Secondary power operations and the Brown-Peterson spectrum at the prime 2

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Outline

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Cup-i products

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- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$
- for $i \geq 0$, $\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y) + x \smile_{i-1} y + y \smile_{i-1} x$.

Some facts about cup-i products

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- Baas-Suillvan theory
- Get E_n-structures by killing the obstructions in E_{n-1}-structures.
- If not? E_∞-structures support power operations
- Find a (secondary) operation such that H_*BP is not close under this operation.

Ingredient

- Operations on (mod-p) homology: Dyer-Lashof operations

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- Calculating some secondary operations on it by some spectral sequences.

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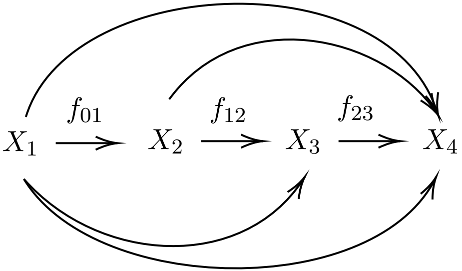
Secondary opeartions



Secondary opeartions

The associated secondary composite is the element of $\pi_1(Map_{\mathcal{D}}(X_0, X_3), f_{03})$ represented by the path composite $(h_{023})^{-1} \cdot (f_{23}h_{012})^{-1} \cdot (h_{123}f_{01}) \cdot h_{013}$

Secondary opeartions

Secondary opeartions

Definition

A tethering of this composite is a homotopy class of nullhomotopy of gf: a homotopy class of path $h : gf \Rightarrow *$ in $Map_{\mathcal{C}}(X_0, X_2)$. We will write $h : g \leftrightsquigarrow f$ to indicate such a tethering.

Secondary opeartions

Definition

For the above diagram, if we have a tethering $h : f_{12} \circ f_{23} \leftrightsquigarrow 0$ and $f_{12} \circ f_{01}$ is nullhomotopic, we write

$$\langle f_{23} \leftrightsquigarrow f_{12}, f_{01} \rangle \subset \pi_1(Map_{\mathcal{C}}(X_0, X_3), *)$$

for the set of all elements $\langle f_{23} \leftrightsquigarrow f_{12} \leftrightsquigarrow f_{01} \rangle$, where the later tethering k ranges over possible tetherings. This also decides an operation for f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic, which is called the secondary operation determined by the tethering.
The set of maps f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic is referred to as the domain of definition of this secondary operation, and the possibly multivalued nature of this function is referred to as the indeterminacy of the secondary operation.

Zero and Indeterminacy

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- *A secondary operation* $\langle f_{23} \leftrightsquigarrow f_{12}, - \rangle$ *determines a well-defined map* Φ *on* $ker\, f_{12} \subset \pi_0 Map_{\mathcal{C}}(X_0, X_1)$ *whose values are right cosets:*

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- *If two tetherings* h, h' *give rise to operations* Φ, Φ', *then there exists an element* $u \in \pi_1 Map_{\mathcal{C}}(X_1, X_3)$ *such that* $\Phi x = \Phi' x \cdot (ux)$ *for all* $x \in ker\, f_{12} \subset \pi_0 Map_{\mathcal{C}}(X_0, X_1)$.

Example: A secondary operation on $\mathbb{C}P^\infty$

Homotopy operations

Homotopy operations can be represented by maps between spheres and their dot unions.

Definition

Given a commutative ring spectrum A, we let $\mathbb{P}_A^{E_n}$ be the left adjoint to the forgetful functor from E_n A-algebras to spectra; if $n = \infty$, we simply write $\mathbb{P}_A$, and if $A = \mathbb{S}$, then we will omit A from the notation. What's more,

$$\mathbb{P}_A^{E_n}(X) \cong \bigvee A \wedge (E_n(k)_+ \wedge_{\Sigma_k} X^{\wedge k})$$

where the spaces $E_n(k)$ are the terms in our chosen E_n-operad

Homotopy operations

Definition

A homotopy operation on E_n A-algebras is a natural transformation of functors

$$\prod \pi_{k_i}(-) \to \pi_j(-)$$

represented by a homotopy class of map of E_n A-algebras

$$\mathbb{P}_A^{E_n}(S^j) \to \mathbb{P}_A^{E_n}(\vee S^{k_i})$$

.

Homotopy operations

Dyer-Lashof operations

Theorem

For any commutative H-algebra A, there are homotopy operations

$$Q^s : \pi_k \to \pi_{k+s}$$

for E_n A-algebras when $s < k + n - 1$, called the Dyer-Lashof operations. These satisfy the following relations:

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- *the Cartan formula: $Q^s(xy) = \Sigma_{p+q=s} Q^p(x) Q^q(y)$;*

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- *the Adem relations: If $r > 2s$, then $Q^r Q^s(x) = \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$.*

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- *the constant operation associated to an element $\alpha \in \pi_n A$ which takes no arguments and whose value on C is the image of α under the map $\pi_* A \to \pi_* C$.*

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- *the Dyer-Lashof operations $Q^s : \pi_n(C) \to \pi_{n+s}(C)$*

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- *the Dyer-Lashof operations $Q^s : \pi_n(C) \to \pi_{n+s}(C)$*
- *the binary addition operations $\pi_n(C) \times \pi_n(C) \to \pi_n(C)$*

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- *the binary addition operations $\pi_n(C) \times \pi_n(C) \to \pi_n(C)$*
- *the binary multiplication operations $\pi_n(C) \times \pi_m(C) \to \pi_{n+m}(C)$.*

Dyer-Lashof operations

Theorem

The suspension operator , on homotopy operations for E_n A-algebras under B, takes zero-preserving homotopy operations $\prod \pi_{l_s} \to \pi_k$ to homotopy operations $\prod \pi_{l_s+1} \to \pi_{k+1}$. Suspension preserves addition, composition, and multiplication by scalars from B. Suspension also takes Q^s to Q^s and takes the binary multiplication operation $\pi_p \times \pi_q \to \pi_{p+q}$ to the trivial operations.

Geometric realization of some secondary operations

Some secondary operations can be detected by spectral sequences.

Theorem

Suppose X, Y, and Z are spectra, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is nullhomotopic, and that $\alpha \in ker(f) \subset \pi_n(X)$ is represented by a map $S^n \to X$. Given any extension $X \to Cf \xrightarrow{h} Z$ from the mapping cone representing a tethering, the secondary operation $\langle g \leftrightsquigarrow f, \alpha \rangle$ is (up to sign) the set $h(\partial^{-1}\alpha)$, where $\partial : \pi_{n+1}Cf \to \pi_n X$ is the connecting homomorphism in the long exact sequence of homotopy groups.

Geometric realization of some secondary operations

Corollary

Suppose that X_* *is a simplicial spectrum with geometric realization* X *and that* F *is the homotopy fiber in the sequence* $F \xrightarrow{j} X_1 \xrightarrow{d_0} X_0$. *Then the composite* $F \xrightarrow{d_1 j} X_0 \xrightarrow{i} |X_*|$ *has a canonical tethering. If* $\alpha \in \pi_n(F) \subset \pi_n X_1$ *is in the kernel of* d_1, *then in the geometric realization spectral sequence*

$$H_p(\pi_q X_*) \Rightarrow \pi_{p+q}|X_*|$$

the secondary operation $\langle i \leftrightsquigarrow d_1 j, \alpha\rangle$ *is represented (up to sign) by the element* $[\alpha] \in H_1(\pi_n X)$ *in the spectral sequence.*

Geometric realization of some secondary operations

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Theorem

Suppose $f : R \to S$ is a map of commutative ring spectra, and let $i = 1 \wedge f : S \wedge R \to S \wedge S$. Then, in the (pointed) category of augmented commutative S-algebras, there is a canonical tethering $p \leftrightsquigarrow i$ for the composite

$$S \wedge R \xrightarrow{i} S \wedge S \xrightarrow{p} S \wedge_R S$$

Let $x \in \pi_n(S \wedge R)$ map to zero in $\pi_n(S \wedge S)$, so that $\sigma x = \langle p \leftrightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ is defined. Then σx is detected by the image of x under $\pi_n(S \wedge R) \to \pi_n(S \wedge R \wedge S)$ in the two-sided bar construction spectral sequence

$$H_p(\pi_q(S \wedge R^{\wedge *} \wedge S)) \Rightarrow \pi_{p+q}(S \wedge_R S)$$

Geometric realization of some secondary operations

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If $\pi_*(S \wedge R)$ is flat over S, we can identify the E_2-term in the two-sided bar construction spectral sequence:

$$E^2_{**} = Tor^{\pi_*(S\wedge R)}_{**}(\pi_*(S \wedge S), \pi_*(S)) \Rightarrow \pi_*(S \wedge_R S)$$

The element x gives rise to the corresponding element in $Tor_{1,n}$. In particular, we have the following result when the target is the mod-2 Eilenberg-Mac Lane spectrum.

Geometric realization of some secondary operations

Theorem

Suppose $R \to H$ is a map of E_∞-algebras and $x \in H_n R$ maps to zero in the dual Steenrod algebra $H_* H$. Then there is an element $\sigma x = \langle p \leftrightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ in the R-dual Steenrod algebra $\pi_*(H \wedge_R H)$ that is detected by the image of x in homological filtration 1 of the spectral sequence

$$Tor_{**}^{H_* R}(H_*, H_* H) \Rightarrow \pi_*(H \wedge_R H)$$

Geometric realization of some secondary operations

We now specialize this result to the case where MU is the complex bordism spectrum.

Theorem

Let n be an integer that is not of the form $2^k - 1$ for any k, so that the corresponding generator $b_n \in H_{2n}MU \cong \mathbb{F}_2[b_1, b_2, \cdots]$ in mod-2 homology is the Hurewicz image of the generator $x_n \in \pi_{2n}MU \cong \mathbb{Z}[x_1, x_2, \cdots]$. Then the diagram of E_∞ H-algebras

$$\mathbb{P}_H S^{2n} \xrightarrow{b_n} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

determines a bracket, and $\sigma x_n \equiv \langle p \leftrightsquigarrow i, b_n \rangle$ mod decomposables.

Outline

Actions of Dyer-Lashof operations on MU

Theorem

The Dyer-Lashof operations in $H_*MU = H_*BU$ are determined by the following identity:

$$\sum Q^j b_k = (\sum_{n=k}^{\infty}\sum_{u=0}^{k}\binom{n-k+u-1}{u} b_{n+u}b_{k-u})(\sum_{n=0}^{\infty} b_n)^{-1}$$

Here $b_0 = 1$ by convention. In particular, we have

$$\sum Q^j b_1 = (\sum_{n=1}^{\infty}(b_n b_1 + (n-1)b_{n+1}))(\sum_{n=0}^{\infty} b_n)^{-1}$$

Actions of Dyer-Lashof operations on MU

Actions of Dyer-Lashof operations on H

Theorem

The 2-primary Dyer-Lashof operations in the dual Steenrod algebra satisfy the following identities:

$$1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \cdots = (1 + \xi_1 + \xi_2 \cdots)^{-1}$$

$$Q^s\bar{\xi_i} = Q^{s+2^i-2}\xi_1 \quad if \quad s \equiv 0, -1 mod 2^i \quad 0 \quad otherwise$$

$$Q^{2^i}\bar{\xi_i} = \bar{\xi_{i+1}}$$

Functional operations for $MU \to \text{ß}H\mathbb{Z}/2$

Theorem

Consider the maps

$$\mathbb{P}_H(x, z_{14}) \xrightarrow{\bar{Q}} \mathbb{P}_H(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

in the category $\mathcal{C}$, where $\bar{Q}$ sends z_{14} to $Q^{10}y_4 + x^2Q^6y_4$ and f sends (x, y_4) to (b_1, b_2). Then a functional homotopy operation $\langle p, f, \bar{Q}\rangle$ is defined in $\mathbb{P}_H(x)$-algebras and satisfies $\langle p, f, \bar{Q}\rangle \equiv \xi_4$ mod decomposables.

Functional operations for $MU \to \text{ß}H\mathbb{Z}/2$

A secondary operation in the dual Steenrod algebra

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In the dual Steenrod algebra, any element in the bracket $\langle \xi_1^2, Q, R \rangle$ is congruent to ξ_5 mod decomposables.

A secondary operation in the dual Steenrod algebra

Theorem

The Brown–Peterson spectrum BP is connective, with $\pi_0 BP \cong \mathbb{Z}_{(2)}$. The map $BP \to H\mathbb{F}_2$ induces an inclusion $H_*BP \hookrightarrow H_*H\mathbb{F}_2$ whose image is the subalgebra

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots] \subset \mathbb{F}_2[\xi_1, \xi_2, \cdots]$$

of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables.

The End