

# On the "Secondary power operations and the Brown-Peterson spectrum at the prime 2"

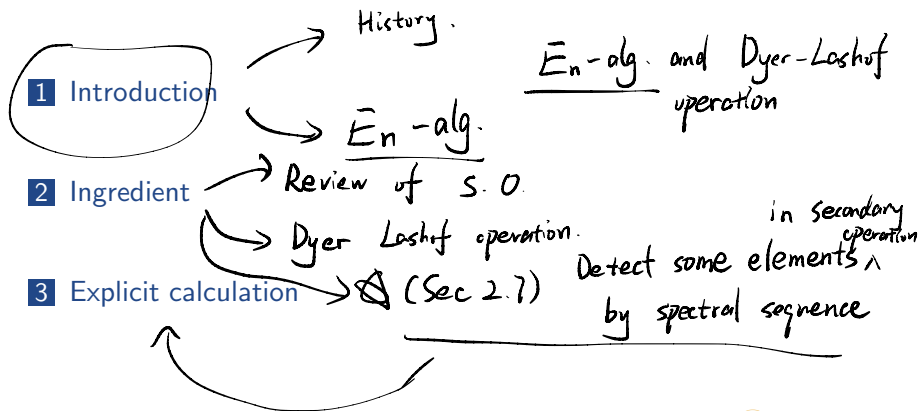
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June 1~~8~~, 2024



# Outline



# Outline

**1** Introduction

2 Ingredient

3 Explicit calculation



# Cup- $i$ products

Cup- $i$  products encode the communications of the coherently commutative multiplication structures.

## Definition

A cochain complex  $C_*$  has cup- $i$  products if it is equipped with operations  $(x, y) \rightarrow x \smile_i y$  for  $i \geq 0$  such that

$$x \smile y \stackrel{\cong}{=} y \smile x$$

↑  
cup-1 prod.

$$x \smile_i y \stackrel{\cong}{=} y \smile_i x$$

↑  
cup- $i$  prod.



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$$x \otimes y \rightarrow x \smile_i y$$



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- if  $x \in C^p$ ,  $y \in C^q$ , then  $x \smile_i y \in C^{p+q-i}$
- $(x + x') \smile_i y = x \smile_i y + x' \smile_i y$  and the same result is true for  $y$ .
- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$

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- for  $i \geq 0$ ,

$$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y)$$

cup-0 prod

= cup prod.

$$x \smile_{i-1} y + y \smile_{i-1} x.$$

Some facts about cup- $i$  products

$$\textcircled{1}. D_0: C_* \rightarrow C_* \times C_*$$

(cup product).

(Any homology group).

$$(U \cup_0 V)(G) = (U \otimes V)(D_0 G)$$

$$D_{i-1} \Rightarrow D_i \quad \underline{D_i \partial + \partial D_i} = D_{i-1} + e D_{i-1}$$

(e: switch two elements)

$$(U \cup_i V)(G) \triangleq (U \otimes V)(D_i G).$$

$$\textcircled{T^2}:$$

$$H_*(T^2)$$

$$\alpha \cup \beta = (\pm) (\alpha + \beta)$$

$$a \xrightarrow{\Delta} a \otimes a.$$

$$\alpha, \beta \in H_*(T^2).$$



Some facts about cup- $i$  products

$$(1) \quad Sq^i(X) = X \cup_{m-i} X \quad m \rightarrow m+i$$

$$\quad \quad \quad \underline{\deg X = m} \quad m+m-(m-i) :$$

(2)  $\bar{E}_1$  : asso. prod.  $\hookrightarrow \bar{E}_n$  ?  
 $\bar{E}_0$  : commu. prod

---


$$(3) \quad \bar{E}_{n+1}\text{-alg.} \cong \bar{E}_1\text{-algs in (Cort. of } \bar{E}_n\text{-alg.)}$$

# Some history about the possible $E_\infty$ -structures on $BP$

## ■ Baas-Suïllvan theory

↳ Give some geo. realization of some spectra.

(with  $E_1$ -str)

Rob 89

La2 01

⋮

BP supports an  $E_\infty$  str.

# Some history about the possible $E_\infty$ -structures on $BP$

- Baas-Suïlvan theory
- Get  $E_n$ -structures by killing the obstructions in  $E_{n-1}$ -structures.

[Rie 06]  $BP \sim \underline{E}_n$  M-Basterra -  
M.A. Mandell.

Moy.  $E_4$   
[BM13]  $\circ BP \hookrightarrow MU_{(p)}$ , summand  
 $\circ BP$  supports an  $\underline{E}_4$ -str.

TAQ thy

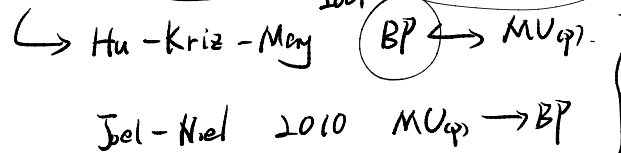


# Some history about the possible $E_\infty$ -structures on $BP$

- Baas-Suillvan theory
- Get  $E_n$ -structures by killing the obstructions in  $E_{n-1}$ -structures.

■ If not?  $E_\infty$ -structures support power operations

BMMS 86  
Hoo Ring spectra  
and their applications



# Some history about the possible $E_\infty$ -structures on $BP$

relation in Dyer-Lashof operation

$$\langle \mathcal{J}_1^2, \mathcal{Q}, \mathcal{R} \rangle = \mathcal{J}_5 \pmod{\text{sth.}}$$

- Baas-Sullivan theory
- Get  $E_n$ -structures by killing the obstructions in  $E_{n-1}$ -structures.
- If not?  $E_\infty$ -structures support power operations
- Find a (secondary) operation such that  $H_*BP$  is not close under this operation.

$$\begin{array}{l}
 (\varphi=2) \quad H_*BP = \mathbb{F}_2[\mathcal{J}_1^2, \mathcal{J}_2^2, \dots] \quad \mathcal{Q}(\mathcal{J}_{a_1}^{2a_1}, \mathcal{J}_{a_2}^{2a_2}, \dots) \\
 \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
 H_*H\mathbb{F}_2 = \mathbb{F}_2[\mathcal{J}_1, \mathcal{J}_2, \dots] \quad \quad \quad \mathcal{I}_{2k+1}
 \end{array}$$



# Ingredient

■
Operations
 on
 (mod- $p$ ) homology
:
 (
 Dyer-Lashof operations
)
  
+
( . . . )

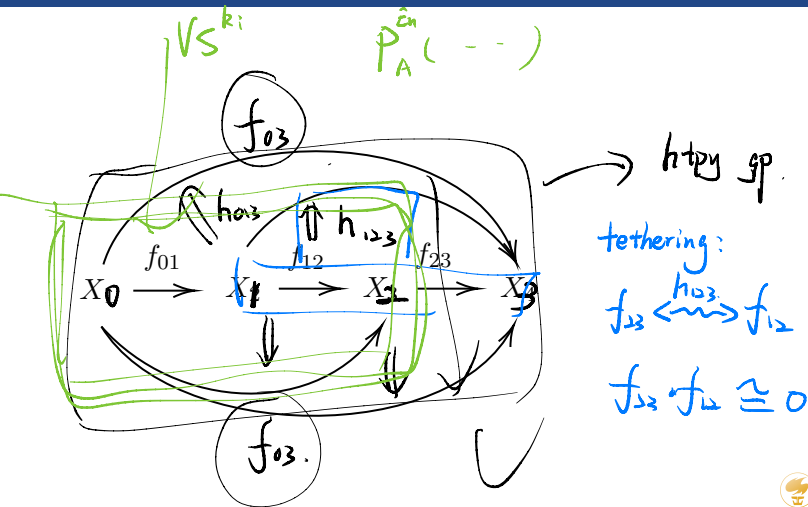


# Ingredient

- Operations on  $(\text{mod-}p)$  homology: Dyer-Lashof operations
- Calculating some secondary operations on it by some spectral sequences.



# Secondary operations

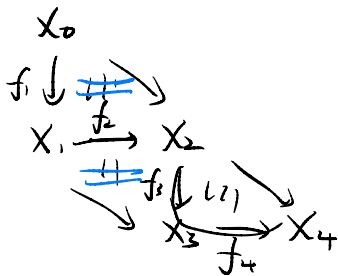


# Secondary operations

$f_{03}$  : base pt.

The associated secondary composite is the element of  $\pi_1(\text{Map}_{\mathcal{D}}(X_0, X_3), f_{03})$  represented by the path composite  $(h_{023})^{-1} \cdot (f_{23} h_{012})^{-1} \cdot (h_{123} f_{01}) \cdot h_{013}$

## Secondary operations



(Juggling).

$$\frac{\langle f_4 f_3, f_2, f_1 \rangle}{f_4 \langle f_3, f_2, f_1 \rangle}$$



# Secondary operations

## Definition

A tethering of this composite is a homotopy class of nullhomotopy of  $gf$ : a homotopy class of path  $h : gf \Rightarrow *$  in  $Map_C(X_0, X_2)$ . We will write  $h : g \rightsquigarrow f$  to indicate such a tethering.

# Secondary operations

## Definition

For the above diagram, if we have a tethering  $h : f_{12} \circ f_{23} \xrightarrow{\sim} 0$  and  $f_{12} \circ f_{01}$  is nullhomotopic, we write

$$\langle f_{23} \xrightarrow{\sim} f_{12}, f_{01} \rangle \subset \pi_1(\text{Map}_{\mathcal{C}}(X_0, X_3), *)$$

for the set of all elements  $\langle f_{23} \xrightarrow{\sim} f_{12} \xrightarrow{\sim} f_{01} \rangle$ , where the later tethering  $k$  ranges over possible tetherings. This also decides an operation for  $f_{01}$  such that  $f_{12} \circ f_{01}$  is nullhomotopic, which is called the secondary operation determined by the tethering.

The set of maps  $f_{01}$  such that  $f_{12} \circ f_{01}$  is nullhomotopic is referred to as the domain of definition of this secondary operation, and the possibly multivalued nature of this function is referred to as the indeterminacy of the secondary operation.



# Zero and Indeterminacy

## Theorem

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- *A secondary operation  $\langle f_{23} \rightsquigarrow f_{12}, - \rangle$  determines a well-defined map  $\Phi$  on  $\ker f_{12} \subset \pi_0 \text{Map}_{\mathcal{C}}(X_0, X_1)$  whose values are right cosets:*

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- $\ker f_{12} \rightarrow (f_{23} \pi_1 \text{Map}_C(X_0, X_2)) \setminus \pi_1 \text{Map}_C(X_0, X_3)$

base pt. : 0

# Zero and Indeterminacy

## Theorem

- *Changing the tethering and homotopy class of maps alters the value of a secondary composite by multiplication by loops.*
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- $\ker f_{12} \rightarrow (f_{23} \pi_1 \text{Map}_C(X_0, X_2)) \backslash \pi_1 \text{Map}_C(X_0, X_3)$
- *If two tetherings  $h, h'$  give rise to operations  $\Phi, \Phi'$ , then there exists an element  $u \in \pi_1 \text{Map}_C(X_1, X_3)$  such that  $\Phi x = \Phi' x \cdot (ux)$  for all  $x \in \ker f_{12} \subset \pi_0 \text{Map}_C(X_0, X_1)$ .*

Example: A secondary operation on  $\mathbb{C}P^\infty$ 

$$\begin{array}{ccc} \phi: H^*(\mathbb{C}P^\infty) & : & x \in H^*(\mathbb{C}P^\infty) \\ \downarrow & & \downarrow \\ \text{zero} & & x^3 \end{array}$$

$\phi(H^2(\mathbb{C}P^\infty)) \text{ zero} = \langle x, 0 \rangle$

indeterminacy: 0

$$S_9^4 S_9^1 + S_9^2 S_9^3 + S_9^1 S_9^4$$

$$S_9^4(x) = 0 \quad \checkmark$$

$S_9^4(x^2) \rightarrow \text{Cartan formula} = 0$

$$[x^3] \times \begin{matrix} 0 \\ 1 \end{matrix}$$

$$S_9^4(x^2) = x^4$$



# Homotopy operations

↑ cohomology operations:  $X_1 \rightarrow X_2$ .  $X_i \in Kp$

$$\oplus \pi_{k_j}(-) \rightarrow \pi_a(-)$$

$$\downarrow \vee \left[ S^{k_j}, - \right] \quad \left[ S^a, - \right]$$

$$S^a \rightarrow \vee S^{k_j}$$

# Homotopy operations

Homotopy operations can be represented by maps between spheres and their dot unions.

## Definition

Given a commutative ring spectrum  $A$ , we let  $\mathbb{P}_A^{E_n}$  be the left adjoint to the forgetful functor from  $E_n$   $A$ -algebras to spectra; if  $n = \infty$ , we simply write  $\mathbb{P}_A$ , and if  $A = \mathbb{S}$ , then we will omit  $A$  from the notation. What's more,

$$\mathbb{P}_A^{E_n}(X) \cong \bigvee A \wedge (E_n(k)_+ \wedge_{\Sigma_k} X^{\wedge k})$$

$\mathbb{P}_A^{E_n} \text{Spec} \rightleftarrows E_n A\text{-alg. } \mathbb{F}$

where the spaces  $E_n(k)$  are the terms in our chosen  $E_n$ -operad

# Homotopy operations

## Definition

A homotopy operation on  $E_n$   $A$ -algebras is a natural transformation of functors

$$\prod \pi_{k_i}(-) \rightarrow \pi_j(-)$$

represented by a homotopy class of map of  $E_n$   $A$ -algebras

$$\mathbb{P}_A^{E_n}(S^j) \rightarrow \mathbb{P}_A^{E_n}(\vee S^{k_i})$$

## Homotopy operations

$B$ :  $E_n$   $A$ -alg.

hty op. on  $E_n$   $A$ -alg. under  $B$

$$B \parallel P_A^{\tilde{E}_n}(s^j) \rightarrow \dots$$

$E_n$ :  $E_n$   $B$ -Alg.

curr. secondary operation.

$$B \parallel P_A^{\tilde{E}_n}(s^j) \xrightarrow{R} B \parallel P_A^{\tilde{E}_n}(V_i s^{k_i})$$

$$\downarrow \quad \nearrow \quad \downarrow Q$$

$$B \rightarrow B \parallel P_A^{\tilde{E}_n}(V_i s^{l_i})$$

$$\leftarrow -, (Q \leftarrow R)$$

$$Q \cdot R = 0$$

cuprod.  
(Lin Alg).

## Dyer-Lashof operations

$A: \mathbb{H}\mathbb{F}_2$

## Theorem

For any commutative  $H$ -algebra  $A$ , there are homotopy operations

$$Q^s : \pi_k \rightarrow \pi_{k+s}$$

for  $E_n$   $A$ -algebras when  $s < k + n - 1$ , called the Dyer-Lashof operations. These satisfy the following relations:

# Dyer-Lashof operations

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- *the Cartan formula:  $Q^s(xy) = \sum_{p+q=s} Q^p(x)Q^q(y)$ ;*



## Dyer-Lashof operations

Theorem

Section 10 → Rezk Lecture notes of power op.

For any commutative  $H$ -algebra  $A$ , there are homotopy operations

Kuhn

$$Q^s : \pi_k \rightarrow \pi_{k+s}$$

① constr. of DL. operation

② DL-operation ↔ Ste. Alg.

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  - the instability relations:  $Q^s x = x^2$  when  $|x| = s$ ,  $Q^s x = 0$  when  $|x| > s$ ;  
(Peter May)
  - the Cartan formula:  $Q^s(xy) = \sum_{p+q=s} Q^p(x)Q^q(y)$ ;
  - the Adem relations: If  $r > 2s$ , then  
 $Q^r Q^s(x) = \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$ .
- homology op. on infinite loop space  
(forgetful map)  
 $m \leq n$ .  $E_n \text{ alg} \xrightarrow{f} E_m \text{ alg}$   
preserve Dyer-Lashof operation.

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Dyer-Lashof operations  $\nearrow A$ .  $H_*: H\mathbb{F}_2$ .

## Theorem

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For any commutative  $H$ -algebra  $A$ , all homotopy operations for  $E_\infty$   $A$ -algebras  $C$  are composites of the following types:

- the constant operation associated to an element  $\alpha \in \pi_n A$  which takes no arguments and whose value on  $C$  is the image of  $\alpha$  under the map  $\pi_* A \rightarrow \pi_* C$ .

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- *the Dyer-Lashof operations  $Q^s : \pi_n(C) \rightarrow \pi_{n+s}(C)$*

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- *the Dyer-Lashof operations  $Q^s : \pi_n(C) \rightarrow \pi_{n+s}(C)$*
- *the binary addition operations  $\pi_n(C) \times \pi_n(C) \rightarrow \pi_n(C)$*

# Dyer-Lashof operations

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- the Dyer-Lashof operations  $Q^s : \pi_n(C) \rightarrow \pi_{n+s}(C)$
- the binary addition operations  $\pi_n(C) \times \pi_n(C) \rightarrow \pi_n(C)$
- the binary multiplication operations  $\pi_n(C) \times \pi_m(C) \rightarrow \pi_{n+m}(C)$ .

*Handwritten note:*  $\mathbb{H}$ : Ho ring - - -  
↳ [IX.2]

# Dyer-Lashof operations

## Theorem

The suspension operator, on homotopy operations for  $E_n$   $A$ -algebras under  $B$  takes zero-preserving homotopy operations  $\prod \pi_{l_s} \rightarrow \pi_k$  to homotopy operations  $\prod \pi_{(l_s+1)} \rightarrow \pi_{k+1}$ . Suspension preserves addition, composition, and multiplication by scalars from  $B$ . Suspension also takes  $Q^s$  to  $Q^s$  and takes the binary multiplication operation  $\pi_p \times \pi_q \rightarrow \pi_{p+q}$  to the trivial operations.

$$\Sigma S^j \rightarrow \Sigma \cup S^{k_i}$$



# Geometric realization of some secondary operations

Some secondary operations can be detected by spectral sequences.

## Theorem

Suppose  $X, Y$ , and  $Z$  are spectra,  $X \xrightarrow{f} Y \xrightarrow{g} Z$  is nullhomotopic, and that  $\alpha \in \ker(f) \subset \pi_n(X)$  is represented by a map  $S^n \rightarrow X$ . Given any extension  $\mathcal{C}f \xrightarrow{h} Z$  from the mapping cone representing a tethering, the secondary operation  $\langle g \rightsquigarrow f, \alpha \rangle$  is (up to sign) the set  $h(\partial^{-1}\alpha)$ , where  $\partial: \pi_{n+1}\mathcal{C}f \rightarrow \pi_n X$  is the connecting homomorphism in the long exact sequence of homotopy groups.

$$S^n \rightarrow X \rightarrow \mathcal{C}f \rightarrow Z$$



# Geometric realization of some secondary operations

Corollary  $X_i$ :  $i$ -skeleton.

Suppose that  $X_*$  is a simplicial spectrum with geometric realization  $X$  and that  $F$  is the homotopy fiber in the sequence  $F \xrightarrow{j} X_1 \xrightarrow{d_0} X_0$ . Then the composite  $F \xrightarrow{d_1 j} X_0 \xrightarrow{i} |X_*|$  has a canonical tethering. If  $\alpha \in \pi_n(F) \subset \pi_n X_1$  is in the kernel of  $d_1$ , then in the geometric realization spectral sequence

$$H_p(\pi_q X_*) \Rightarrow \pi_{p+q} |X_*|$$

the secondary operation  $\langle i \leftarrow d_1 j, \alpha \rangle$  is represented (up to sign) by the element  $[\alpha] \in H_1(\pi_n X)$  in the spectral sequence.



## Geometric realization of some secondary operations

 $|X_*|$  1-skeleton.

$$X_i \vee X_i \rightarrow X_i \wedge [0, 1]_+$$

$$d_{i0} \vee d_{i1} \downarrow \qquad \qquad \qquad \downarrow$$

$$X_0 \rightarrow SK^{(1)}(|X_*|)$$

a htpy  $\wedge$   $(id_0) \sim (id)$  is given by the above diagram.  
between

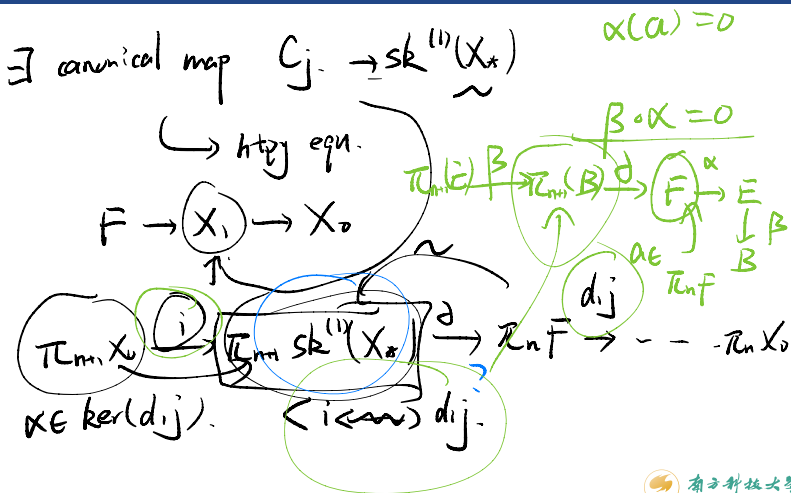
$$d_{0j} \Rightarrow 0$$

$$id_{ij} \Rightarrow id_{0j}$$

$$0 \swarrow$$



# Geometric realization of some secondary operations



# Geometric realization of some secondary operations

$$\pi_* sk^{(i)}(X_*) \rightarrow \pi_* sk^{(i+v)}(X_*)$$

$\left[ \pi_* sk^{(i+v)} / sk^{(i)} \right]$

# Geometric realization of some secondary operations

## Theorem

Suppose  $f : R \rightarrow S$  is a map of commutative ring spectra, and let  $i = 1 \wedge f : S \wedge R \rightarrow S \wedge S$ . Then, in the (pointed) category of augmented commutative  $S$ -algebras, there is a canonical tethering  $p \rightsquigarrow i$  for the composite

$$\boxed{S^n \rightarrow S \wedge R \xrightarrow{i} S \wedge S} \xrightarrow{p} S \wedge_R S$$

Let  $x \in \pi_n(S \wedge R)$  map to zero in  $\pi_n(S \wedge S)$ , so that  $\sigma x = \langle p \rightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$  is defined. Then  $\sigma x$  is detected by the image of  $x$  under  $\pi_n(S \wedge R) \rightarrow \pi_n(S \wedge R \wedge S)$  in the two-sided bar construction spectral sequence

$$H_p(\pi_q(S \wedge R^{\wedge*} \wedge S)) \Rightarrow \pi_{p+q}(S \wedge_R S)$$

## Geometric realization of some secondary operations

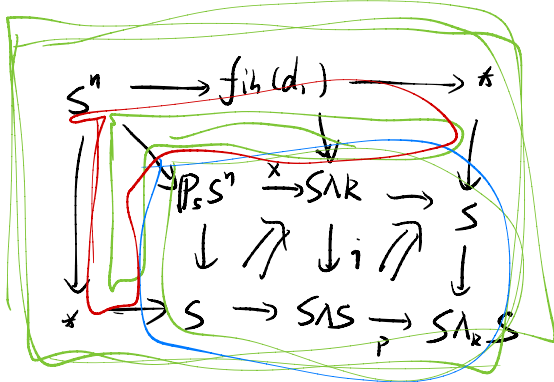
$$\boxed{SARAS} \begin{matrix} \Rightarrow \\ \Rightarrow \end{matrix} SAS \rightarrow S\lambda_2 S.$$

$$d_j: SAR \rightarrow SARAS \rightarrow SAS$$

$$j=0: SAR \rightarrow S \xrightarrow{\eta} SAS \quad \text{null map}$$

$$j=1 \quad SAR \rightarrow SAS$$

# Geometric realization of some secondary operations



$\langle p, \langle \sim \rangle i, X \rangle$



# Geometric realization of some secondary operations

If  $\pi_*(S \wedge R)$  is flat over  $S$ , we can identify the  $E_2$ -term in the two-sided bar construction spectral sequence:

$$E_{**}^2 = \text{Tor}_{**}^{\pi_*(S \wedge R)}(\pi_*(S \wedge S), \pi_*(S)) \Rightarrow \pi_*(S \wedge_R S)$$

The element  $x$  gives rise to the corresponding element in  $\text{Tor}_{1,n}$ . In particular, we have the following result when the target is the mod-2 Eilenberg-Mac Lane spectrum.

$$(S = H\mathbb{F}_2)$$



# Geometric realization of some secondary operations

## Theorem

*Suppose  $R \rightarrow H$  is a map of  $E_\infty$ -algebras and  $x \in H_n R$  maps to zero in the dual Steenrod algebra  $H_* H$ . Then there is an element  $\sigma x = \langle p \rightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$  in the  $R$ -dual Steenrod algebra  $\pi_*(H \wedge_R H)$  that is detected by the image of  $x$  in homological filtration 1 of the spectral sequence*

$$\mathrm{Tor}_{**}^{H_* R}(H_*, H_* H) \Rightarrow \pi_*(H \wedge_R H)$$

$$R \cong MU$$

## Geometric realization of some secondary operations

We now specialize this result to the case where  $MU$  is the complex bordism spectrum.

$\sigma \in SL$   
Sec 3.4: Some DL operation act on the  $MU$ .

## Theorem

Let  $n$  be an integer that is not of the form  $2^k - 1$  for any  $k$ , so that the corresponding generator  $b_n \in H_{2n}MU \cong \mathbb{F}_2[b_1, b_2, \dots]$  in mod-2 homology is the Hurewicz image of the generator  $x_n \in \pi_{2n}MU \cong \mathbb{Z}[x_1, x_2, \dots]$ . Then the diagram of  $E_\infty$   $H$ -algebras

$$\mathbb{P}_H S^{2n} \xrightarrow{b_n} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

determines a bracket, and  $\sigma x_n \equiv \langle p \rightsquigarrow i, b_n \rangle \text{ mod decomposables.}$

$\sigma x_n \rightarrow$  可被更进一步处理  
 $Q^S(\sigma x_n) = ? \text{ (mod } \dots)$

# Outline

- 1 Introduction
- 2 Ingredient
- 3 Explicit calculation**



# Actions of Dyer-Lashof operations on $MU$

[Pr: 75] Dyer-Lashof operations for the classifying space of certain matrix groups.

## Theorem

The Dyer-Lashof operations in  $H_*MU = H_*BU$  are determined by the following identity:

$$\sum Q^j b_k = \left( \sum_{n=k}^{\infty} \sum_{u=0}^k \binom{n-k+u-1}{u} b_{n+u} b_{k-u} \right) \left( \sum_{n=0}^{\infty} b_n \right)^{-1}$$

Here  $b_0 = 1$  by convention. In particular, we have

$$\sum Q^j b_1 = \left( \sum_{n=1}^{\infty} (b_n b_1 + (n-1)b_{n+1}) \right) \left( \sum_{n=0}^{\infty} b_n \right)^{-1}$$

Remark: Find out how to cal. explicit actions of DL. op. on some explicit homology sp. DL. up. Ste-up 大学

# Actions of Dyer-Lashof operations on $MU$

# Actions of Dyer-Lashof operations on $H$

## Theorem

*The 2-primary Dyer-Lashof operations in the dual Steenrod algebra satisfy the following identities:*

$$1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \cdots = (1 + \xi_1 + \xi_2 \cdots)^{-1}$$

$$Q^s \bar{\xi}_i = Q^{s+2^i-2} \xi_1 \quad \text{if } s \equiv 0, -1 \pmod{2^i} \quad 0 \quad \text{otherwise}$$

$$Q^{2^i} \bar{\xi}_i = \xi_{i+1}^-$$

Functional operations for  $MU \rightarrow H\mathbb{Z}/2$ 

## Theorem

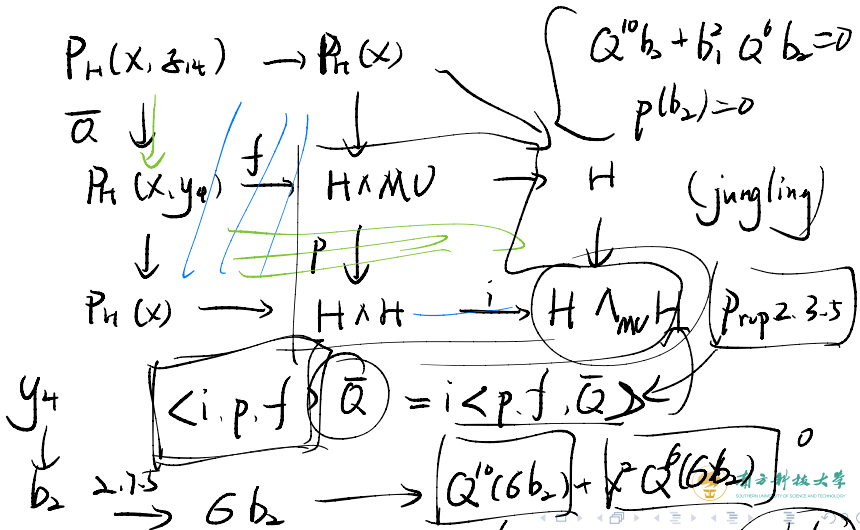
Consider the maps

$$\mathbb{P}_H(x, z_{14}) \xrightarrow{\bar{Q}} \mathbb{P}_H(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

in the category  $\mathcal{C}$ , where  $\bar{Q}$  sends  $z_{14}$  to  $Q^{10}y_4 + x^2Q^6y_4$  and  $f$  sends  $(x, y_4)$  to  $(b_1, b_2)$ . Then a functional homotopy operation  $\langle p, f, \bar{Q} \rangle$  is defined in  $\mathbb{P}_H(x)$ -algebras and satisfies  $\langle p, f, \bar{Q} \rangle \equiv \xi_4 \pmod{\text{decomposables}}$ .



# Functional operations for $MU \rightarrow \mathbb{Z}/2$



## A secondary operation in the dual Steenrod algebra

$$\begin{array}{c}
 \textcircled{6X_1} \equiv \textcircled{i(\mathcal{I}_4)} \\
 H_* H \xrightarrow{i} H_* MU \\
 \textcircled{i \langle p.f, \bar{Q} \rangle} \equiv \textcircled{\langle i.p.f \rangle \bar{Q}} \equiv \textcircled{i(\mathcal{I}_4)} \\
 \textcircled{\delta \bar{Q}} \quad \textcircled{(\text{mod decomp})} \\
 \text{indeterminacy?} \swarrow \searrow \\
 \langle p.f, \bar{Q} \rangle \equiv \mathcal{I}_4 \text{ (mod decomp.)}
 \end{array}$$

## A secondary operation in the dual Steenrod algebra

$(Q, R)$        $\langle x, (Q \rightarrow R) \rangle$

$HR = \bar{Q}v + \beta\alpha$

$\langle \mathcal{S}_i^2, (Q, R) \rangle = \langle p b_i, (Q, R) \rangle$

$\quad \quad \quad \cdot \langle p, b_i, (Q, R) \rangle$

$\quad \quad \quad = \langle p, f \mu, (R) \rangle$

$\quad \quad \quad \rightarrow \langle p, f, HR \rangle$

$\xrightarrow{f} \equiv$

$\langle p, f, \bar{Q} \rangle v$

$\langle p, f, \beta \rangle \alpha$

mod (decomp)

## A secondary operation in the dual Steenrod algebra

$$\langle \mathcal{J}_1^2 Q, R \rangle \equiv \langle p, f, \bar{Q} \rangle \nu + \langle p, f, \beta \rangle \alpha$$

$$\langle p, f, \bar{Q} \rangle (\nu \mathcal{J}_{30}) = Q^{16}(\dots) = \mathcal{J}_5$$

$\equiv 0 \pmod{\text{decamp}}$

$$\Rightarrow \langle \mathcal{J}_1^2 Q, R \rangle \equiv \mathcal{J}_5 \pmod{\text{decamp}}$$

# A secondary operation in the dual Steenrod algebra

In the dual Steenrod algebra, any element in the bracket  $\langle \xi_1^2, Q, R \rangle$  is congruent to  $\xi_5$  mod decomposables.

# A secondary operation in the dual Steenrod algebra

## Theorem

The Brown–Peterson spectrum  $BP$  is connective, with  $\pi_0 BP \cong \mathbb{Z}_{(2)}$ . The map  $BP \rightarrow H\mathbb{F}_2$  induces an inclusion  $H_* BP \hookrightarrow H_* H\mathbb{F}_2$  whose image is the subalgebra

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] \subset \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables.

# The End

Surgery thy.



E. Medsen?

(红皮书)

The Five

(Surgery)

→ Classifying manifold  $\cong$  " principle bundle