The Functoriality of The Postnikov Tower

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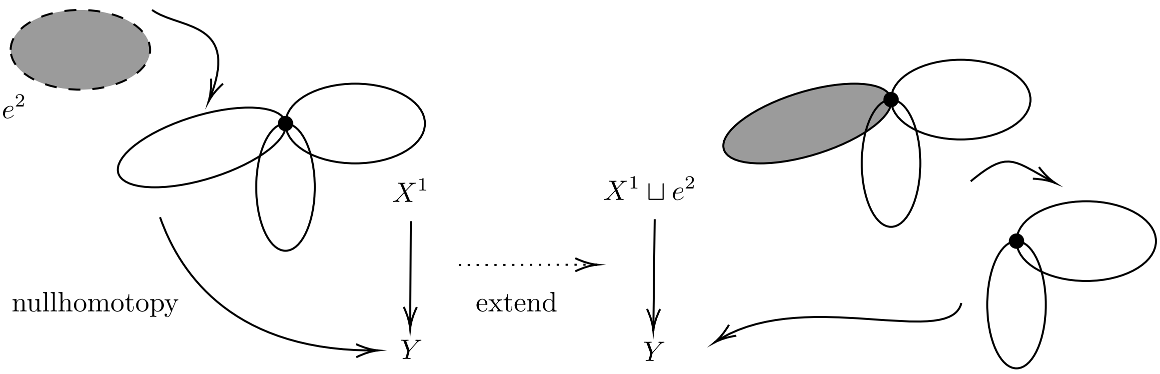
Here I will introduce the Moore tower, Whitehead tower and particular the Postnikov tower. I will show how to construct a Postnikov tower of a 0-connected CW complex and discuss the functoriality of the Postnikov tower. On the other hand, I will give an example to show that the Moore tower has no functoriality.

The extension lemma

Let (X, A) be a CW pair, Y be path-connected. If $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n , then $f : A \longrightarrow Y$ can be extended to a map $X \longrightarrow Y$.

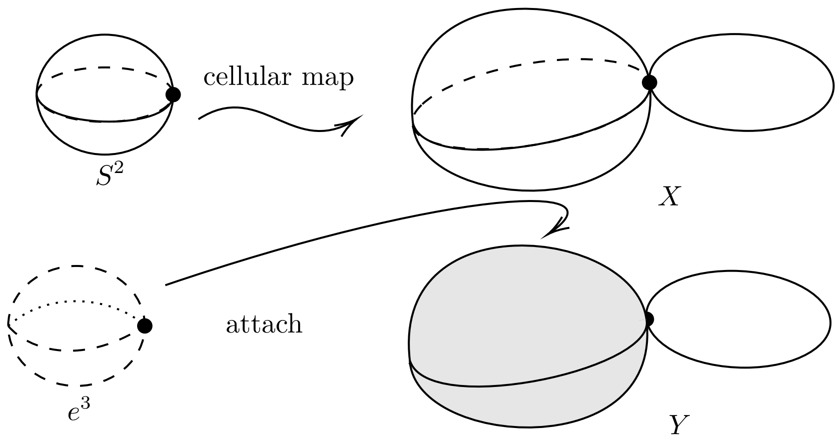
Proof :

Assume inductively that f has been extended over the $(n-1)$-skeleton X^{n-1} (containing A) . Then an extension over an n-cell exists if and only if the composition of this n-cell attaching map $S^{n-1} \longrightarrow X^{n-1}$ with $f : X^{n-1} \longrightarrow Y$ is nullhomotopic.



The Postnikov Towers

For a 0-connected CW complex X , one can construct a sequence $\widetilde{X_n}$ such that $\pi_i(X) \cong \pi_i(\widetilde{X_n})$ for $i \leq n$ and $\pi_i(\widetilde{X_n}) = 0$ for $i > n$.

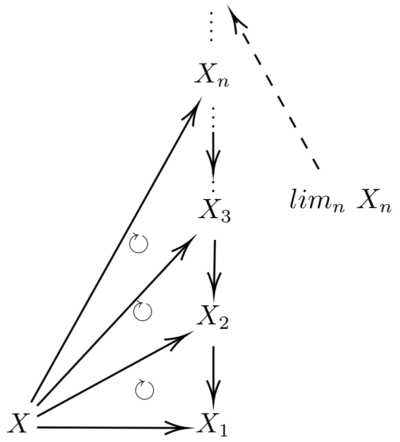


For every generator of $\langle S^{n+1}, X\rangle$ in $\pi_{n+1}(X)$, by the cellular approximation theorem one can make it to be cellular. If we attach a e^{n+2} to X by this cellular map, then one has :

(1) $\pi_i(Y) \cong \pi_i(X)$ for $i \le n$ since the i-skeletons of Y and X are the same, by the cellular approximation theorem, one can make $\langle S^i, Y\rangle$ homotopic to $\langle S^i, X\rangle$.

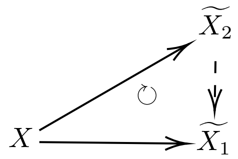
(2) $\pi_{n+1}(Y) = 0$ since the generators in X are nullhomotopy in Y .

Let $X = Y$ and repeat this process, one can get a CW complex $\widetilde{X_n}$ such that $\pi_k(i) : \pi_k(X) \longrightarrow \pi_k(\widetilde{X_n})$ is an isomorphism for $k \le n$ and $\pi_k(\widetilde{X_n}) = 0$ for $k > n$.



(1) By the extension lemma :

$X \longrightarrow \widetilde{X_1}$ can be extended to a map $\widetilde{X_2} \longrightarrow \widetilde{X_1}$.



($X \sqcup e^4_\beta \sqcup \cdots$ retracts to X_2)

($X \sqcup e^3_\alpha \sqcup e^4_\beta \sqcup \cdots$ retracts to X_1)

(2) Map $\widetilde{X_n} \longrightarrow X_{n-1}$ factors as $\widetilde{X_n} \longrightarrow X_n \longrightarrow X_{n-1}$.

(any map can be turned into a fibration up to homotopy)

$\varprojlim_n X_n = \{(x_1, x_2, \cdots) \mid x_2 \longmapsto x_1\ ,\ x_3 \longmapsto x_2\ ,\ \cdots\}$ is a subgroup of $\prod_n X_n$.

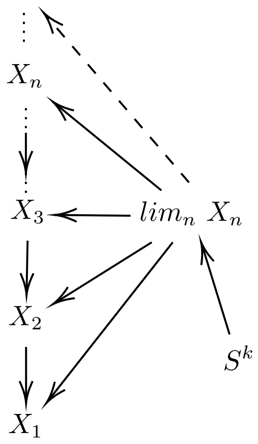
Propositon

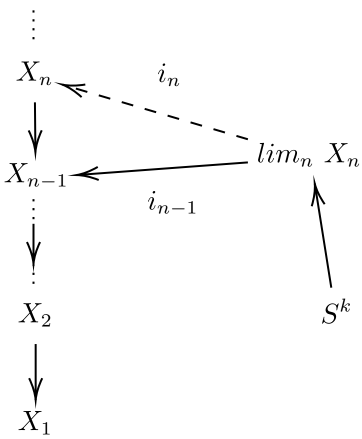
(1) Consider $\mathcal{X} : (\mathbb{N}, \le) \longrightarrow (\textbf{Top})$, if for each n , $i_n : X_n \longrightarrow X_{n+1}$ is a closed inclusion and X_n is a T_1 space, then for the functor $C_\bullet : (\textbf{Top}) \longrightarrow (\textbf{Comp})$ one has $\varinjlim_n C_\bullet(X_n) = C_\bullet(\varinjlim_n X_n)$, moreover :

$$\varinjlim_n H_k(X_n) = \varinjlim_n (H_k(C_\bullet(X_n))) = H_k(\varinjlim_n C_\bullet(X_n)) = H_k(C_\bullet(\varinjlim_n X_n)) = H_k(\varinjlim_n X_n) .$$

Similarly, $\varinjlim_n \pi_1(X_n) = \pi_1(\varinjlim_n X_n)$ since S^1 is also T_1 .

(2) Consider $(\mathbb{N}, \ge) \xrightarrow{\mathcal{X}} (\textbf{Top}) \xrightarrow{\pi_k} (\textbf{Gp})$, if $p_n : X_{n+1} \longrightarrow X_n$ is a fibration for each n , then the unique map $\pi_k(\varprojlim_n X_n) \longrightarrow \varprojlim_n \pi_k(X_n)$ is surjective. And it is injective if the map $\pi_{k+1}(X_n) \longrightarrow \pi_{k+1}(X_{n-1})$ is surjective for n sufficiently large.



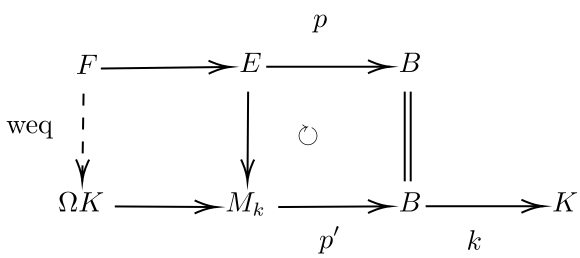


Proposition

The unique map $X \longrightarrow \varprojlim_n X_n$ is a weak homotopy equivalence, X is a CW approximation to $\varprojlim_n X_n$, since $\pi_k(X) \longrightarrow \pi_k(\varprojlim_n X_n) \longrightarrow \varprojlim_n \pi_k(X_n)$ is an isomorphism for n sufficiently large.

Principal fibrations

A fibration $p : E \longrightarrow B$ with fibre F is called equivalent to a principal fibration if there is a homotopy equivalence $E \longrightarrow M_k$ where $k : B \longrightarrow K$ such that the diagram commutes.



Thus one must have a weak homotopy equivalence $F \longrightarrow \Omega K$.

The induced fibration $p' : M_k \longrightarrow B$ is called the principal fibration induced by $p : E \longrightarrow B$.

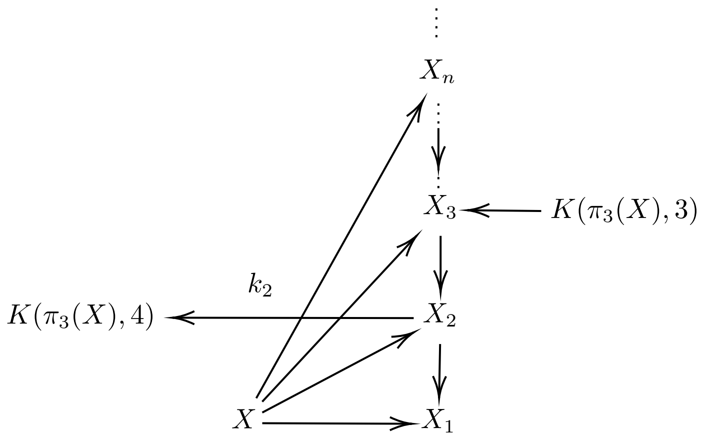
Proposition

A 0-connected CW complex X has a Postnikov tower of principal fibrations.
$\Longleftrightarrow$ $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n \geq 2$.

Any 1-conected CW complex X has a Postnikov tower of principal fibrations.

k-invariants

If the fibration $p : X_{n+1} \longrightarrow X_n$ is principal in the Postnikov tower, then one has an induced fibration $k_n : X_n \longrightarrow K(\pi_{n+1}(X), n+2)$ with fibre M_{k_n} (homotopy equivalent to X_{n+1}) .

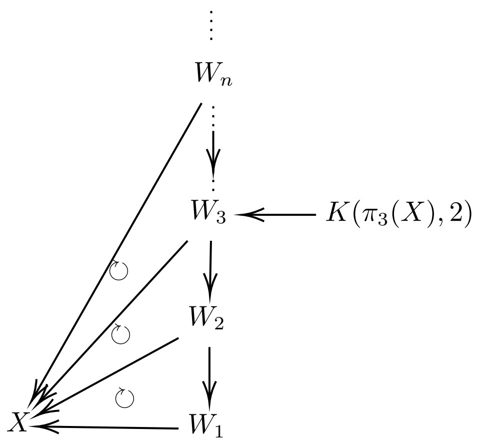


Thus there is a fibre sequence $K(\pi_n(X), n) \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow K(\pi_n, n+1)$.

$k_n : X_n \longrightarrow K(\pi_{n+1}(X), n+2)$ is a class in $H^{n+2}(X_n; \pi_{n+1}(X))$ called the n-th k-invariant (Postnikov invariant) of X (By the Brown representation theorem, $H^{n+2}(X_n; \pi_{n+1}(X)) \cong \langle X_n, K(\pi_{n+1}(X), n+2) \rangle$) .

The Whitehead tower

For a 0-connected CW complex X , one has the commutative diagram such that $W_n \longrightarrow W_{n-1}$ is a fibration with fibre $K(\pi_n(X), n-1)$ for each n



where $\pi_k(W_n) \longrightarrow \pi_k(X)$ is an isomorphism for $k \geq n+1$ and $\pi_k(W_n) = 0$ for $k \leq n$.

Proposition

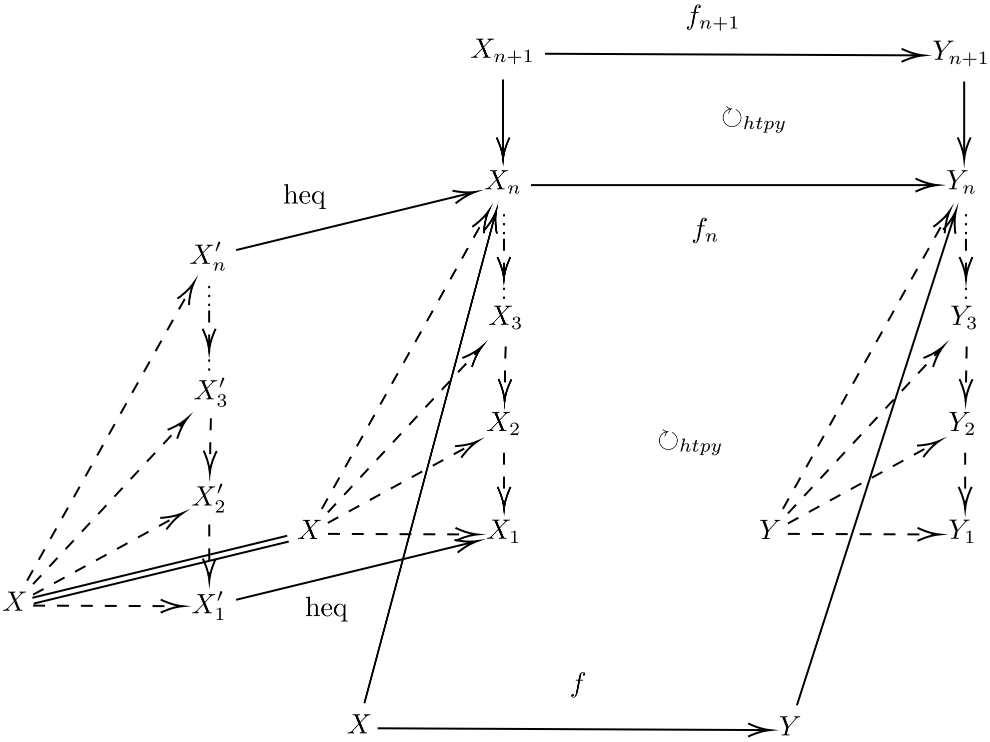
For CW pairs (X, A) where cells in $X \setminus A$ have dimension $k \geq n+2$, then there is an induced map $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$.

If $\pi_m(Y) = 0$ for $n \geq n+2$, then $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$ is injective.

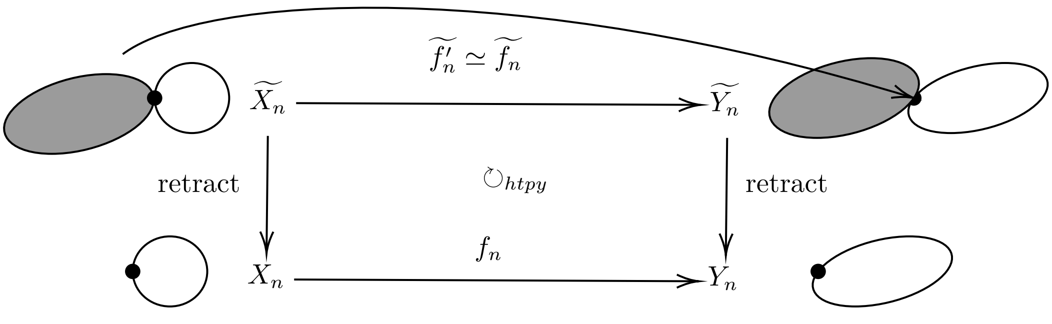
If $\pi_m(Y) = 0$ for $n \geq n+1$, then $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$ is surjective.

The functoriality of the Postnikov tower in $\mathbf{Ho}(\mathbf{Top}_*)$

Consider the category of the tower-like diagrams, the object is the Postnikov tower $\mathcal{P}(X)$ of space X , the morphism is $f \prod_n f_n$ where $f : X \longrightarrow Y$, $f_n : X_n \longrightarrow Y_n$ (assume that all are 1-connected) .



Consider the inclusion $i_n : X \longrightarrow \widetilde{X_n}$, then there is a unique $[\widetilde{f_n}]$ such that $[i'_n \circ f] \longmapsto [\widetilde{f_n}]$ where $i'_n : Y \longrightarrow \widetilde{Y_n}$, $\widetilde{f_n} : \widetilde{X_n} \longrightarrow \widetilde{Y_n}$, thus $f_n : X_n \longrightarrow Y_n$ is well defined.



Proposition

If f is a homotopy equivalence, then f_n is a homotopy equivalence.

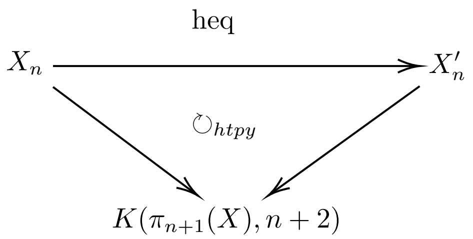
For the homotopy inverse g , one has

$$f_n \circ g_n \simeq (fg)_n \simeq \mathbb{1}_{Y_n} \ , \ g_n \circ f_n \simeq (gf)_n \simeq \mathbb{1}_{X_n} \ .$$

Take $f = \mathbb{1}_X$, $Y = X$, then for two section X_n and X'_n they are homotopy equivalent.

Commutativity with k-invariant

For two Postnikov towers of X , one has X_n and X'_n are homotopy equivalent, then one has the diagram commutes.



Thus $H^{n+2}(X_n; \pi_{n+1}(X)) \cong \langle X_n, K(\pi_{n+1}(X), n+2)\rangle = \langle X'_n, K(\pi_{n+1}(X), n+2)\rangle \cong H^{n+2}(X'_n; \pi_{n+1}(X))$.

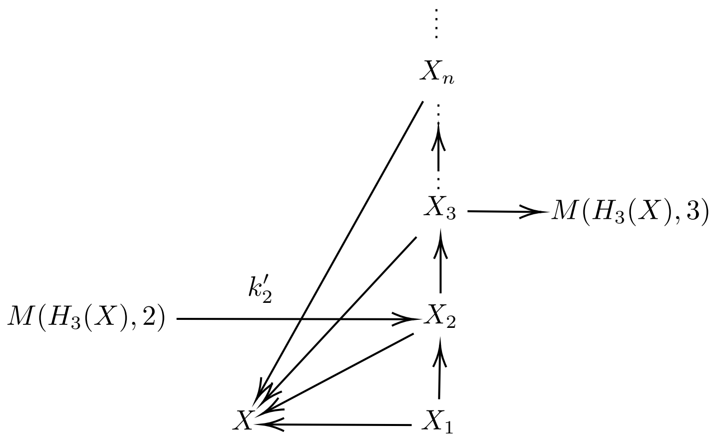
The Universal Coefficient Theorem for Homotopy

Define the homotopy group with coefficient $\pi_n(X; G) = \langle M(G, n), X\rangle$, for $n \geq 2$ there is an exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(G, \pi_{n+1}(X)) \longrightarrow \pi_n(X; G) \longrightarrow \mathrm{Hom}(G, \pi_n(X)) \longrightarrow 0 \ .$$

The Moore tower

If X is 1-connected, then X has a Moore tower (commutative diagram) of principal cofibrations.

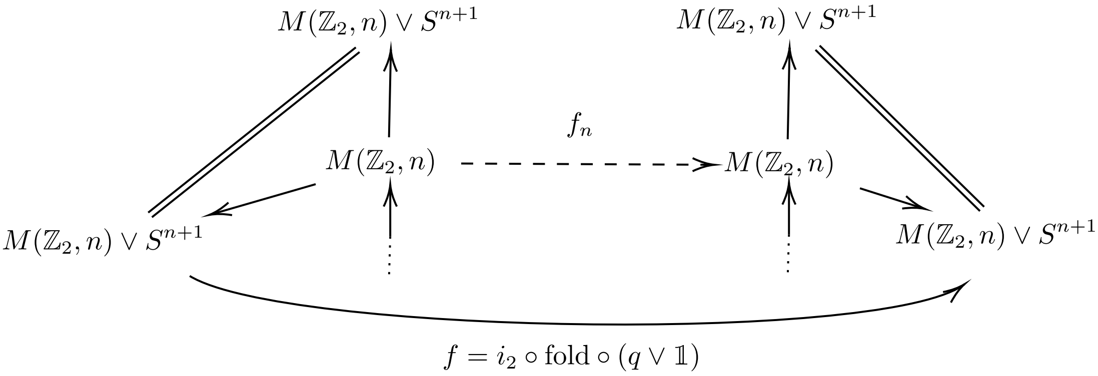


$i_n : X_n \longrightarrow X_{n+1}$ is a principal cofibration inducing the cofibration $k'_n : M(H_{n+1}(X), n) \longrightarrow X_n$ with cofibre $C_{k'_n}$ (homotopy equivalent to X_3) .

Thus there is a cofibre sequence $M(H_{n+1}(X), n) \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow M(H_{n+1}(X), n+1)$.

The Moore tower has no factoriality

Take an 1-connected $X = M(\mathbb{Z}_2, n) \vee S^{n+1}$, take $X_n = M(\mathbb{Z}_2, n)$, $X_{n+1} = M(\mathbb{Z}_2, n) \vee S^{n+1} = X$. By the universal coefficient theorem one has $\langle M(\mathbb{Z}_2, n), S^{n+1} \rangle = \pi_n(X; \mathbb{Z}_2) \cong \mathrm{Ext}^1_{\mathbb{Z}_2}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$ since $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = 0$. Thus there is a nonconstant map $q : M(\mathbb{Z}_2, n) \longrightarrow S^{n+1}$. Consider $f = i_2 \circ \mathrm{fold} \circ (q \vee \mathbb{1}) : X \longrightarrow X$.



If $f \circ i_1 \simeq i_1 \circ f_n$, then $q = (q \vee c) \circ i_2 \circ q = (q \vee c) \circ f \circ i_1 \simeq (q \vee c) \circ i_1 \circ f_n = c$ makes a contradiction.