Motives, Motivic Cohomology and Motivic Homotopy: a historical introduction

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Daydreaming About Algebraic Topology

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At the very beginning, algebraic geometry and algebraic topology are one.
They separated as the theory developed.
Algebraic topology tends to study very flexible and complicate spaces.
Algebraic geometry tends to study very rigidity and simpler spaces.
In algebraic topology world, many pleasure properties are established.
Many powerful tool we have. We have homotopy groups, singular cohomology, spectra,...
Can we have similar things in algebraic geometry?

A famous example is André Weil's conjectured cohomology theory which inspired by algebraic topology. Nowadays, we call it Weil cohomology. His purpose is solving his famous Weil conjecture. The later story is far-reaching.
Grothendieck and his school revolutionized algebraic geometry. Schemes, site, Grothendieck topology, functorial viewpoint,... Finally, Grothendieck's student Deligne proved the Weil conjecture using étale cohomology.
But this is not Grothendieck's original approach. His suggestion is deducing Weil conjecture from standard conjecture on algebraic cycles.
Moreover, Grothendieck envisioned a theory of motives. It should unifies all Weil cohomologies and explain all arithmetic informations in geometric terms. Nowadays, we call it pure motives.

Following Grothendieck, Beilinson formulated his famous conjecture on abelian category of mixed motives and motivic cohomology. Independently, Lichtenbaum conjectured an étale motivic cohomology.
The comparison between Beilinson's Zariski motivic cohomology and Lichtenbaum's étale motivic cohomology is called Beilinson-Lichtenbaum conjecture.
This conjecture(in nice situation) is solved by Voevodsky based on Rost's work. It's a beautiful strike application of motivic homotopy theory.

Motives-Grothendieck's dream

In the early sixties, Grothendieck had developed étale cohomology theory with the help of Artin and Verdier. From that moment on there existed a cohomology theory for every prime number ℓ different from the characteristic p of the underlying field. When $\ell = p$, Grothendieck outline the crystalline cohomology theory. Latter furnished by Berthelot in his thesis. Moreover, in characteristic zero there exist also the classical Betti and de Rham theory.

There was an abundance of good Weil cohomology theories.

Let k be an arbitrary field and $SmProj(k)$ denotes the category of smooth projective reduced schemes over k. Let F be a field of characteristic 0 and let $GrVect_F$ be the category of finite dimensional graded F-vector spaces.

Definition

A Weil cohomology theory is a functor

$$H : SmProj(k)^{op} \to GrVect_F$$

which satisfies the following axioms
(1) there exists a cup product $\cup : H(X) \times H(X) \to H(X)$.
(2) $H^i(X_d) \times H^{2d-i}(X_d) \xrightarrow{\cup} H^{2d}(X_d) \xrightarrow{\sim} F$ is a perfect pairing.
(3) Künneth formula: $H(X) \otimes H(Y) \to H(X \times Y)$.
(4) functorial cycle class maps $\gamma_X : CH^i(X) \to H^{2i}(X)$, compatible with cup product.
(5) Weak Lefschetz: if $i : Y_{d-1} \to X_d$ is a smooth hyperplane section, $H^i(X) \xrightarrow{i^*} H^i(Y)$ is an isomorphism if $i < d-1$, injective for $i = d-1$.
(6) Hard Lefschetz: $L(\alpha) = \alpha \cup \gamma_X(X)$ induces isomorphisms

$$L^{d-i} : H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X), 0 \leq i \leq d.$$

There are famous comparison theorems between them: the famous de Rham isomorphism theorem Betti and de Rham theory: If $k \subset \mathbb{C}$ then

$$H_{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H_{dR}(X_{an}; \mathbb{C}) \xrightarrow{\sim} H_{Betti}(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

and Artin isomorphism between Betti and étale cohomology if $k = \mathbb{C}$:

$$H_{\text{ét}}(X, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{Betti}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

There should be a deeper reason behind this! Grothendieck envisioned a **universal cohomology theory** for algebraic varieties: the theory of **motives**.

Grothendieck expected that there should exists a suitable $\mathbb{Q}$-linear semisimple abelian tensor category with **realisation** functor to all Weil cohomology theories.
Grothendieck has constructed what we now call the category of pure motives.

Definition

An algebraic cycle on a verity X is a formal finite integral linear combination $Z = \sum n_\alpha Z_\alpha$ of k-irreducible sub-varieties Z_α of X. If all the Z_α have the same codimension i, we say that Z is a codimension i cycle. We introduce the abelian group

$$Z^i(X) := \{\text{codim } i \text{ cycles on } X\}$$

The group of codimension 1 cycles is cycles, divisors, is also written $Div(X)$.
Grothendieck's method is theory of algebraic cycles modulo a suitable equivalence relation.

Rational equivalence is the finest adequate relation d'équivalence. It's a generalisation of linear equivalence for divisors. The divisor of a rational function $f \in k(X)$ is defined as follows:

$$\operatorname{div}(f) := \sum_{Y \subset X} \operatorname{ord}_Y(f) \cdot Y, \, Y \text{ of codim } 1,$$

where the order homomorphism $\operatorname{ord}_Y : k(X)^* \to \mathbb{Z}$ is defined as follows. The local ring $A = \mathcal{O}_{X,Y}$ is one dimensional and for $f \in A$ one puts $\operatorname{ord}_Y(f) = \ell_A(A/(f))$, where ℓ_A the length of an A-module. For $f \in k(X)^*$, $f = f_1/f_2$, put $\operatorname{ord}_Y(f) = \operatorname{ord}_Y(f_1) - \operatorname{ord}_Y(f_2)$.

The divisor div(f) for a function $f \in K(Y)^*$ on an irreducible sub-variety $Y \subset X$ is a codimension 1 cycle on Y. Hence, if Y is of codimension $i-1$ in X, div$(f) \in Z^i(X)$ and by definition $Z^i_{rat}(X)$
For a codimenison i-cycle $Z \sim_{rat} 0$ if and only if there is a finite collection of pairs (Y_α, f_α) of codimension $(i-1)$ irreducible varieties and non-zero functions on them such that $Z = \sum \mathrm{div}(f_\alpha)$. If $X^{(i)}$ stands for the collection of irreducible codimension i subvarieties of X we have

$$Z^i_{rat}(X) = \mathrm{Im}(\oplus_{Y \in X^{(i-1)}} k(Y)^* \xrightarrow{div} \oplus_{Z \in X^{(i)}} \mathbb{Z})$$

Definition

The **Chow group** are the cokernel of these maps:

$$CH^i(X) := Z^i(X)/Z^i_{rat}(X)$$

Calculation of Chow group is very difficult.

Besides rational equivalence, we will introduce two more equivalence relations. Recall that we have cycle class map:

$$\gamma_X : CH^i(X) \to H^{2i}(X).$$

Definition

We say a cycle $Z \sim_{hom} 0$ if $\gamma_X(Z) = 0$.

It looks like that homological equivalence depends on the choice of a Weil cohomology theory. But standard conjecture says no.

Definition

Let $X_d \in SmProj(k)$. For $Z \in Z^i(X_d)$ we put $Z \sim_{num} 0$ if for every $W \in Z^{d-i}(X_d)$ such that $Z \cdot W$ is defined we have $\deg(Z \cdot W) = 0$.

Standard Conjecture (D)

Suppose $k = \bar{k}$. Then $Z^i_{hom}(X) = Z^i_{num}(X)$.
This conjecture is known for divisors in arbitrary characteristic. In characteristic zero, it is also known for $i = 2$, for dimension 1, and for abelian varieties.

Grothendieck's Construction

Since Jannsen's result, only numerical equivalence should be considered. If we want motives. And we shall use rational instead of integral coefficients.

Definition

A correspondence from X to Y is a cycle on the product $X \times Y$.

$$Cor(X, Y) := CH(X \times Y) \otimes \mathbb{Q}$$

For $f \in Cor(X, Y)$ and $g \in Cor(Y, Z)$ we define the composition $g \circ f \in Cor(X, Z)$ by the formula

$$g \circ f := pr_{XZ}\{(f \times Z) \cdot (X \times g)\}$$

Composition gives a map

$$Cor(X, Y) \times Cor(Y, Z) \to Cor(X, Z)$$

A projector for X is an element $p \in Cor(X, X)$ for which $p \circ p = p$. Let X_d and Y be varieties. Put

$$Cor^r(X_d, Y) := C^{d+r}(X \times Y; \mathbb{Q})$$

Note that $Cor^0(X, X) \subset Cor(X, X)$ is a subring and that if p is a projector, then p has degree 0.

The construction of the category $Mot(k)$ of motives proceeds in several steps

$$SmProj(k)^{op} \to CSmProj(k) \to Mot^{eff}(k) \to Mot(k).$$

Recall that $SmProj(k)$ is the category of smooth projective varieties defined over k and morphisms are the usual morphisms between varieties.
CSmProj(k) has the same objects but the morphisms are the degree zero correspondences and the composition is the composition of correspondences. Note that $CSmProj(k)$ is an additive category.
$Mot^{eff}(k)$ category of effective motives. The objects are pairs (X, p) with $X \in SmProj(k0$, p a projector, and where the morphisms $(X, p) \to (Y, q)$ are of the form $f = q \circ f' \circ p$ with f' a degree 0 correspondence.
Finally, the motives. Objects are triples (X, p, m) with $X \in SmProj(k)$, p a projector, $m \in \mathbb{Z}$, The morphisms are as follows

$$\mathrm{Hom}_{Mot}((X, p, m), (Y, q, n)) = q \circ Cor^{n-m}(X, Y) \circ p$$

Algebraic K-theory

The history of motivic cohomology date back to Grothendieck's work on Riemann-Roch theorem(1957). Grothendieck introduced what we called algebraic K-group today.(Actually, Grothendieck defined G-theory)
Let R be a ring.

$$K_0(R) := \mathbb{Z}[\text{isomorphic classes of f.g. proj } R\text{-mod}]/[M \oplus N] = [M] + [N]$$

Algebraic K-theory is very difficult to calculate. It has deep connections with **number theory**, **geometric topology**, **algebraic geometry**.

example

Relation to ζ-function. $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, s \in \mathbb{C}$. For X a scheme of finite type over $\mathbb{Z}$,

$$\zeta_X(s) = \prod_{x \in X, \text{ closed}} \frac{1}{1 - |\kappa(x)|^{-s}}$$

converge for $Re(s) > \dim X$.
Soulé's conjecture: $\mathrm{ord}_{s=n}\zeta_X(s) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}}(K_i(X) \otimes \mathbb{Q})^{(n)}$.
Soulé's conjecture is know when $n \geq \dim X$. For $n = \dim X - 1$, $\Rightarrow$ Birch-Swinnerton-Dyer conjecture.

Geometric Topology

Let Y be a finite CW complex and $X \subset Y$ with a retract. Is X homotopy equivalent to a finite CW complex?
Not always, Wall defined an obstruction in $K_0(\mathbb{Z}[\pi_1 X])$. $\tilde{X} \to X$ a universal cover. $C_*(\tilde{X})$ is a chain complex of $\mathbb{Z}[\pi_1 X]$-module.
$\chi(C_*(\tilde{X})) := \sum_i (-1)^i [C_i(\tilde{X})]$ in $K_0(\mathbb{Z}[\pi_1 X])$. (Wall): X is homotopy equivalent to a finite CW complex if and only if $\chi(C_*(\tilde{X})) = 0$ in $K_0(\mathbb{Z}[\pi_1 X])$.

Whitehead torsion

M, N smooth compact manifold of dimension n. A cobordism between M and N is a $(n+1)$-dimension manifold W s.t. $\partial W = M \sqcup N$. W is an h-cobordism if $M \to W$ and $N \to W$ are homotopy equivalent.
Is every h-cobordism trivial? No, there is an obstruction in $K_1(\mathbb{Z}[\pi_1 M])$.
(s-cobordism theorm) $n = \dim M \geq 5$. Different class of h-cobordism $\cong K_1(\mathbb{Z}[\pi_1 M]) / \pm \pi_1 M$
$n = 4$: this is trivial.
$n = 3$: open problem(equivalent to existence of exotic smooth structure on S^4).
$n = 2$: equivalent to Poincaré conjecture (theorem of Perelman).

(1959) Atiyah and Hirzebruch imitating Grothendieck's method constructed topological K-theory in algebraic topology. Besides K_0^{top}, they constructed higher topological K-theory. Moreover, the celebrated Atiyah-Hirzebruch spectral sequence:

$$H^p(X; K_q^{top}(*)) \Rightarrow K_{p+q}^{top}(X)$$

Further, there is an operation called Adams operation on topological K-theory.

Inspiration from algebraic topology

Relation between singular cohomology and topological K-theory is given by Chern character:

Chern Character

For a finite space X, there is an equivalence

$$K_0^{top}(X) \otimes \mathbb{Q} \cong \bigoplus_{n \in \mathbb{Z}} H^{2n}(X; \mathbb{Q})$$

Imagine a world in which K-theory of a topological space X had been defined, but the ordinary cohomology had not yet been defined. Algebraic geometry is such a world. Although étale cohomology theory behave like singular cohomology, it only works very-well with torsion coefficient. And it lacked some good properties when deal with p-torsion in characteristic p.(This is why we need crystalline cohomology)

Algebraic cycles and algebraic K-theory

Unlike topological side. K-theory in algebraic geometry is much more involved. Calculations and definitions are complicated. Notably, definition of higher algebraic K-theory is not purely algebraic. It used homotopy theory. Adams operation and multiplicity structure in algebraic K-theory are also difficult.

The following fact indicates that singular cohomology in algebraic geometry should contain Chow group:

SGA6

For X a integral smooth of finite type scheme over a field. We have

$$, K_0(X) \otimes \mathbb{Q} \cong \oplus_{p \geq 0} \mathrm{CH}^p(X) \otimes \mathbb{Q}$$

After Quillen's seminal work on higher algebraic K-theory. It's natural to ask what $K_n(X) \otimes \mathbb{Q}$ is.
The answer is rational motivic cohomology.

Thank You