On the "Secondary power operations and the Brown-Peterson spectrum at the prime 2"

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Outline

1. Introduction
2. Ingredient
3. Explicit calculation

History.

E_n-alg.

E_n-alg. and Dyer-Lashof operation

Review of S.O.

Dyer Lashof operation.

☆ (Sec 2.7)

Detect some elements in secondary operation by spectral sequence

Outline

Cup-i products

Cup-i products encode the communications of the coherently commutative multiplication structures.

Definition

A cochain complex C_* has cup-i products if it is equipped with operations $(x, y) \to x \smile_i y$ for $i \geq 0$ such that

$x \cup y \hat{=} y \cup x$

cup-1 prod.

(?)

$x \cup_1 y \hat{=} y \cup_1 x$

cup-2 prod.

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- if $x \in C^p$, $y \in C^q$, then $x \smile_i y \in C^{p+q-i}$

$x \otimes y \longrightarrow x \cup_i y$

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- $(x + x') \smile_i y = x \smile_i y + x' \smile_i y$ and the same result is true for y.

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- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$

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cup-0 prod = cup prod.

- for $i \geq 0$,
  $\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y) + x \smile_{i-1} y + y \smile_{i-1} x.$

Some facts about cup-i products

① $D_0 : C_* \to C_* \times C_*$ (Any homology group).

(cup roduct).

$$(U \cup_0 V)(\sigma) = (U \otimes V)(D_0 \sigma)$$

$$D_{i-1} \Rightarrow D_i \quad \underline{D_i \partial + \partial D_i} = D_{i-1} + e D_{i-1}$$

(e : switch two elements)

$$(u \cup_i v)(\sigma) \triangleq (u \otimes v)(D_i \sigma).$$

T^2:

$$\alpha \cup_1 \beta = (\pm)(\alpha + \beta) \qquad a, b \in H_1(T^2).$$

$$H_*(T^2) \qquad a \xrightarrow{\Delta} a \otimes a.$$

Some facts about cup-i products

① $Sq^i(x) = x \cup_{m-i} x$. $m \to m+i$.

$\underline{\deg x = m}$ $m+m-(m-i)$:

② E_1 : asso. prod. E_n ?

E_∞ : commu. prod

③ E_{n+1}-alg. $\simeq$ E_1-algs in (Cat. of E_n-alg)

Some history about the possible E_∞-structures on BP

- Baas-Suillvan theory

BP supports an E_∞ str.

$\hookrightarrow$ Give some geo. realization of some spectra.

(with E_1-str)

Rob 89

Laz 01

⋮

Some history about the possible E_∞-structures on BP

- Baas-Suillvan theory
- Get E_n-structures by killing the obstructions in E_{n-1}-structures.

M-Basterra.

[Rie 06] $BP \sim E_n$ M.A. Mandell.

TAQ thy

May.

[BM13] ① $BP \xhookrightarrow{E_4} MU_{(2)}$, summand

② BP supports an E_4-str.

Some history about the possible E_∞-structures on BP

- Baas-Suillvan theory
- Get E_n-structures by killing the obstructions in E_{n-1}-structures.
- If not? E_∞-structures support power operations

BMMS 86

H_∞ Ring spectra and their applications

2001

$\hookrightarrow$ Hu-Kriz-May

$BP \longleftrightarrow MU_{(2)}$.

not E_∞-map.

Joel-Noel 2010 $MU_{(2)} \longrightarrow BP$

Some history about the possible E_∞-structures on BP

relation in Dyer-Lashof operation

$$\langle \xi_1^2, Q, R \rangle = \xi_5 \pmod{\text{sth}}.$$

- Baas-Suillvan theory
- Get E_n-structures by killing the obstructions in E_{n-1}-structures.
- If not? E_∞-structures support power operations
- Find a (secondary) operation such that H_*BP is not close under this operation.

$(p=2)$

$$H_*BP = \mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots]$$

$$\downarrow$$

$$H_*H\mathbb{F}_2 = \mathbb{F}_2[\xi_1, \xi_2, \cdots]$$

$$Q(\xi_{a_1}^{2a}\xi_{a_2}^{2b}\cdots)$$

$$\downarrow$$

$$\xi_m^{2k+1}$$

Ingredient

- Operations on (mod-p) homology: (Dyer-Lashof operations)

$+($ $)$

Ingredient

- Operations on (mod-p) homology: Dyer-Lashof operations
- Calculating some secondary operations on it by some spectral sequences.

Outline

Secondary opeartions

Vs^{k_i} $\quad P^{\hat{e}_n}_A(- -)$

$$
\begin{tikzcd}
X_0 \arrow[r, "f_{01}"] \arrow[rr, bend left, "h_{013}"] \arrow[rrr, bend left=60, "f_{03}"] \arrow[rr, bend right] & X_1 \arrow[r, "f_{12}"] \arrow[rr, bend left, "h_{123}"] \arrow[rr, bend right] & X_2 \arrow[r, "f_{23}"] & X_3
\end{tikzcd}
$$

f_{03}.

$\to$ htpy gp

tethering:

$f_{23} \overset{h_{123}}{\rightsquigarrow} f_{12}$

$f_{23} \cdot f_{12} \simeq 0$

Secondary opeartions

f_{03} : base pt.

The associated secondary composite is the element of $\pi_1(Map_{\mathcal{D}}(X_0, X_3), f_{03})$ represented by the path composite $(h_{023})^{-1} \cdot (f_{23}h_{012})^{-1} \cdot (h_{123}f_{01}) \cdot h_{013}$

Secondary oeartions

$$\begin{tikzcd}
X_0 \arrow[d, "f_1"'] \arrow[dr] \arrow[d, phantom, shift left=3ex, "="] & & \\
X_1 \arrow[r, "f_2"] \arrow[dr] \arrow[r, phantom, shift right=3ex, "="] & X_2 \arrow[d, "f_3"'] \arrow[dr] \arrow[d, phantom, shift left=3ex, "(2)"] & \\
& X_3 \arrow[r, "f_4"'] & X_4
\end{tikzcd}$$

(Juggling).

$\underline{\langle f_4 f_3, f_2, f_1 \rangle}$

$\cup$

$\underline{f_4 \langle f_3, f_2, f_1 \rangle}$

Secondary opeartions

Definition

A tethering of this composite is a homotopy class of nullhomotopy of gf: a homotopy class of path $h : gf \Rightarrow *$ in $Map_{\mathcal{C}}(X_0, X_2)$. We will write $h : g \leftrightsquigarrow f$ to indicate such a tethering.

Secondary opeartions

Definition

For the above diagram, if we have a tethering $h : f_{12} \circ f_{23} \leftrightsquigarrow 0$ and $f_{12} \circ f_{01}$ is nullhomotopic, we write

$$\langle f_{23} \leftrightsquigarrow f_{12}, f_{01} \rangle \subset \pi_1(Map_{\mathcal{C}}(X_0, X_3), *)$$

for the set of all elements $\langle f_{23} \leftrightsquigarrow f_{12} \leftrightsquigarrow f_{01} \rangle$, where the later tethering k ranges over possible tetherings. This also decides an operation for f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic, which is called the secondary operation determined by the tethering.
The set of maps f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic is referred to as the domain of definition of this secondary operation, and the possibly multivalued nature of this function is referred to as the indeterminacy of the secondary operation.

Zero and Indeterminacy

Theorem

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- *Changing the tethering and homotopy class of maps alters the value of a secondary composite by multiplication by loops.*

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- *A secondary operation* $\langle f_{23} \leftrightsquigarrow f_{12}, -\rangle$ *determines a well-defined map* Φ *on* $ker f_{12} \subset \pi_0 Map_{\mathcal{C}}(X_0, X_1)$ *whose values are right cosets:*

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base pt. : 0

- $ker\, f_{12} \to (f_{23}\pi_1 Map_{\mathcal{C}}(X_0, X_2)) \backslash \pi_1 Map_{\mathcal{C}}(X_0, X_3)$

Zero and Indeterminacy

Theorem

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- $ker\, f_{12} \to (f_{23}\pi_1 Map_{\mathcal{C}}(X_0, X_2))\backslash \pi_1 Map_{\mathcal{C}}(X_0, X_3)$
- *If two tetherings* h, h' *give rise to operations* Φ, Φ', *then there exists an element* $u \in \pi_1 Map_{\mathcal{C}}(X_1, X_3)$ *such that* $\Phi x = \Phi' x \cdot (ux)$ *for all* $x \in ker\, f_{12} \subset \pi_0 Map_{\mathcal{C}}(X_0, X_1)$.

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Example: A secondary operation on $\mathbb{C}P^\infty$

$\phi : H^*(\mathbb{C}P^\infty) : \quad x \in H^*(\mathbb{C}P^\infty)$

$\downarrow$

x^3

$\phi(x^2) \to H^6$

$\phi(H^2(\mathbb{C}P^\infty))$ zero: $\langle x, 0 \rangle$

indeterminacy : 0

$Sq^4 Sq^1 + Sq^2 Sq^3 + Sq^1 Sq^4$

$Sq^4(x) = 0$ ✓

$Sq^2(x^2) \longrightarrow$ Cartan formula $= 0$.

$[x^3] \times \begin{matrix} 0 \\ 1 \end{matrix}$

$Sq^4(x^2) = x^4$

Homotopy operations

↑ cohomology operations: $X_1 \to X_2$. $X_i \in K_p$

$$\bigoplus \pi_{k_j}(-) \longrightarrow \pi_a(-)$$

$$\| \qquad\qquad\qquad\qquad$$

$$\bigvee [S^{k_j}, -] \qquad [S^a, -]$$

$$S^a \longrightarrow \bigvee S^{k_j}$$

Homotopy operations

Homotopy operations can be represented by maps between spheres and their dot unions.

Definition

Given a commutative ring spectrum A, we let $\mathbb{P}_A^{E_n}$ be the left adjoint to the forgetful functor from E_n A-algebras to spectra; if $n = \infty$, we simply write $\mathbb{P}_A$, and if $A = \mathbb{S}$, then we will omit A from the notation. What's more,

$\mathbb{P}_A^{E_n}$ Spec. $\underset{F}{\overset{\mathbb{P}_A^{E_n}}{\rightleftarrows}}$ E_n A-alg.

$$\mathbb{P}_A^{E_n}(X) \cong \bigvee A \wedge (E_n(k)_+ \wedge_{\Sigma_k} X^{\wedge k})$$

where the spaces $E_n(k)$ are the terms in our chosen E_n-operad

Homotopy operations

Definition

A homotopy operation on E_n A-algebras is a natural transformation of functors

$$\prod \pi_{k_i}(-) \to \pi_j(-)$$

represented by a homotopy class of map of E_n A-algebras

$$\mathbb{P}_A^{E_n}(S^j) \to \mathbb{P}_A^{E_n}(\vee S^{k_i})$$

.

Homotopy operations

B: E_n A-alg.

htpy op. on E_n A-alg. under B $\langle -, Q \rightsquigarrow R \rangle$

$B \amalg P_A^{E_n}(S^j) \longrightarrow \cdots$

E_∞: E_∞ B-Alg. L

corr. secondary operation.

$Q \circ R = 0$

coprod. (in Alg).

$$\begin{tikzcd}
B \amalg P_A^{E_n}(S^j) \arrow[r, "R"] \arrow[d] & B \amalg P_A^{E_n}(\vee_i S^{k_i}) \arrow[d, "Q"] \\
B \arrow[r] \arrow[ur, phantom, "\Nearrow"] & B \amalg P_A^{E_n}(\vee_{i,s} S^{l_{i,s}})
\end{tikzcd}$$

Dyer-Lashof operations

$A : H\mathbb{F}_2$

Theorem

For any commutative H-algebra A, there are homotopy operations

$$Q^s : \pi_k \to \pi_{k+s}$$

for E_n A-algebras when $s < k + n - 1$, called the Dyer-Lashof operations. These satisfy the following relations:

Dyer-Lashof operations

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- *the additivity relation: $Q^s(x + y) = Q^s(x) + Q^s(y)$;*

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- *the instability relations: $Q^s x = x^2$ when $|x| = s$, $Q^s x = 0$ when $|x| > s$;*

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- *the Cartan formula: $Q^s(xy) = \Sigma_{p+q=s} Q^p(x) Q^q(y)$;*

Dyer-Lashof operations

Theorem Section 5, 10 ⟶ Rezk Lecture notes of power op.

For any commutative H-algebra A, there are homotopy operations

Kuhn ① constr. of DL. operation

$$Q^s : \pi_k \to \pi_{k+s}$$

② DL-operation ⟷ Ste. Alg ?

for E_n A-algebras when $s < k + n - 1$, called the Dyer-Lashof operations. These satisfy the following relations:

homology op. on infinite loop space (Peter May)

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- *the instability relations:* $Q^s x = x^2$ *when* $|x| = s$, $Q^s x = 0$ *when* $|x| > s$;
- *the Cartan formula:* $Q^s(xy) = \Sigma_{p+q=s} Q^p(x)Q^q(y)$;
- *the Adem relations: If* $r > 2s$, *then* $Q^r Q^s(x) = \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$.

(forgetful map)

$m \leq n$. $E_n\text{alg} \xrightarrow{f} E_m\text{-alg}$

preserve Dyer-Lashof operation.

Dyer-Lashof operations

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Dyer-Lashof operations ?A. $H: H\mathbb{F}_2$.

Theorem

For any commutative H-algebra A, all homotopy operations for E_∞ A-algebras C are composites of the following types:

Dyer-Lashof operations

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For any commutative H-algebra A, all homotopy operations for E_∞ A-algebras C are composites of the following types:

- *the constant operation associated to an element $\alpha \in \pi_n A$ which takes no arguments and whose value on C is the image of α under the map $\pi_* A \to \pi_* C$.*

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- *the Dyer-Lashof operations $Q^s : \pi_n(C) \to \pi_{n+s}(C)$*

Dyer-Lashof operations

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- *the binary addition operations $\pi_n(C) \times \pi_n(C) \to \pi_n(C)$*

Dyer-Lashof operations

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- *the Dyer-Lashof operations $Q^s : \pi_n(C) \to \pi_{n+s}(C)$*
- *the binary addition operations $\pi_n(C) \times \pi_n(C) \to \pi_n(C)$*
- *the binary multiplication operations $\pi_n(C) \times \pi_m(C) \to \pi_{n+m}(C)$.*

併: H_∞ ring - - -

$\hookrightarrow [IX.2]$

Dyer-Lashof operations

Theorem

The suspension operator, on homotopy operations for E_n A-algebras under B takes zero-preserving homotopy operations $\prod \pi_{l_s} \to \pi_k$ to homotopy operations $\prod \pi_{l_s+1} \to \pi_{k+1}$. Suspension preserves addition, composition, and multiplication by scalars from B. Suspension also takes Q^s to Q^s and takes the binary multiplication operation $\pi_p \times \pi_q \to \pi_{p+q}$ to the trivial operations.

$$\Sigma S^j \to \Sigma \vee S^{k_i}$$

Geometric realization of some secondary operations

Some secondary operations can be detected by spectral sequences.

Theorem

Suppose X, Y , and Z are spectra, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is nullhomotopic, and that $\alpha \in ker(f) \subset \pi_n(X)$ is represented by a map $S^n \to X$. Given any extension $\cancel{X}$ X $\to Cf \xrightarrow{h} Z$ from the mapping cone representing a tethering, the secondary operation $\langle g \leftrightsquigarrow f, \alpha \rangle$ is (up to sign) the set $h(\partial^{-1}\alpha)$, where $\partial : \pi_{n+1}Cf \to \pi_n X$ is the connecting homomorphism in the long exact sequence of homotopy groups.

$S^n \to X \to Y \to Z$

Geometric realization of some secondary operations

Corollary

X_i : i-skeleton.

Suppose that X_* is a simplicial spectrum with geometric realization X and that F is the homotopy fiber in the sequence $F \xrightarrow{j} X_1 \xrightarrow{d_0} X_0$. Then the composite $F \xrightarrow{d_1 j} X_0 \xrightarrow{i} |X_*|$ has a canonical tethering. If $\alpha \in \pi_n(F) \subset \pi_n X_1$ is in the kernel of d_1, then in the geometric realization spectral sequence

$$H_p(\pi_q X_*) \Rightarrow \pi_{p+q}|X_*|$$

the secondary operation $\langle i \leftrightsquigarrow d_1 j, \alpha \rangle$ is represented (up to sign) by the element $[\alpha] \in H_1(\pi_n X)$ in the spectral sequence.

Geometric realization of some secondary operations

$|X_*|$ 1-skeleton.

$$\begin{tikzcd} X_1 \vee X_1 \arrow[r] \arrow[d, "d_0 \vee d_1"'] & X_1 \wedge [0,1]_+ \arrow[d] \\ X_0 \arrow[r] & sk^{(1)}|X_*| \end{tikzcd}$$

a htpy $\wedge$ between $id_0 \sim id_1$ is given by the above diagram.

$d_0 j \Rightarrow 0$ $\quad$ $i d_1 j \Rightarrow i d_0 j$

$\Rightarrow 0$.

Geometric realization of some secondary operations

$\exists$ canonical map $C_j \to sk^{(1)}(X_*)$

$\hookrightarrow$ htpy equ.

$$F \to X_1 \to X_0$$

$$\pi_{n+1}X_0 \xrightarrow{i} \pi_{n+1}\, sk^{(1)}(X_*) \xrightarrow{\partial} \pi_n F \to \cdots \pi_n X_0$$

$x \in \ker(d_1 j)$.

$\langle i(x) \rangle\, d_1 j$.

$\alpha(a) = 0$

$\beta \circ \alpha = 0$

$$\pi_{n+1}(E) \xrightarrow{\beta} \pi_{n+1}(B) \xrightarrow{\partial} (F) \xrightarrow{\alpha} E$$

$\downarrow \beta$

B

a_E

$\pi_n F$

$d_1 j$

Geometric realization of some secondary operations

$$\begin{tikzcd}
\pi_* sk^{(i)}|X_*| \arrow[r] & \pi_* sk^{(i+1)}|X_*| \arrow[dl] \\
\pi_* sk^{(i+1)}/sk^{(i)} \cdots \arrow[u] &
\end{tikzcd}$$

Geometric realization of some secondary operations

Theorem

Suppose $f : R \to S$ is a map of commutative ring spectra, and let $i = 1 \wedge f : S \wedge R \to S \wedge S$. Then, in the (pointed) category of augmented commutative S-algebras, there is a canonical tethering $p \leftrightsquigarrow i$ for the composite

$$\boxed{S^n \to S \wedge R \xrightarrow{i} S \wedge S} \xrightarrow{p} S \wedge_R S$$

Let $x \in \pi_n(S \wedge R)$ map to zero in $\pi_n(S \wedge S)$, so that $\sigma x = \langle p \leftrightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ is defined. Then σx is detected by the image of x under $\pi_n(S \wedge R) \to \pi_n(S \wedge R \wedge S)$ in the two-sided bar construction spectral sequence

$$H_p(\pi_q(S \wedge R^{\wedge *} \wedge S)) \Rightarrow \pi_{p+q}(S \wedge_R S)$$

Geometric realization of some secondary operations

$$S \wedge R \wedge S \rightrightarrows S \wedge S \longrightarrow S \wedge_R S.$$

$$d_j : S \wedge R \longrightarrow S \wedge R \wedge S \longrightarrow S \wedge S$$

$j = 0$: $S \wedge R \to S \xrightarrow{\eta_L} S \wedge S$. null map

$j = 1$ $S \wedge R \to S \wedge S$

Geometric realization of some secondary operations

$$
\begin{tikzcd}
S^n \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{fib}(d_1) \arrow[r] \arrow[d] & * \arrow[d] \\
& \mathbb{P}_s S^n \arrow[r, "x"] \arrow[d] & S\wedge k \arrow[r] \arrow[d, "i"] & S \arrow[d] \\
* \arrow[r] & S \arrow[r] \arrow[ur] & S\wedge S \arrow[r, "p"'] \arrow[ur] & S\wedge_k S
\end{tikzcd}
$$

$\langle p, \langle\sim\rangle i, x\rangle$

Geometric realization of some secondary operations

If $\pi_*(S \wedge R)$ is flat over S, we can identify the E_2-term in the two-sided bar construction spectral sequence:

$$E^2_{**} = Tor^{\pi_*(S\wedge R)}_{**}(\pi_*(S \wedge S), \pi_*(S)) \Rightarrow \pi_*(S \wedge_R S)$$

The element x gives rise to the corresponding element in $Tor_{1,n}$. In particular, we have the following result when the target is the mod-2 Eilenberg-Mac Lane spectrum.

$(S = H\mathbb{F}_2)$

Geometric realization of some secondary operations

Theorem

Suppose $R \to H$ *is a map of* E_∞*-algebras and* $x \in H_n R$ *maps to zero in the dual Steenrod algebra* H_*H*. Then there is an element* $\sigma x = \langle p \leftrightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ *in the* R*-dual Steenrod algebra* $\pi_*(H \wedge_R H)$ *that is detected by the image of* x *in homological filtration* 1 *of the spectral sequence*

$$Tor_{**}^{H_*R}(H_*, H_*H) \Rightarrow \pi_*(H \wedge_R H)$$

$R \triangleq MU$

Geometric realization of some secondary operations

σ ← SL_1 (

We now specialize this result to the case where MU is the complex bordism spectrum.

Sec 3.4: Some DL operation act on the MU.

Theorem

Let n be an integer that is not of the form $2^k - 1$ for any k, so that the corresponding generator $b_n \in H_{2n}MU \cong \mathbb{F}_2[b_1, b_2, \cdots]$ in mod-2 homology is the Hurewicz image of the generator $x_n \in \pi_{2n}MU \cong \mathbb{Z}[x_1, x_2, \cdots]$. Then the diagram of E_∞ H-algebras

σX_n → 可被更进一步处理

$Q^s(\sigma X_n) = ?$ (mod ⋯)

$$\mathbb{P}_H S^{2n} \xrightarrow{b_n} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

determines a bracket, and $\sigma x_n \equiv \langle p \leftrightsquigarrow i, b_n \rangle$ mod decomposables.

Outline

Actions of Dyer-Lashof operations on MU

[Pri: 73] Dyer-Lashof operations for the classifying space of of certain matrix groups.

Theorem

The Dyer-Lashof operations in $H_*MU = H_*BU$ are determined by the following identity:

$$\sum Q^j b_k = (\sum_{n=k}^{\infty} \sum_{u=0}^{k} \binom{n-k+u-1}{u} b_{n+u} b_{k-u})(\sum_{n=0}^{\infty} b_n)^{-1}$$

Here $b_0 = 1$ by convention. In particular, we have

$$\sum Q^j b_1 = (\sum_{n=1}^{\infty} (b_n b_1 + (n-1) b_{n+1}))(\sum_{n=0}^{\infty} b_n)^{-1}$$

on some explicit homology gp.

Remark: Find out how to cal. explicit actions of DL. op. / Ste.-op.

Actions of Dyer-Lashof operations on MU

Actions of Dyer-Lashof operations on H

Theorem

The 2-primary Dyer-Lashof operations in the dual Steenrod algebra satisfy the following identities:

$$1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \cdots = (1 + \xi_1 + \xi_2 \cdots)^{-1}$$

$$Q^s\bar{\xi_i} = Q^{s+2^i-2}\xi_1 \quad if \quad s \equiv 0, -1 mod 2^i \quad 0 \quad otherwise$$

$$Q^{2^i}\bar{\xi_i} = \bar{\xi_{i+1}}$$

Functional operations for $MU \to HZ/2$

Theorem

Consider the maps

$$\mathbb{P}_H(x, z_{14}) \xrightarrow{\bar{Q}} \mathbb{P}_H(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

in the category $\mathcal{C}$, where $\bar{Q}$ sends z_{14} to $Q^{10}y_4 + x^2Q^6y_4$ and f sends (x, y_4) to (b_1, b_2). Then a functional homotopy operation $\langle p, f, \bar{Q}\rangle$ is defined in $\mathbb{P}_H(x)$-algebras and satisfies $\langle p, f, \bar{Q}\rangle \equiv \xi_4$ mod decomposables.

Functional operations for $MU \to BHZ/2$

$$\begin{array}{ccccc}
P_H(X, z_{14}) & \longrightarrow & P_H(X) & & \\
\bar{Q} \downarrow & & \downarrow & & \\
P_H(X, y_4) & \xrightarrow{f} & H \wedge MU & \longrightarrow & H \\
\downarrow & & p \downarrow & & \downarrow \\
P_H(X) & \longrightarrow & H \wedge H & \xrightarrow{i} & H \wedge_{MU} H
\end{array}$$

$Q^{10} b_3 + b_1^2 Q^6 b_2 = 0$

$p(b_2) = 0$

(jingling)

Prop 2.3.5

$\boxed{\langle i, p, f \rangle} \bar{Q} = i \langle p, f, \bar{Q} \rangle$

y_4

$\downarrow$

$b_2 \xrightarrow{2.1.5} 6b_2 \longrightarrow \boxed{Q^{10}(6b_2)} + \boxed{b_1^2 Q^6(6b_2)} = 0$

4.4.6 $6X_7$ (mod decomp)

A secondary operation in the dual Steenrod algebra

$\sigma x_7 \equiv i(\zeta_4)$

$H_* H \xrightarrow{i} H_* MU$

$i\langle p, f, \bar{Q}\rangle \equiv \langle i, p, f\rangle \bar{Q} \equiv i(\zeta_4)$

$\sigma \bar{Q}$

(mod decomp)

indeterminacy?

$\langle p, f, \bar{Q}\rangle \equiv \zeta_4$ (mod decomp)

A secondary operation in the dual Steenrod algebra

Q, R $\quad \langle x, Q \to R \rangle$

mod decomp

$\mu R = \bar{Q}\nu + \beta\alpha$ $\quad \equiv$

$$\langle \xi_1^2, Q, R \rangle = \langle p b_1, Q, R \rangle$$

$$\subset \langle p, b, Q, R \rangle$$

$$= \langle p, f, \mu, R \rangle$$

$$\supset \langle p, f, \mu R \rangle$$

$$\langle p, f, \bar{Q} \rangle \nu + \langle p, f, \beta \rangle \alpha$$

A secondary operation in the dual Steenrod algebra

$\langle \zeta_1^2, Q, R \rangle \equiv \langle p, f, \bar{Q} \rangle v + \langle p, f, \beta \rangle \alpha$.

$\langle p, f, \bar{Q} \rangle (v \xi_{30}) = Q^{16}(\cdots) = \zeta_5$.

$\equiv 0$. mod decomp

$\Rightarrow \langle \zeta_1^2, Q, R \rangle \equiv \zeta_5$ mod decomp.

A secondary operation in the dual Steenrod algebra

In the dual Steenrod algebra, any element in the bracket $\langle \xi_1^2, Q, R \rangle$ is congruent to ξ_5 mod decomposables.

A secondary operation in the dual Steenrod algebra

Theorem

The Brown–Peterson spectrum BP is connective, with $\pi_0 BP \cong \mathbb{Z}_{(2)}$. The map $BP \to H\mathbb{F}_2$ induces an inclusion $H_*BP \hookrightarrow H_*H\mathbb{F}_2$ whose image is the subalgebra

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots] \subset \mathbb{F}_2[\xi_1, \xi_2, \cdots]$$

of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables.

The End

Surgery thy.

$\downarrow$

(红皮书)

E. Medsen?

The End

(Surgery)

Classifying manifold $\simeq$ "principle" bundle.