_1(k) & \coprod & \mathrm{BGL}_2(k) & & \cdots & & &
\end{array}
$$

This colimit is $\mathrm{BGL}(k) \times \mathbf{Z}$. For general R it gives $\mathrm{BGL}(R) \times \mathrm{K}_0(R)$.

We now have two candidates for the group completion of $M = \mathrm{Proj}(R)^{\simeq}$:

$$\Omega \mathrm{B} M \quad \text{and} \quad \mathrm{K}_0(R) \times \mathrm{BGL}(R).$$

These turn out to be different. In fact, $\mathrm{BGL}(R)$ is often not an infinite loop space: $\pi_1 \mathrm{BGL}(R) \simeq \mathrm{GL}(R)$, which is far from commutative in general.

In a sense, the $+$-construction is a fix for this:

$$\mathrm{K}_0(R) \times \mathrm{BGL}(R)^+ \simeq \Omega \mathrm{B} M \simeq \Omega^\infty \mathrm{K}(R).$$

Higher K-theory for rings VI

The following can be obtained from the "group completion theorem":

Theorem (McDuff–Segal, [3])

There is a map $\mathrm{BGL}(R) \to (\Omega \mathrm{B} M)_0$ *inducing a homology equivalence.*

As a corollary,

$$\begin{aligned} \pi_1((\Omega \mathrm{B} M)_0) &\cong \mathrm{H}_1((\Omega \mathrm{B} M)_0) \\ &\cong \mathrm{H}_1(\mathrm{BGL}(R)) \\ &\cong \pi_1(\mathrm{BGL}(R))/[\pi_1(\mathrm{BGL}(R)), \pi_1(\mathrm{BGL}(R))] \\ &\cong \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]. \end{aligned}$$

We conclude that $(\Omega BM)_0 \simeq \mathrm{BGL}(R)^+$: using the universal property of $(-)^+$, there is a map $(\Omega \mathrm{B} M)_0 \to \mathrm{BGL}(R)^+$ inducing a homology equivalence between these simple spaces.

Brauer lifting I

Recall that

$$\mathrm{BU}^{\psi^q} := \mathrm{hofib}(\mathrm{BU} \xrightarrow{\psi^q - 1} \mathrm{BU}).$$

We need a map

$$\mathrm{Br} : \mathrm{BGL}(k) \to \mathrm{BU}^{\psi^q}$$

inducing an equivalence in (co)homology groups.

Brauer lifting II

Idea. To find a map

$$\mathrm{Br} : \mathrm{BGL}(k) \to \mathrm{BU}^{\psi^q},$$

it suffices to find a compatible family of virtual complex representations

$$\mathrm{GL}_n(k) \to \mathrm{U}$$

fixed by $\psi^q \circlearrowright \mathrm{Rep}_{\mathbf{C}}(\mathrm{GL}_n(k))$, and then take colimit as $n \to \infty$.

Quillen found such a family using **Brauer lifting** to lift the standard representations $\mathrm{GL}_n(k) \circlearrowright k^n$ to *complex* representations.

Brauer lifting III

Make a **choice** of embedding of multiplicative groups

$$\rho : \bar{k}^* \hookrightarrow \mathbf{C}^*$$

where $\bar{k}$ is the algebraic closure of k and the choice of ρ determines a (non-canonical) isomorphism

$$\bar{k}^* \cong \bigoplus_{\ell \neq p} \mathbf{Q}_\ell / \mathbf{Z}_\ell.$$

Brauer lifting IV

For E a finite-dimensional representation of a finite group G over $\bar{k}$, define the **Brauer character** of E to be the $\mathbf{C}$-valued function

$$\chi_E(g) := \sum_i \rho(\lambda_i).$$

where $\{\lambda_i\}$ is the set of eigenvalues counted with multiplicity.

Theorem (Green)

The Brauer character χ_E is the character of a unique virtual complex representation $\mathrm{Br}(E) \in \mathrm{Rep}_{\mathbf{C}}(G)$.

This is the Brauer lifting. It gives a homomorphism

$$\mathrm{Rep}_{\bar{k}}(G) \to \mathrm{Rep}_{\mathbf{C}}(G).$$

Brauer lifting V

If E is a representation of G over $k < \bar{k}$, let

$$\bar{E} := E \otimes_k \bar{k}$$

be its "base-change" to $\bar{k}$. Thus we have the bottom maps in

$$\begin{array}{ccccc} & & & & \mathrm{Rep}_{\mathbf{C}}(G)^{\psi^q} \\ & & & \nearrow & \downarrow \\ \mathrm{Rep}_k(G) & \xrightarrow[E \mapsto \bar{E}]{} & \mathrm{Rep}_{\bar{k}}(G) & \xrightarrow[\mathrm{Br}]{} & \mathrm{Rep}_{\mathbf{C}}(G) \end{array}$$

Brauer lifting VI

$$\begin{array}{ccccc} & & & & \mathrm{Rep}_{\mathbf{C}}(G)^{\psi^q} \\ & & & \nearrow & \downarrow \\ \mathrm{Rep}_k(G) & \xrightarrow[E \mapsto \bar{E}]{} & \mathrm{Rep}_{\bar{k}}(G) & \xrightarrow[\mathrm{Br}]{} & \mathrm{Rep}_{\mathbf{C}}(G) \end{array}$$

We claim that the bottom map factors through $\mathrm{Rep}_{\mathbf{C}}(G)^{\psi^q}$. Namely, $\psi^q \circlearrowright \mathrm{Rep}_{\mathbf{C}}(G)$ fixes $\mathrm{Br}(\bar{E})$ for any $E \in \mathrm{Rep}_k(G)$. Indeed,

$$\chi_{\psi^q(\mathrm{Br}(\bar{E}))}(g) = \chi_{\mathrm{Br}(\bar{E})}(g^q) = \sum_i \rho(\lambda_i)^q = \sum_i \rho(\lambda_i^q) = \sum_i \rho(\lambda_i)$$

since the eigenvalues $\{\lambda_i\}$ of g are in k, hence stable under the Frobenius $\mathrm{Fr} : x \mapsto x^q$.

Brauer lifting VII

We get a map

$$\mathrm{Rep}_k(G) \to \mathrm{Rep}_{\mathbf{C}}(G)^{\psi^q}.$$

which is in fact an isomorphism. Composing with[3]

$$\mathrm{Rep}_{\mathbf{C}}(G)^{\psi^q} \to [\mathrm{B}G, \mathrm{BU}]^{\psi^q} \to [\mathrm{B}G, \mathrm{BU}^{\psi^q}]$$

we obtain

$$\mathrm{Rep}_k(G) \to [\mathrm{B}G, \mathrm{BU}^{\psi^q}].$$

Brauer lifting VIII

Take $G = \mathrm{GL}_n$ and use the standard representations

$$\mathrm{GL}_n(k) \circlearrowright k^n.$$

It gives a compatible family of maps

$$\mathrm{BGL}_n(k) \to \mathrm{BU}^{\psi^q},$$

hence a map $\mathrm{Br} : \mathrm{BGL}(k) \to \mathrm{BU}^{\psi^q}$ as $n \to \infty$. This is the desired map.

[3] The second map is an isomorphism since $[\mathrm{B}G, \mathrm{U}] = 0$ for G compact Lie.

Cohomology of BU^{ψ^q} I

Since $\pi_{\geq 1}(\mathrm{BU}^{\psi^q})$ are torsion with order prime relative to p ($q^i - 1$ or 0), the homology groups of this space with $\mathbf{Q}$ or $\mathbf{F}_p$ coefficients are all 0. So it suffices to compute $\mathrm{H}_*(\mathrm{BU}^{\psi^q}; \mathbf{F}_\ell)$.

Denote $\mathrm{H}_*(X; \mathbf{F}_\ell)$ by $\mathrm{H}_*(X)$ in this section. Recall that

$$r := \text{the least integer} \geq 1 \text{ such that } q^r \equiv 1 \bmod \ell.$$

We will assume that $\ell \neq 2$ so that we are in the **typical case** (the **exceptional case** $\ell = 2$ needs more care)[4].

Cohomology of BU^{ψ^q} II

First a crude computation. Using that BU^{ψ^q} is the homotopy pullback

$$\begin{array}{ccc} \mathrm{BU}^{\psi^q} & \longrightarrow & \mathrm{BU} \\ \downarrow & & \downarrow \Delta \\ \mathrm{BU} & \xrightarrow{(\mathrm{id},\psi^q)} & \mathrm{BU} \times \mathrm{BU} \end{array}$$

we obtain

$$\begin{array}{ccc} \mathrm{H}^*(\mathrm{BU}^{\psi^q}) & \longleftarrow & \mathrm{H}^*(\mathrm{BU}) \\ \uparrow & & \uparrow \Delta^* \\ \mathrm{H}^*(\mathrm{BU}) & \xleftarrow{(\mathrm{id},\psi^q)^*} & \mathrm{H}^*(\mathrm{BU})^{\otimes 2}. \end{array}$$

This is not a pushout square, but we have the Eilenberg–Moore spectral sequence with E_2-page

$$E_2^{s,t} = \mathrm{Tor}^{s,t}_{\mathrm{H}^*(\mathrm{BU})^{\otimes 2}}(\mathrm{H}^*(\mathrm{BU}), \mathrm{H}^*(\mathrm{BU})) \Rightarrow \mathrm{H}^*(\mathrm{BU}^{\psi^q}).$$

Cohomology of BU^{ψ^q} III

Recall that

$$\mathrm{H}^*(\mathrm{BU}) \cong \mathrm{P}[c_1, c_2, \dots]$$

We completely understand all rings and maps in the E_2-page (by the splitting principle, $(\psi^q)^*(c_i) = q^i c_i$). It follows that

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{\mathrm{H}^*(\mathrm{BU}\times\mathrm{BU})}(\mathrm{H}^*(\mathrm{BU}), \mathrm{H}^*(\mathrm{BU})) \cong \mathrm{P}[c_r, c_{2r}, \dots] \otimes \Lambda[e_r, e_{2r}, \dots]$$

where $|c_{jr}| = (0, 2jr)$ and $|e_r| = (-1, 2jr)$. The spectral sequence degenerates at the E_2-page for degree reasons.

Cohomology of BU^{ψ^q} IV

Lemma

For a suitable filtration of $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$,

$$\mathrm{Gr}(\mathrm{H}^*(\mathrm{BU}^{\psi^q})) \cong \mathrm{P}[c_r, c_{2r}, \ldots] \otimes \Lambda[e_r, e_{2r}, \ldots]$$

where $|c_{jr}| = 2jr$ *and* $|e_{jr}| = 2jr - 1$ *for all* $j \geq 1$.

This determines $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$ as an $\mathbf{F}_\ell$-vector space, but not as an algebra. Explicit generators $\{c_{jr}\}$ and $\{e_{jr}\}$ are found in [6] so that this determines the **algebra** structure on $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$.

- The generators $\{c_{jr} \in \mathrm{H}^*(\mathrm{BU}^{\psi^q})\}$ are pullbacks of the mod-ℓ Chern classes $\{c_{jr}\}$ of BU. (We abuse notation to denote both by $\{c_{jr}\}$.)
- The generators $\{e_{jr} \in \mathrm{H}^*(\mathrm{BU}^{\psi^q})\}$ are the reduction mod ℓ of certain classes $\tilde{e}_i \in \mathrm{H}^{2i-1}(\mathrm{BU}^{\psi^q}; \mathbf{Z}/(q^i - 1))$.

Cohomology of BU^{ψ^q} V

We sketch a proof that $\{c_{jr}\}$ and $\{e_{jr}\}$ generate $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$. Denote

- $C := \mathbf{Z}/(q^r - 1)$,
- $\zeta : C \to \mathbf{C}^*,\ 1 \mapsto e^{\frac{2\pi i}{p^r - 1}}$,
- $W := \zeta \oplus \zeta^{\otimes q} \oplus \cdots \oplus \zeta^{\otimes q^{r-1}} \in \mathrm{Rep}_{\mathbf{C}}(C)^{\psi^q}$.

Idea. Taking $\bigoplus_{i=1}^m W_i$ gives a **detection map**

$$\mathrm{H}^*(\mathrm{BU}^{\psi^q}) \xrightarrow{\bigoplus_{i=1}^m W_i} \mathrm{H}^*(\mathrm{B}C^m)$$

We completely understand $\mathrm{H}^*(\mathrm{B}C^m)$ and the image of c_{jr} and e_{jr} under this map (denote these by $\bar{c}_{jr}$ and $\bar{e}_{jr}$)[5].

Cohomology of BU^{ψ^q} VI

Lemma

1. $\bar{c}_{jr} = \bar{e}_{jr} = 0$ *for* $j > m$.
2. $(\bar{e}_{jr})^2 = 0$.
3. *The monomials* $(\alpha_j \geq 0,\ 0 \leq \beta_j \leq 1)$

$$\bar{c}_r^{\alpha_1} \bar{c}_{2r}^{\alpha_2} \cdots \bar{c}_{mr}^{\alpha_m} \bar{e}_r^{\beta_1} \bar{e}_{2r}^{\beta_2} \cdots \bar{e}_{mr}^{\beta_m}$$

are linearly independent.

Taking $m \to \infty$ we conclude:

Theorem

$$\mathrm{P}[c_r, c_{2r}, \dots] \otimes \Lambda[e_r, e_{2r}, \dots] \xrightarrow{\cong} \mathrm{H}^*(\mathrm{BU}^{\psi^q}).$$

Cohomology of BU^{ψ^q} VII

Conclusion. We have

$$\mathrm{H}^*(\mathrm{BU}^{\psi^q};\kappa) = \begin{cases} 0, & \kappa = \mathbf{Q} \text{ or } \kappa = \mathbf{F}_p, \\ \mathrm{P}[c_r, c_{2r}, \ldots] \otimes \Lambda[e_r, e_{2r}, \ldots], & \kappa = \mathbf{F}_\ell. \end{cases}$$

We next turn to the (co)homology of the space $\mathrm{BGL}(k)$.

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[4] The essential difference between these two cases is in the ring $\mathrm{H}^*(\mathrm{B}C)$.

[5] $\mathrm{H}^*(\mathrm{B}C) \cong \mathrm{P}[u] \otimes \Lambda(v)$ for $|u| = 2$ and $|v| = 1$ since we are in the typical case. Let $x := (-1)^{r-1}u^r$ and $y := (-1)^{r-1}u^{r-1}v$. We have

$$\bar{c}_{jr} = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j} = \sigma_j, \quad \bar{e}_{jr} = \sum_{i_1 < \cdots < i_j} \sum_{1 \le k \le j} x_{i_1} \cdots \hat{x}_{i_k} \cdots x_{i_j} y_{i_k} = d\sigma_j$$

where σ_j is the j^{th} symmetric polynomial in the variables $\{x_i\}$.

Cohomology of $\mathrm{BGL}(k)$ I

Rational (co)homology of finite groups are trivial. We have

$$\tilde{\mathrm{H}}_*(\mathrm{BGL}(k); \mathbf{Q}) \cong \operatorname{colim} \tilde{\mathrm{H}}_*(\mathrm{BGL}_n(k); \mathbf{Q}) = 0.$$

So it suffices to determine the cohomology of $\mathrm{BGL}(k)$ with $\mathbf{F}_\ell$ and $\mathbf{F}_p$ coefficients.

Cohomology of $\mathrm{BGL}(k)$ II

We deal with $\mathbf{F}_p$-coefficients first. The cohomology of $\mathrm{GL}_n(k)$ with $\mathbf{F}_p$ coefficients is hard in general. However, in the present situation we have:

Theorem (Quillen, [6])

Let $k = \mathbf{F}_q = \mathbf{F}_{p^d}$.

1. $\mathrm{H}^i(\mathrm{BGL}_n(k)) = 0$ *for* $0 < i < d(p-1)$ *and all* n.
2. $\mathrm{H}^i(\mathrm{BGL}(k)) = 0$ *for all* $i > 0$.

The first item is obtained by a comparison map

$$\mathrm{H}^*(\mathrm{BGL}_n(k)) \hookrightarrow \mathrm{H}^*(B_n)^{T_n}.$$

where $B_n < \mathrm{GL}_n(k)$ denotes the upper-triangular matrices, and T_n denotes the diagonal matrices. The second item follows from the first using a transfer argument (tensor up $E : G \to \mathrm{GL}(k)$ to some $E' = E \otimes_k k'$ where $[k' : k] = d'$ is large enough so that the corresponding characteristic class vanish and $(d', p) = 1$. Restrict back to k conclude).

Cohomology of $\mathrm{BGL}(k)$ III

We will next focus on mod-ℓ cohomology. Denote $\mathrm{H}^*(-;\mathbf{F}_\ell)$ by $\mathrm{H}^*(-)$ from now on.

Recall that in computing $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$ we crucially used

$$W := \zeta \oplus \zeta^{\otimes q} \oplus \cdots \oplus \zeta^{\otimes q^{r-1}} \in \mathrm{Rep}_{\mathbf{C}}(C)^{\psi^q},$$

and detected $\mathrm{H}^*(\mathrm{BU}^{\psi^q})$ by

$$\mathrm{B}C^m \xrightarrow{\bigoplus_{i=1}^m W_i} \mathrm{BU}^{\psi^q}.$$

It turns out that we can explicitly find a modular representation L of C whose Brauer lift is W! We will detect $\mathrm{H}^*(\mathrm{BGL}(k))$ by

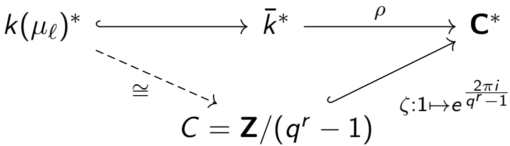
$$\mathrm{B}C^m \xrightarrow{\bigoplus_{i=1}^m L_i} \mathrm{BGL}(k) \xrightarrow{\mathrm{Br}} \mathrm{BU}^{\psi^q}.$$

Let $k(\mu_\ell)$ denote k with a primitive ℓth root of unity adjoined. Recall that

$$r := \text{the least integer} \geq 1 \text{ such that } q^r \equiv 1 \bmod \ell.$$

Then $[k(\mu_\ell) : k] = r$ since $\mathrm{Gal}(k(\mu_\ell)/k)$ is generated by $\mathrm{Fr}(x) = x^q$ and has order r.

The group $k(\mu_\ell)^*$ is **non-canonically** isomorphic to $C := \mathbf{Z}/(q^r - 1)$. But we made a chocie $\rho : \bar{k}^* \hookrightarrow \mathbf{C}^*$.



There is thus a **canonical** isomorphism $C \cong k(\mu_\ell)^*$ making the above diagram commute.

Denote by $L \in \mathrm{Rep}_k(C)$ the modular representation

$$C \cong k(\mu_\ell)^* \circlearrowright k(\mu_\ell),$$

where $k(\mu_\ell)^*$ acts by multiplication.

Lemma

The Brauer lift of L equals W, making the following diagram commute:

$$\begin{array}{ccc} \mathrm{B}C & \xrightarrow{W=\bigoplus_{i=0}^{r-1}\zeta^{q^i}} & \mathrm{BU}^{\psi^q} \\ & {\scriptstyle L}\searrow \quad \nearrow{\scriptstyle \mathrm{Br}} & \\ & \mathrm{BGL}(k) & \end{array}.$$

Cohomology of $\mathrm{BGL}(k)$ VI

We show that $\mathrm{Br}(L) \cong W$ by directly computing its character. There is a ring isomorphism

$$L \otimes_k \bar{k} = k(\mu_\ell) \otimes_k \bar{k} \cong \bar{k}^r, \quad z \otimes w \mapsto (zw, z^q w, \dots, z^{q^{r-1}} w).$$

Therefore, the $C \cong k(\mu_\ell)^*$-action on $L \otimes_k \bar{k}$ has Brauer character

$$z \mapsto \sum_{i=0}^{r-1} \rho(z^{q^i})$$

which is equal to the character of

$$W = \zeta \oplus \zeta^q \oplus \cdots \oplus \zeta^{q^{r-1}}.$$

Cohomology of $\mathrm{BGL}(k)$ VII

We may now detect $\mathrm{H}^*(\mathrm{BGL}(k))$ by L. Let $n = mr + e$ with $0 \leq e < r$. Then $L^{\oplus m}$ extends to a representation of

$$(C \rtimes \pi)^m \rtimes \Sigma_m$$

where $\pi := \mathrm{Gal}(k(\mu_\ell)/k)$. Further taking e copies of the trivial representation, this gives an embedding

$$L^{\oplus m} \oplus k^{\oplus e} : (C \rtimes \pi)^m \rtimes \Sigma_m \hookrightarrow \mathrm{GL}_n(k),$$

and hence a map[6]

$$\mathrm{H}^*(\mathrm{BGL}_n(k)) \to \mathrm{H}^*(C^m)^{\pi^m \rtimes \Sigma_m}.$$

Cohomology of $\mathrm{BGL}(k)$ VIII

Theorem

We have

$$\mathrm{H}^*(\mathrm{BGL}_n(k)) \cong \mathrm{H}^*(C^m)^{\pi^m \rtimes \Sigma_m}$$

in the typical case. Furthermore, $\mathrm{H}^*(C^m)^{\pi^m \rtimes \Sigma_m}$ *is also the image of*

$$\mathrm{H}^*(\mathrm{BU}^{\psi^q}) \xrightarrow{(\bigoplus_i W_i)^*} \mathrm{H}^*(C^m).$$

Take $n, m \to \infty$ in

$$\mathrm{H}^*(\mathrm{BU}^{\psi^q}) \to \mathrm{H}^*(\mathrm{BGL}(k)) \to \mathrm{H}^*(\mathrm{BGL}_n(k)) \xrightarrow{\cong} \mathrm{H}^*(C^m)^{\pi^m \rtimes \Sigma_m}.$$

Note that this composition is the same as $(\bigoplus_i W_i)^*$. We get

$$\mathrm{H}^*(\mathrm{BGL}(k)) \cong \mathrm{H}^*(\mathrm{BU}^{\psi^q}).$$

This concludes our computation of $\mathrm{K}_*(\mathbf{F}_q)$.

[6] We use that inner automorphisms induce identity map on group (co)homology.

Historical remarks I

How could one guess that

$$\mathrm{H}^*(\mathrm{BGL}(k); \mathbf{Z}) \cong \mathrm{H}^*(\mathrm{BU}^{\psi^q}; \mathbf{Z})$$

in the first place? Here are some speculations.
Quillen considered the map

$$\mathrm{BGL}(\bar{k}) \to \mathrm{BU}$$

in proving the Adams conjecture. He showed that this induces an isomorphism in cohomology with $\mathbf{F}_\ell$ coefficients.

Historical remarks II

The Adams operation ψ^q is a "Frobenius lift". The Frobenius Fr acts on $\mathrm{GL}(\bar{k})$ and we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{BGL}(k) & \longrightarrow & \mathrm{BGL}(\bar{k}) & \xrightarrow{\mathrm{B}(\mathrm{Fr})-1} & \mathrm{BGL}(\bar{k}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BU}^{\psi^q} & \longrightarrow & \mathrm{BU} & \xrightarrow{\psi^q-1} & \mathrm{BU} \end{array}$$

It is perhaps a natural next step to think about what the left vertical map does to (co)homology groups.

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