Braid groups are algebraic objects that can be used to decompose knots and links into some simple building blocks corresponding to crossings.

Definition 2 (Braid groups)**.** *The group*

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} : \begin{array}{ll} \textit{if} \quad |i-j| > 1, & \sigma_i \sigma_j = \sigma_j \sigma_j \\ \textit{if} \quad i = 1, \ldots, n-2 & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle, \tag{1.1}$$

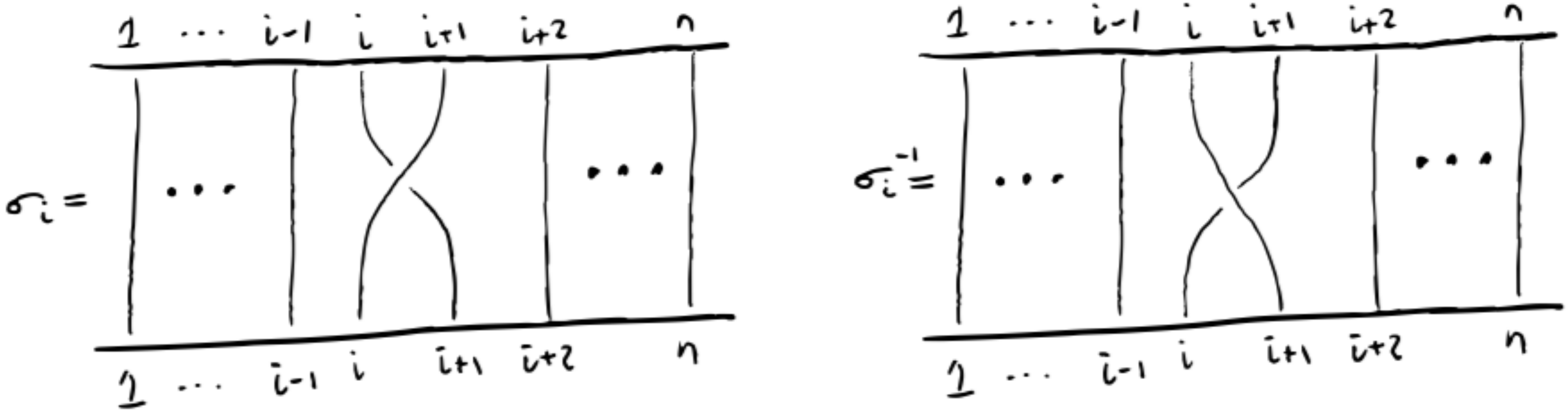


Figure 1.9: Generators of the braid group from crossings.

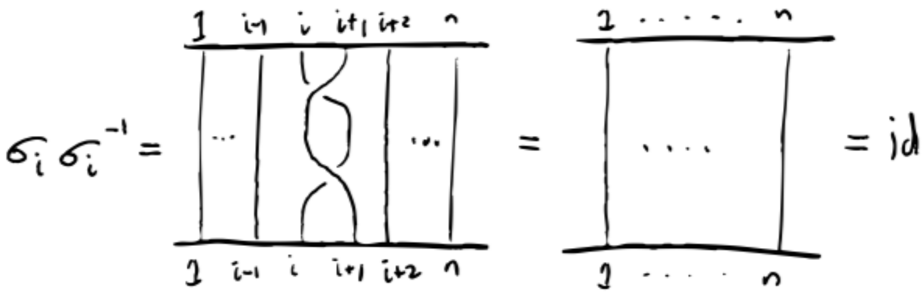


Figure 1.10: Inverse in the braid group corresponds to Reidemeister II in Figure 1.3.

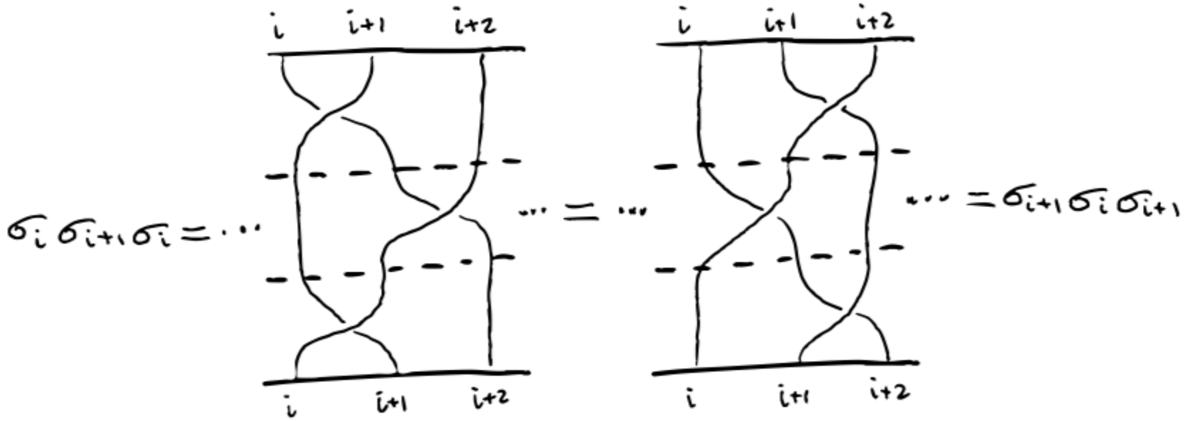


Figure 1.11: Relations in the braid group corresponds to Reidemeister III in Figure 1.3. The top left crossing is passed under.

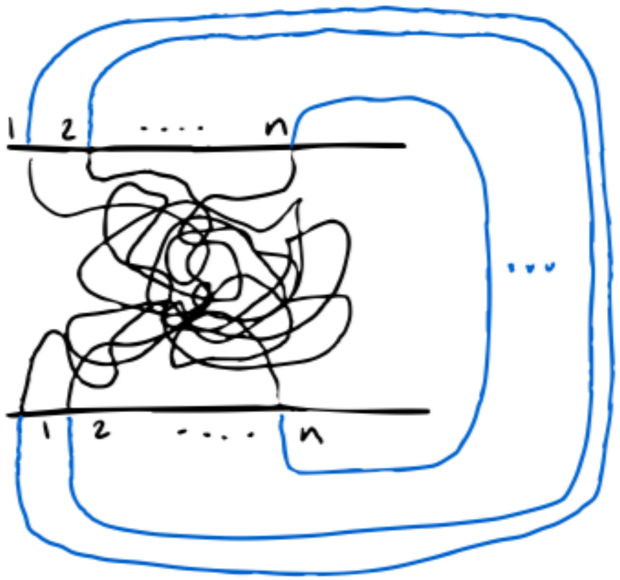


Figure 1.13: The closure of a braid.

Theorem A–1. *[109, Thm. 2.3]. Every oriented link in S^3 is isotopic to a braid closure.*

This shows that we have a surjection from the set of braids onto the set of links. The question is then of course what equivalence relation this surjection gives *i.e.* how are braids in the preimage of a link related. To describe this equivalence relation, we need to understand how the closure can swap a braid element from the top to the bottom, and how Reidemeister I in Figure 1.3 can be realised with braids.

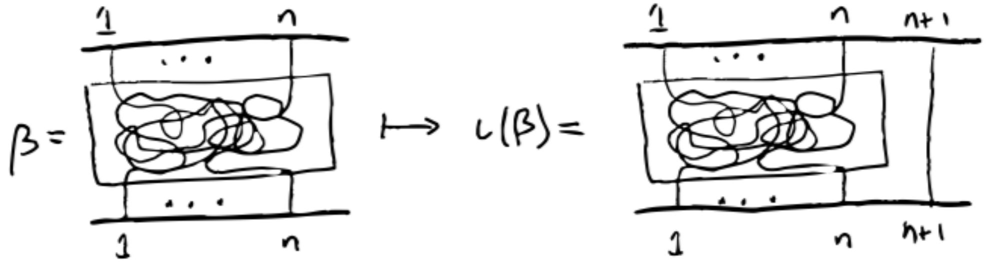


Figure 1.12: Embeddings of the braid groups in each other.

$$\iota_n : B_n \to B_{n+1} \quad s.t. \quad \iota(\sigma_i) \;=\; \sigma_i\,.$$

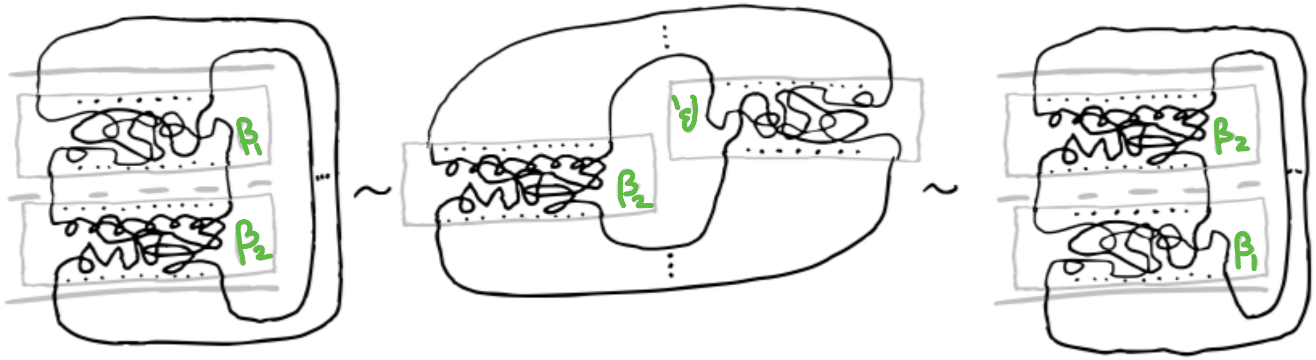


Figure 1.14: The first Markov move implied by isotoping along the closure.

$\beta_1\beta_2 \sim \beta_2\beta_1 \quad \Rightarrow \quad \beta_1 \sim \beta_2\beta_1\beta_2^{-1}$

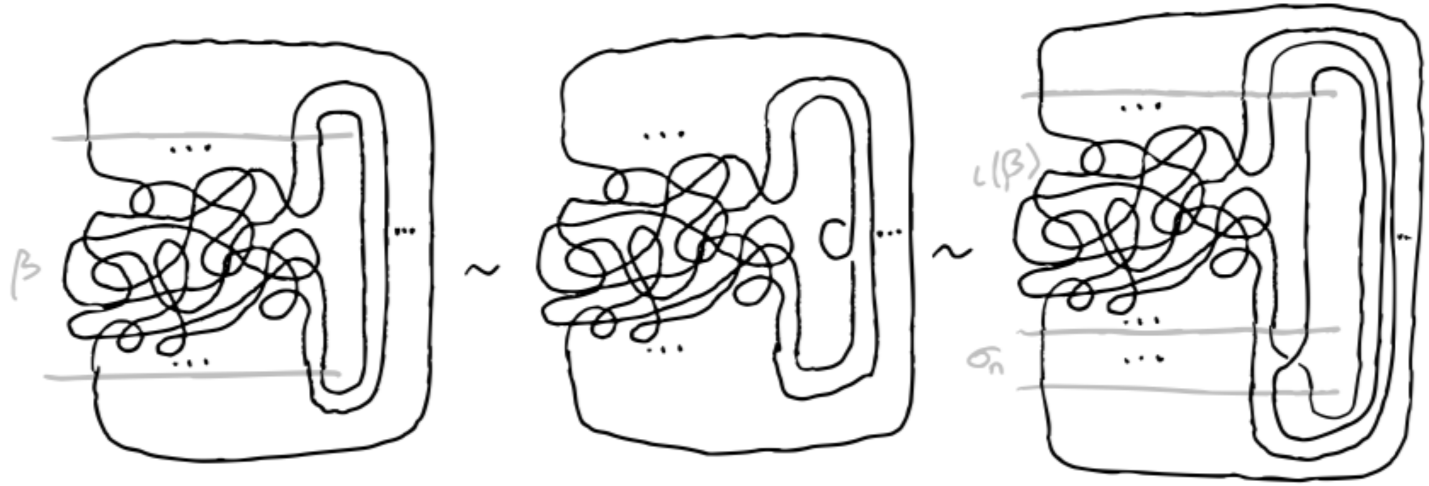


Figure 1.15: The second Markov move implied by Reidemeister I in Figure 1.3.

$\beta \sim \iota_n(\beta)\sigma_n^{\pm}$

Theorem [Markov] The closure of two braids give isotopic links if and only if they are related by Markov moves.

$$\bigsqcup_{n\geq 1} B_n \xrightarrow{\text{closure}} \{\text{Links.}\}.$$

$$\bigsqcup_{n\geq 1} B_n \Big/ \sim \;\longleftrightarrow\; \{\text{Links}\}$$

$$\boxed{\begin{aligned} \sim :\quad & \beta_1 = \beta_2 \beta_1 \beta_2^{-1} \quad \text{for } \beta_1, \beta_2 \in B_n. \\ & \beta \sim \iota_n(\beta)\, \sigma_n^{\pm} \end{aligned}}$$

Markov moves

Definition 7 (Enhanced Yang-Baxter operator [185])**.** *Let V be an N dimensional vector space over $\mathbb{C}$, $R : V \otimes V \cong V \otimes V$, $\mu : V \cong V$, and $a, b \in \mathbb{C}$. Then (R, μ, a, b) is an* enhanced Yang-Baxter operator *if it satisfies*

$$\begin{aligned} (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V) &= (\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R) && (2.8a) \\ R(\mu \otimes \mu) &= (\mu \otimes \mu)R && (2.8b) \\ \mathrm{Tr}_2(R^{\pm}(\mathrm{Id}_V \otimes \mu)) &= a^{\pm} b \mathrm{Id}_V && (2.8c) \end{aligned}$$

where $\mathrm{Tr}_k : \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes (k-1)})$ such that for $f \in \mathrm{End}(V^{\otimes k})$ and e_i a basis of V with $f(e_I) = \sum_J f_I^J e_J$ we have $\mathrm{Tr}_k(f)(e_I) = \sum_{J,j} f_{I,j}^{J,j} e_J$.

$$\omega : B_n \to \mathbb{Z}$$

$$\sigma_i \mapsto 1$$

Theorem 1.6 (185, Thm. 3.1.2)**.** *If (R, μ, a, b) is an enhanced Yang-Baxter operator, then we get a representation of the braid group,*

$$\rho_n : B_n \to \operatorname{End}\left(V^{\otimes n}, V^{\otimes n}\right) \quad s.t. \quad \rho\left(\sigma_i\right) = \underset{1}{\mathrm{Id}_V} \otimes \cdots \otimes \underset{i,i+1}{R} \otimes \cdots \otimes \underset{n}{\mathrm{Id}_V}$$

and

$$T_S : \sigma \mapsto a^{-w(\sigma)} b^{-n} \operatorname{Tr}(\rho_n \circ \mu^{\otimes n})$$

for $\sigma \in B_n$. It's invariant under the Markov moves.

$$T_S(\sigma) = T_s(\sigma\sigma_n^{\pm}) = T_s(\eta^{-1}\sigma\eta)$$

Theorem 1.5. (*R-matrix for* $\mathfrak{sl}_2$). *Let* $m, n \in \frac{1}{2}\mathbb{Z}_{>0}$, $R : V_n \otimes V_m \to V_m \otimes V_n$ *where* $V_m = \mathrm{Span}\left\{e_{-m}, e_{-m+1}, \cdots, e_{m-1}, e_m\right\}, V_n = \mathrm{Span}\left\{e_{-n}, e_{-n+1}, \cdots, e_{n-1}, e_n\right\}$, *such that*

$$R\left(e_k \otimes e_\ell\right) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,j+n)} \delta_{\ell,i+p}\delta_{k+p,j} \\ \times (-1)^p q^{ij-\frac{E}{2}(m+n)-(i-j)p-p(p+1)/2} \frac{(q;q)_{m+\ell}(q;q)_{n-k}}{(q;q)_{m+i}(q;q)_p(q;q)_{n-j}} e_i \otimes e_j.$$

and $\mu : V_n \to V_n$ *such that* $\mu\left(e_j\right) = q^j e_j$. *Then* $\left(R, \mu, q^{n(n+1)}, 1\right)$ *is an enhanced Yang-Baxter operator when restricted to* V_n *for some* n. *More generally, this gives a representation of a ribbon Hopf algebra associated to* $U_q\left(\mathfrak{sl}_2\right)$.

Lemma 1. The inverse of the R matrix in Theorem 14 is given explicitly by

$$R^{-1}\left(e_k \otimes e_\ell\right) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,j+n)} \delta_{\ell,i-p}\delta_{k-p,j}$$
$$\times\, q^{-ij-\frac{p}{2}(m+n)} \frac{(q;q)_{m-\ell}(q;q)_{n+k}}{(q;q)_{m-i}(q;q)_p(q;q)_{n+j}} e_i \otimes e_j.$$

If L is the closure of σ,the for $\left(R, \mu, q^{n(n+1)}, 1\right)$, define $J_{L,n}(q) = T_s(\sigma)$.

The Jones polynomial $J_L(q) \in \mathbb{Z}[q^{\pm 1/2}]$ of an oriented link L in 3-space is uniquely determined by the linear relations [41]

$$q J_{\times}(q) - q^{-1} J_{\times}(q) = (q^{1/2} - q^{-1/2}) J_{)(}(q) \qquad J_{\bigcirc}(q) = q^{1/2} + q^{-1/2} \,.$$

The Jones polynomial has a unique extension to a polynomial invariant $J_{L,c}(q)$ of links L together with a coloring c of their components that are colored by positive natural numbers that satisfy the following rules

$$\begin{aligned} J_{L\cup K, c\cup\{N+1\}}(q) &= J_{L\cup K^{(2)}, c\cup\{N,2\}}(q) - J_{L\cup K, c\cup\{N-1\}}(q), \qquad N \geq 2, \\ J_{L\cup K, c\cup\{1\}}(q) &= J_{L,c}(q), \\ J_{L,\{2,\ldots,2\}}(q) &= J_L(q), \end{aligned}$$

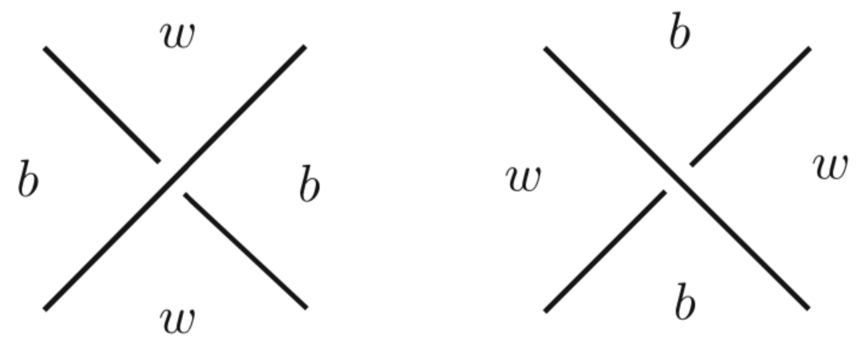
where $(L \cup K, c \cup \{N\})$ denotes a link with a distinguished component K colored by N and $K^{(2)}$ denotes the 2-parallel of K with zero framing. Here, a natural number N attached to a component of a link indicates the N-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

2.2 From alternating links to planar (Tait) graphs

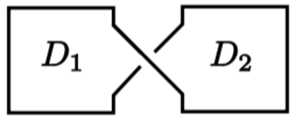
Given a diagram D of a *reduced alternating non-split* link L, its Tait graph can be constructed as follows: the diagram D gives rise to a polygonal complex of $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Since D is alternating, it is possible to label each polygon by a color b (black) or w (white) such that at every crossing, the coloring looks as follows in Fig. 3.

There are exactly two ways to color the regions of D with black and white colors. In this note, we will work with the one whose unbounded region has color w. In each

Fig. 3 The checkerboard coloring of a link diagram

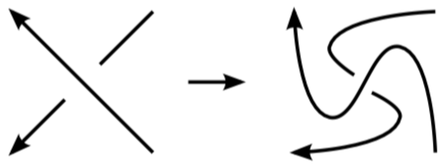


2.1. **Downward link diagram.** Recall that a link diagram $D \subset \mathbb{R}^2$ is *alternating* if walking along it, the sequence of crossings alternates from overcrossings to undercrossings. A diagram D is *reduced* if it is not of the form



where D_1 and D_2 are diagrams with at least one crossing.

A *downward link diagrams of links* is an oriented link diagram in the standard plane in general position (with its height function) such that at every crossing the orientation of both strands of the link is downward. A usual link diagram may not satisfy the downward requirement on the orientation at a crossing. However, it is easy to convert a link diagram into a downward one by rotating the non-downward crossings as follows:



2.2. **Link diagrams and states.** Fix a downward link diagram D of an oriented link K with c_D crossing. Considering D as a 4-valent graph, it has $2c_D$ edges. A *state* of D is a map

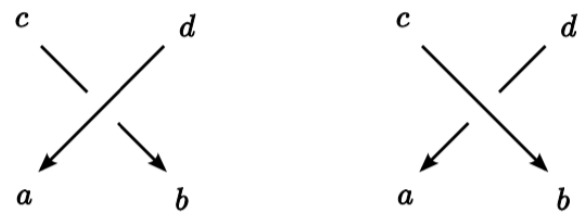
$$r : \{\text{edges of } D\} \to \mathbb{R}$$

such that at every crossing we have

$$a + b = c + d,$$

where a, b, c, d are the values of s of the edges incident to the crossing as in the following figure

(11)



The set $S_{D,\mathbb{R}}$ of all states of D is a vector space. For a state $r \in S_{D,\mathbb{R}}$ and a crossing v of D define

$$r(v) = \operatorname{sign}(v)\,(a - d)\,,$$

where as usual the sign of the crossing on the left hand side of (11) is positive and the sign of the one on the right hand side is negative. For a positive integer n, a state $r \in S_{D,\mathbb{R}}$ is called *n-admissible* if the values of r are integers in $[0, n]$ and $r(v) \geq 0$ for every crossing v. Let $S_{D,n}$ be the set of all n-admissible states.

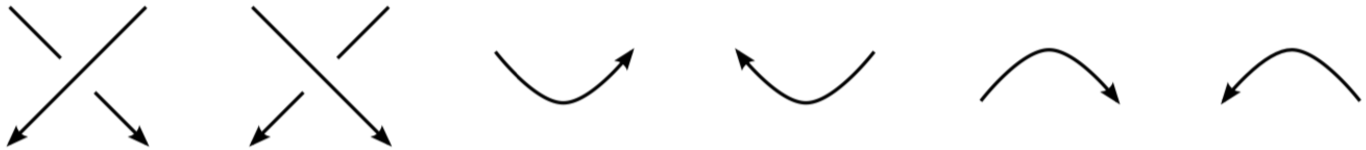
2.4. **Local weights, the colored Jones polynomial, and their factorization.** Consider the monoid

$$\mathbb{Z}_{\succ}[q] = 1 + q\mathbb{Z}[q]\,.$$

Fix a natural number $n \geq 1$ and a downward link diagram D.

A local part of D is a small neighborhood of a crossing or a local extreme of D. There are six types of local parts of D: two types of crossings (positive or negative) and four types of local extrema (minima or maxima, oriented clockwise, or counterclockwise):

(13)

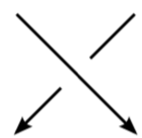


For an n-admissible state r and a local part X, the weight $w(X,r)$ is defined by

$$w(X,r) = w_{\mathrm{lt}}(X,r)\,w_{\succ}(X,r)\,,$$

where $w_{\mathrm{lt}}(X,r) \in \{\pm q^{m/4} \,|\, m \in \mathbb{Z}\}$ is a monomial, $w_{\succ}(X,r) \in \mathbb{Z}_{\succ}[q]$, and $w_{\mathrm{lt}}(X,r)$ and $w_{\succ}(X,r)$ are given by Table 1.

Table 1. The local weights w_{lt} and $w_{\succ}$ of a state.

						
w_{lt}	$q^{(n+nd+nb-ab-dc)/2}$	$(-1)^{b-c}q^{(-n-nb-nd+bd+ac-b+c)/2}$	$q^{-(2a-n)/4}$	$q^{-(2a-n)/4}$	$q^{(2a-n)/4}$	$q^{(2a-n)/4}$
$\omega_{\succ}$	$(q;q)_{c-b}\binom{n-d}{a-d}_q\binom{c}{c-b}_q$	$(q;q)_{b-c}\binom{n-c}{b-c}_q\binom{d}{d-a}_q$	1	1	1	1

Let the weight of a state be defined by

$$w(r) = \prod_X w(X, r)\,,$$

where the product is over all the local parts of D. Then the unframed version of the colored Jones polynomial of the link K, each component of which is colored by the $n+1$-dimensional sl_2-module, is given by

$$J_{K,n}(q) = \sum_{r \in S_{D,n}} w(r)\,, \tag{14}$$

where $S_{D,n}$ is the set of all n-admissible states of D. For example, the value of the unknot is

$$J_{\mathrm{Unknot},n}(q) = [n+1] := \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}}\,.$$

Note that $J_{K,0}(q) = 1$ for all links and $J_{K,1}(q^{-1})/J_{\mathrm{Unknot},1}(q^{-1})$ is the Jones polynomial of K [Jon87]. Since we could not find a reference for the state sum formula (14) in the literature, we will give a proof in the Appendix.

Nahm sum is of the form

$$\Phi(q) = \sum_{n \in C \cap \mathbb{N}^r} (-1)^{a \cdot n} \frac{q^{\frac{1}{2} n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \cdots (q)_{n_r}} \tag{1}$$

where C is a rational polyhedral cone in $\mathbb{R}^r$, $b, a \in \mathbb{Z}^r$ and A is a symmetric (possibly indefinite) symmetric matrix. We will say that the generalized Nahm sum (1) is *regular* if the function

$$n \in C \cap \mathbb{N}^r \mapsto \frac{1}{2} n^t \cdot A \cdot n + b \cdot n$$

is proper and bounded below. Regularity ensures that the series (1) is a well-defined element of the Novikov ring

$$\mathbb{Z}((q)) = \{ \sum_{n \in \mathbb{Z}} a_n q^n \,|\, a_n = 0, \, n \ll 0 \}$$

of power series in q with integer coefficients and bounded below minimum degree. In the remaining of the paper, by Nahm sum we will mean a generalized Nahm sum. The paper is concerned with a new source of Nahm sums that originate in Quantum Knot Theory.

1.2. **Stability of a sequence of polynomials.** For $f(q) = \sum a_j q^j \in \mathbb{Z}((q))$ let $\mathrm{mindeg}_q f(q)$ denote the smallest j such that $a_j \neq 0$ and let $\mathrm{coeff}(f(q), q^j) = a_j$ denote the coefficient of q^j in $f(q)$.

Definition 1.1. Suppose $f_n(q), f(q) \in \mathbb{Z}((q))$. We write that

$$\lim_{n\to\infty} f_n(q) = f(q)$$

if

- there exists C such that $\mathrm{mindeg}_q(f_n(q)) \geq C$ for all n, and
- for every $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathrm{coeff}(f_n(q), q^j) = \mathrm{coeff}(f(q), q^j). \tag{2}$$

Since Equation (2) involves a limit of integers, the above definition implies that for each j, there exists N_j such that

$$f_n(q) - f(q) \in q^j \mathbb{Z}[[q]]$$

(and in particular, $\mathrm{coeff}(f_n(q), q^j) = \mathrm{coeff}(f(q), q^j)$) for all $n > N_j$.

Remark 1.2. Although for every integer j we have $\lim_{n\to\infty} \mathrm{coeff}(q^{-n^2}, q^j) = 0$, it is not true that $\lim_{n\to\infty} q^{-n^2} = 0$.

Definition 1.3. A sequence $f_n(q) \in \mathbb{Z}[[q]]$ is *k-stable* if there exist $\Phi_j(q) \in \mathbb{Z}((q))$ for $j = 0, \dots, k$ such that

$$\lim_{n\to\infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0 \tag{3}$$

We say that $(f_n(q))$ is *stable* if it is k-stable for all k. Notice that if $f_n(q)$ is k-stable, then it is k'-stable for all $k' < k$ and moreover $\Phi_j(q)$ for $j = 0, \dots, k$ is uniquely determined by $f_n(q)$. We call $\Phi_k(q)$ the k-limit of $(f_n(q))$. For a stable sequence $(f_n(q))$, its associated series is given by

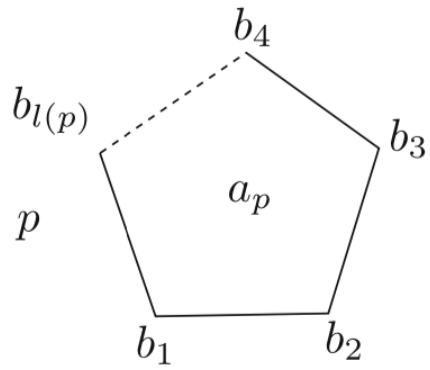
$$F_f(x, q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]] \,.$$

It is easy to see that the pointwise sum and product of k-stable sequences are k-stable.

Theorem 4.1. *For every alternating link K, the sequence $\left(\hat{J}_{K,n}(q)\right)$ is stable and its associated k-limit $\Phi_{K,k}(q)$ and series $F_K(x, q)$ can be effectively computed from any reduced, alternating diagram D of K.*

An *admissible state* (a, b) of G is an integer assignment a_p for each face p of G and b_v for each vertex v of G such that $a_p + b_v \geq 0$ for all pairs (v, p), where v is a vertex of p. For the unbounded face p_∞, we set $a_\infty = 0$, and thus $b_v = a_\infty + b_v \geq 0$ for all $v \in p_\infty$. We also set $b_v = 0$ for a fixed vertex v of p_∞. In the formulas below, v and w will denote vertices of G, and p is the face of G and p_∞ is the unbounded face. We also write $v \in p$, $vw \in p$ if v is a vertex and vw is an edge of p.

For a polygon p with $l(p)$ edges and vertices $b_1, \ldots, b_{l(p)}$ in counterclockwise order,



we define

$$\gamma(p) = l(p)a_p^2 + 2a_p(b_1 + b_2 + \cdots + b_{l(p)}) .$$

Let

$$A(a, b) = \sum_p \gamma(p) + 2 \sum_{e=(v_i v_j)} b_{v_i} b_{v_j} , \tag{1}$$

where the p-summation (here and throughout the paper) is over the set of *bounded* faces of G and the e-summation is over the set of edges $e = (v_i v_j)$ of p, and

$$B(a,b) = 2\sum_{v} b_v + \sum_{p} (l(p) - 2) a_p, \tag{2}$$

where the v-summation is over the set of vertices of G and the p-summation is over the set of bounded faces of G.

Definition 1.1 [13] With the above notation, we define

$$\Phi_G(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod_{(p,v):v\in p} (q)_{a_p+b_v}}, \tag{3}$$

where the sum is over the set of all admissible states (a,b) of G, and in the product $(p,v) : v \in p$ means a pair of face p and vertex v such that p contains v. Here, c_2 is the number of edges of G and

$$(q)_\infty = \prod_{n=1}^{\infty} (1-q)^n = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \ldots$$

When L is an alternating link, the colored Jones polynomial $J_{L,n}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ (normalized to be 1 at the unknot, and colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$ [13]) has the lowest q-monomial with coefficient ± 1, and after dividing by this monomial, we obtain the *shifted* colored Jones polynomial $\hat{J}_{L_G,n}(q) \in 1 + q\mathbb{Z}[q]$.

Theorem 2.1 *[13, Theorem 1.10] Let L be an alternating link projection and G be its Tait graph. Then, the following limit exists:*

$$\lim_{n\to\infty} \hat{J}_{L,n}(q) = \Phi_G(q) \in \mathbb{Z}[[q]]. \tag{6}$$

Application in Rogers–Ramanujan Identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \tag{0.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \tag{0.2}$$

In recent years, q-series identities have arisen in topology; specifically in knot theory.

For example, in [GL15], Stavros Garoufalidis and Thang T. Q. Lê prove the multisum q-series to infinite product identity

$$\sum_{i,j,k\geqq 0} \frac{(-1)^i q^{i(3i+1)/2+ij+ik+jk+j+k}}{(q)_i(q)_j(q)_k(q)_{i+j}(q)_{i+k}} = \frac{1}{(q;q)_\infty^2}, \tag{6.21}$$

which is related to the 3_1 knot, and the identity

$$\sum_{\substack{i,j,k,l,m\geqq 0\\ i+j=l+m}} \frac{(-1)^{j+l} q^{j^2/2+l^2/2+jk+ik+il+jm+i/2+k+m/2}}{(q)_{j+k}(q)_i(q)_j(q)_k(q)_l(q)_m(q)_{k+l}} = \frac{1}{(q;q)_\infty^3}, \tag{6.22}$$

which is related to the 4_1 knot, where we use the standard abbreviation $(q)_n$ for $(q;q)_n$, in order to save space. For the amphicheiral knot 6_3, they conjectured (and Andrews proved in [And14]) the identity