Picard ∞-groupoids, Picard groups of $\mathbb{E}_\infty$–rings and generalized Thom spectra

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April 23, 2024

Picard categories and Picard spaces

Let $\mathcal{C}$ be an (ordinary) monoidal category.

Definition (invertible objects)

An object X is invertible iff there exists $Y \in \mathcal{C}$ such that $X \otimes Y \simeq 1 \simeq Y \otimes X$.

The monoidal structure makes the equivalence classes $\pi_0\mathcal{C}$ a monoid set. So equivalently, $X \in \mathcal{C}$ is invertible iff $[X] \in \pi_0\mathcal{C}$ is an invertible element.

Definition (Classical)

(i) The Picard category $\mathcal{PIC}(\mathcal{C})$ is the full monoidal subcategory that consists of invertible objects in $\mathcal{C}$.
(ii) The Picard groupoid $\mathcal{P}ic(\mathcal{C})$ is the maximal groupoid (also named the core) $\mathcal{PIC}(\mathcal{C})^{\simeq} \subset \mathcal{PIC}(\mathcal{C})$. The $\mathcal{P}ic(\mathcal{C})$ also admits a natural monoidal structure.
(iii) The Picard group of $\mathcal{C}$ is the $\pi_0\mathcal{P}ic(\mathcal{C})$, which is exactly the maximal subgroup $\pi_0\mathcal{P}ic(\mathcal{C}) \subset \pi_0\mathcal{C}$.

Apparently, when $\mathcal{C}$ is symmetric monoidal, the $\pi_0\mathcal{P}ic(\mathcal{C})$ and $\pi_0\mathcal{C}$ are abelian.

Picard ∞-groupoids

Let $\mathcal{C}^{\otimes}$ be an $\mathbb{E}_k$-monoidal ∞-category, where $1 \leq k \leq \infty$. Our most cared cases are $k = 1$ and $k = \infty$ which correspond monoidal and symmetric monoidal respectively.

Definition

(i) The Picard category $\mathcal{PIC}(\mathcal{C})^{\otimes}$ is the full $\mathbb{E}_k$-monoidal subcategory that consists of invertible objects in $\mathcal{C}$.
(ii) The Picard ∞-groupoid (Picard space) $\mathcal{P}ic(\mathcal{C})$ is the core of $\mathcal{PIC}(\mathcal{C})$. The $\mathcal{P}ic(\mathcal{C})$ also admits a natural $\mathbb{E}_k$-monoidal structure $\mathcal{P}ic(\mathcal{C})^{\otimes}$.
(iii) The Picard group of $\mathcal{C}^{\otimes}$ is the $\pi_0\mathcal{P}ic(\mathcal{C})$.

Proposition (*)

By straightening-unstraightening equivalence, we have a natural equivalence of ∞-categories $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}) \simeq \mathrm{Grpd}_{\mathbb{E}_k,\otimes}$. That means we can identify $\mathbb{E}_k$-monoidal ∞-groupoids with $\mathbb{E}_k$-spaces!

So we can identify $\mathcal{P}ic(\mathcal{C})^{\otimes}$ as an $\mathbb{E}_k$-space. Even more, it is a **group-like** $\mathbb{E}_k$-space, meaning its π_0 is a group. Besides, when $k \geq 2$ the group $\mathrm{Pic}(\mathcal{C})$ is abelian.

Higher Picard groups

We can also describe the higher homotopy groups of the $\mathcal{P}ic(\mathcal{C})$ for a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$.

Definition

Since $\mathcal{C}$ is symmetric monoidal, the full subcategory $B\,\mathrm{End}(\mathbf{1}) \subset \mathcal{P}IC(\mathcal{C})$ consisting of just one object $\mathbf{1}$ is canonically a symmetric monoidal ∞-category. And $B\,\mathrm{Aut}(\mathbf{1}) = B\,\mathrm{End}(\mathbf{1})^{\simeq} \subset \mathcal{P}ic(\mathcal{C})$ is an $\mathbb{E}_{\infty}$-space .

Since

$$\Omega\mathcal{P}ic(\mathcal{C}) \simeq \mathrm{Aut}(\mathbf{1})$$

we get the relations

$$\pi_1(\mathcal{P}ic(\mathcal{C}), \mathbf{1}) = \pi_0(\mathrm{End}(\mathbf{1}), \mathrm{id}_{\mathbf{1}})^{\times} \quad \text{and} \quad \pi_i(\mathcal{P}ic(\mathcal{C}), \mathbf{1}) = \pi_{i-1}(\mathrm{End}(\mathbf{1}), \mathrm{id}_{\mathbf{1}}) \quad \text{for } i \geq 2.$$

Group-like $\mathbb{E}_k$-spaces

Definition

An $\mathbb{E}_k$-space X is said to be group-like if the underlying H-space is an H-group.

Proposition

Let M be an H-space (i.e. a monoid object in $h\mathcal{S}$). Then it is an H-group iff the monoid $\pi_0 M$ is a group.

So we can equivalently replace the group-like condition above into that π_0 is a group.

Proposition

For an $\mathbb{E}_k$-space X, there is a maximal grouplike subspace $\mathrm{GL}_1 X$. That is, the inclusion

$$\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \underset{GL_1}{\overset{i}{\rightleftharpoons}} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})$$

of grouplike $\mathbb{E}_k$-spaces into $\mathbb{E}_k$-spaces has a right adjoint GL_1 given by passage to the maximal grouplike $\mathbb{E}_k$-space.

$\mathbb{E}_k$-infinite loop space machine

Theorem (Higher Algebra 5.2.6)

Let $0 < k < \infty$, and let $\mathcal{S}_*^{\geq k}$ denote the full subcategory of $\mathcal{S}_*$ spanned by the k-connective pointed spaces. Then The free functor $\beta_k : \mathcal{S}_* \simeq \mathrm{Mon}_{\mathbb{E}_0}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ is fully faithful when restricted to $\mathcal{S}_*^{\geq k}$ and induces an equivalence

$$\mathcal{S}_*^{\geq k} \xrightarrow{\sim} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$$

to the full sub ∞-category spanned by the grouplike $\mathbb{E}_k$-spaces.

Theorem

When $k = \infty$ we can get the classical infinite loop space machine by passage the equivalences above to limit. That is, we have the following natural equivalence.

$$\mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S})$$

Picard spectra and Picard groups of $\mathbb{E}_\infty$-rings

Definition

Let $\mathcal{C}^\otimes$ be a presentably symmetric monoidal ∞-category. Then by infinite loop space machine we can identify the group-like $\mathbb{E}_\infty$ space $\mathcal{P}ic(\mathcal{C})^\otimes$ with a connective spectrum $\mathrm{pic}(\mathcal{C})$, called the Picard spectrum.

Definition

Let R be an $\mathbb{E}_\infty$-ring. We write

$$\mathcal{P}ic(R) \overset{\text{def}}{=} \mathcal{P}ic(\mathrm{Mod}(R)) \text{ and } \mathrm{pic}(R) \overset{\text{def}}{=} \mathrm{pic}(\mathrm{Mod}(R)).$$

In particular, the homotopy groups of $\mathcal{P}ic(R)$ look very much like those of R (with a shift), starting at π_2. In fact, we have natural equivalences of spaces

$$\Omega\mathcal{P}ic(R) \simeq \mathrm{Aut}(R) = \mathrm{GL}_1 R \quad \text{and} \quad \tau_{\geq 1}\mathrm{GL}_1 R \simeq \tau_{\geq 1}\Omega^\infty R.$$

and isomorphisms of abelian groups

$$\pi_1\mathcal{P}ic(R) = \pi_0(R)^\times \quad \text{and} \quad \pi_i\mathcal{P}ic(R) = \pi_{i-1}(R) \quad \text{for } i \geq 2.$$

Fundamental property of Picard spaces of $\mathbb{E}_\infty$-rings

Unlike the group-valued functor $\pi_0\mathcal{P}ic$, both $\mathcal{P}ic$ and pic have the fundamental property that they commute with homotopy limits and hence are compatible with the descent theory.

Definition (faithfully flat)

We say that a map $R \to R'$ of $\mathbb{E}_\infty$-rings is faithfully flat if the map $\pi_0 R \to \pi_0 R'$ is faithfully flat and the natural map $\pi_* R \otimes_{\pi_0 R} \pi_0 R' \to \pi_* R'$ is an isomorphism.

Theorem (DAG VII, Lurie 2011)

Suppose $R \to R'$ is a faithfully flat morphism of $\mathbb{E}_\infty$-rings. Then the symmetric monoidal ∞-category $\mathrm{Mod}(R)$ can be recovered as the limit of the cosimplicial diagram of symmetric monoidal ∞-categories

$$\mathrm{Mod}\,(R') \rightrightarrows \mathrm{Mod}\,(R' \otimes_R R') \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; \cdots$$

Fundamental property of Picard spaces of $\mathbb{E}_\infty$-rings

So when comes to the Picard functor, the theory of faithfully flat descent goes into effect.

Corollary

As a result, $\mathcal{P}ic(R)$ can be recovered as a totalization of spaces,

$$\mathcal{P}ic(R) \simeq \mathrm{Tot}\left(\mathcal{P}ic\left(R'^{\otimes(\bullet+1)}\right)\right).$$

Equivalently, one has an equivalence of connective spectra

$$\mathrm{pic}(R) \simeq \tau_{\geq 0}\,\mathrm{Tot}\left(\mathrm{pic}\left(R'^{\otimes(\bullet+1)}\right)\right).$$

The Picard functor also commutes with filtered colimits.

Theorem (Mathew–Stojanoska 2016)

The functor $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S}p) \to \mathcal{S}$ given by $R \mapsto \mathcal{P}ic(R)$ commutes with filtered colimits. And hence the composition $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S}p) \to \mathrm{N}(\mathcal{S}et)$ given by $R \mapsto \pi_0\mathcal{P}ic(R)$ also commutes with filtered colimits.

Algebraic approximation

Definition

Let R be an $\mathbb{E}_\infty$-ring. There is a monomorphism

$$\Phi : \pi_0 \mathcal{P}ic\,(R_*) \to \pi_0 \mathcal{P}ic\,(R)\,,$$

constructed as follows.

If M_* is an invertible R_*-module, it has to be finitely generated and projective of rank one. Consequently, there is a projection p_* with a section s_* :

$$F_* \underset{p_*}{\overset{s_*}{\leftrightarrows}} M_*$$

where F_* can be realized as a finite wedge sum of copies of R or its suspensions. Let e_* be the idempotent given by composition $s_* \circ p_*$. Since F is free over R, e_* can be realized as an idempotent R-module map $e : F \to F$. Define M to be the colimit of the sequence $F \xrightarrow{e} F \xrightarrow{e} \cdots$.

Picard-algebraic

Definition

Given an $\mathbb{E}_\infty$-ring, when $\Phi : \pi_0\mathcal{P}ic\,(R_*) \to \pi_0\mathcal{P}ic\,(R)$ is an isomorphism, we say that R is **Picard-algebraic**.

Theorem (Mathew–Stojanoska 2016)

1. *Suppose R is a connective $\mathbb{E}_\infty$-ring. Then R is Picard-algebraic.*
2. *Suppose R is a weakly even periodic $\mathbb{E}_\infty$-ring with $\pi_0 R$ a regular noetherian ring. We have the following:*
   (i) The R is Picard-algebraic.
   (ii) If further R is Landweber exact, Let $n \geq 1$ be an integer, and let L_n denote localization with respect to the Lubin-Tate spectrum E_n. Then the Picard group of $L_n R$ is $\mathrm{Pic}\,(L_n R) = \mathrm{Pic}\,(\pi_* R) \times \pi_{-1}\,(L_n R)$, where we denote Pic as $\pi_0\mathcal{P}ic$. Besides, $\mathrm{Pic}\,(\pi_* R)$ sits in an extension $0 \to \mathrm{Pic}\,(\pi_0 R) \to \mathrm{Pic}\,(\pi_* R) \to \mathbb{Z}/2 \to 0$, which is split if R is strongly even periodic.

Examples

Definition

Suppose that $\mathcal{C}$ is a symmetric monoidal stable ∞-category such that the tensor product commutes with finite colimits in each variable. Then one has a natural homomorphism

$$\mathbb{Z} \to \mathrm{Pic}(\mathcal{C})$$

sending $n \mapsto \Sigma^n \mathbf{1}$.

Example (Bott periodicity)

- The KU is an even periodic $\mathbb{E}_\infty$-ring with a regular noetherian π_0, so it is Picard-algebraic. Then $\mathrm{Pic}(KU) = \mathrm{Pic}(\mathbb{Z}[u^{\pm}]_*) \simeq \mathbb{Z}/2$ generated by ΣKU.
- By $KO \simeq KU^{hC_2}$ and the homotopy fixed point spectral sequence $H^s(C_2, \pi_t \operatorname{pic}(KU)) \Rightarrow \pi_{t-s} \operatorname{pic}(KO)$, we can calculate that $\mathrm{Pic}(KO) \simeq \mathbb{Z}/8$ generated by ΣKO.

Examples

Theorem (Sphere spectrum)

Let Sp *be the* ∞*-category of spectra with the smash product. Then* $\mathrm{Pic}(Sp) = \mathrm{Pic}(\mathbb{S}) \simeq \mathbb{Z}$*, generated by the suspension* $\Sigma^1\mathbb{S}$*.*

A quick proof: If $T \in \mathrm{Sp}$ is invertible, so that there exists a spectrum T' such that $T \wedge T' \simeq \mathbb{S}$, then we need to show that T is a suspension of $\mathbb{S}$.

Since the unit object $\mathbb{S} \in \mathrm{Sp}$ is compact, it follows that T is compact: that is, it is a finite spectrum. By suspending or desuspending, we may assume that T is connective, and that $\pi_0 T \neq 0$.

By the Künneth formula, it follows easily that $H_*(T; F)$ is concentrated in the dimension 0 for each field F. Since $H_*(T; \mathbb{Z})$ is finitely generated, an argument with the universal coefficient theorem implies that $H_*(T; \mathbb{Z})$ is torsion-free of rank 1 : i.e. $H_0(T; \mathbb{Z}) \simeq \mathbb{Z}$. By the Hurewicz theorem, $T \simeq \mathbb{S}$.

Corollary

By the Picard-algebraic property of $\mathbb{S}$, we have $\mathrm{Pic}(\pi_*\mathbb{S}) \simeq \mathrm{Pic}(\mathbb{S}) \simeq \mathbb{Z}$.

Calculation of Picard groups of TMF and Tmf

Theorem (Picard descent spectral sequence, Gepner–Lawson 2016)

Suppose that X is a regular Deligne-Mumford stack with a quasi-affine flat map $X \to M_{\mathrm{FG}}$, and suppose $\mathfrak{X}$ is an even periodic realization of X. Then there is a spectral sequence with

$$E_2^{s,t} = \begin{cases} H^s(X, \mathbb{Z}/2) & \text{if } t = 0 \\ H^s\left(X, \mathcal{O}_X^\times\right) & \text{if } t = 1, \\ H^s\left(X, \omega^{(t-1)/2}\right) & \text{if } t \geq 3 \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

whose abutment is $\pi_{t-s} \operatorname{pic} \Gamma(\mathfrak{X}, \mathcal{O}^{top})$. The differentials run $d_r : E_r^{s,t} \to E^{s+r,t+r-1}$.

Calculation of Picard groups of TMF and Tmf

Theorem (Mathew–Stojanoska 2016)

The Picard group of integral TMF is $\mathbb{Z}/576$, generated by ΣTMF .

A quick proof: We can use étale descent to produce the spectral sequence, as TMF is obtained as the global sections of the sheaf $\mathcal{O}^{\text{top}}$ of even-periodic E_∞-rings on the moduli stack of elliptic curves. Namely, by the fact that the map $M_{\text{ell}} \to M_{\text{FG}}$ is affine, the spectral sequence is

$$H^s\left(M_{\text{ell}}, \pi_t \operatorname{pic} \mathcal{O}^{\text{top}}\right) \Rightarrow \pi_{t-s} \operatorname{pic} \Gamma(M_{\text{ell}}, \mathcal{O}^{\text{top}}) = \pi_{t-s} \operatorname{pic}(TMF),$$

and we are interested in π_0. The E_2-page of this spectral sequence is given by (for $t-s \geq 0$)

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & \text{if } t = s = 0, \\ H^s\left(M_{\text{ell}}, \mathcal{O}_{M_{\text{ell}}}^{\times}\right) & \text{if } t = 1, \\ H^s\left(M_{\text{ell}}, \omega^{(t-1)/2}\right) & \text{if } t \geq 3 \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Over a field k of characteristic $\neq 2, 3$, Mumford showed that

$$H^1\left((M_{\text{ell}})_k, \mathcal{O}^{\times}_{M_{\text{ell}}}\right) \simeq \mathbb{Z}/12,$$

generated by the canonical line bundle ω that assigns to an elliptic curve the dual of its Lie algebra. This result is also true over $\mathbb{Z}$ by the work of Fulton and Olsson.

Actually, the differentials involving 3-torsion classes wipe out everything above the $s = 5$ line, and those involving 2-torsion classes wipe out everything above the $s = 7$ line. We conclude that the following are the only groups that can survive:
- at most a group of order 2 in $(t - s, s) = (0, 0)$,
- at most a group of order 12 in $(0, 1)$,
- at most a group of order 12 in $(0, 5)$, and
- at most a group of order 2 in $(0, 7)$.

This gives us an upper bound $2^6 3^2 = 576$ on the cardinality of π_0, which is exactly the periodicity of TMF.

□

Calculation of Picard groups of TMF and Tmf

Theorem (Mathew–Stojanoska 2016)

The Picard group of integral Tmf is $\mathbb{Z} \oplus \mathbb{Z}/24$, generated by ΣTmf and a certain 24-torsion.

The relevant part of the Picard descent spectral sequence is similar to that of TMF, with the following exceptions: the algebraic part $H^1\left(\overline{M}_{\mathrm{ell}}, \mathcal{O}^\times\right)$ is now $\mathbb{Z}$ generated by ω, according to Fulton–Olsson, and we have
- at most a group of order 2 in $(t-s, s) = (0,0)$,
- a subquotient of $\mathbb{Z}$ in $(0,1)$,
- at most a group of order 12 in $(0,5)$, and
- at most a group of order 2 in $(0,7)$

as potential contributions to the $s = t$ line of the E_∞-page.
The rest of the E_∞-filtration now tells us that $\mathrm{Pic}(\mathrm{Tmf})$ sits in an extension

$$0 \to A \to \mathrm{Pic}(\mathrm{Tmf}) \to \mathbb{Z} \to 0,$$

where A is a finite group of order at most 24, actually $\mathbb{Z}/24$. So $\mathrm{Pic}(\mathrm{Tmf}) \simeq \mathbb{Z} \oplus \mathbb{Z}/24$.

Recalling Thom spectra

Definition (Classical Thom spectrum functor)

Let $(f : X \to BO) \in \mathcal{S}_{/BO}$, then the standard filtration $X_n = f^{-1}(BO_n)$ induces a Thom spectrum given by

$$M(X)_n = \mathrm{Th}(E(X_n)) = D(X_n)/S(X_n).$$

In the ∞-categorical view, this process is exactly equivalent to taking the homotopy colimit of the diagram $\mathcal{L}_f : X \to BO \to BGL_1(\mathbb{S}) \subset \mathcal{P}ic(\mathbb{S}) \hookrightarrow \mathrm{Mod}_{\mathbb{S}} \simeq Sp$, namely $M(X) \simeq \mathrm{hocolim}_{\alpha \in X} \mathcal{L}_\alpha$ taken in Sp. That leads to the following ∞-categorical definition.

Definition (∞-categorical definition)

Let $\mathcal{C}^{\otimes}$ be a presentably $\mathbb{E}_k$-monoidal ∞-category. We define the generalized Thom spectrum functor $M : \mathcal{S}_{/\mathcal{P}ic(\mathcal{C})} \to \mathcal{C}$ as given by $(\mathcal{L} : X \to \mathcal{P}ic(\mathcal{C})) \mapsto \mathrm{colim}_{\alpha \in X} \mathcal{L}(\alpha)$ taken in $\mathcal{C}$.

Universal property of the Picard space

We firstly solve a technical problem about the smallness of the Picard group. Let $\mathcal{C}^{\otimes}$ be an $\mathbb{E}_k$-monoidal ∞-category, where $1 \leq k \leq \infty$.

Proposition

If $\mathcal{C}^{\otimes}$ is presentably $\mathbb{E}_k$-monoidal, then there exists an uncountable regular cardinal κ such that $\mathcal{PIC}(\mathcal{C}) \subset \mathcal{C}^{\kappa}$ all invertible objects are κ-compact. Hence $\mathcal{PIC}(\mathcal{C})$ and $\mathcal{P}ic(\mathcal{C})$ are essentially small (∞-categories).

Now we claim that the Picard space functor is a right adjoint to the presheaf functor.

Theorem (Ando–Blumberg–Gepner 2018)

The picard space functor induces the following adjunction

$$\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \underset{\mathcal{P}ic}{\overset{\mathrm{PSh}}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})$$

where the presheaf functor PSh is given by $K \mapsto \mathrm{Fun}(K^{op}, \mathcal{S})$.

A sketch of the proof

The $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \underset{\mathcal{P}ic}{\overset{\mathrm{PSh}}{\rightleftharpoons}} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})$ is an adjunction.

Proof: Let $G^{\otimes} \in \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})$ and $\mathcal{C}^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})$. Then we have

$$Map_{\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})}(\mathrm{PSh}(G)^{\otimes}, \mathcal{C}^{\otimes}) \simeq Map_{\mathrm{Alg}_{\mathbb{E}_k}(\widehat{\mathrm{Cat}}_{\infty})}(G^{\otimes}, \mathcal{C}^{\otimes})$$

by the universal property of Yoneda embedding. Secondly we have

$$Map_{\mathrm{Alg}_{\mathbb{E}_k}(\widehat{\mathrm{Cat}}_{\infty})}(G^{\otimes}, \mathcal{C}^{\otimes}) \simeq Map_{\mathrm{Alg}_{\mathbb{E}_k}(\widehat{\mathcal{S}})}(G^{\otimes}, \mathcal{C}^{\simeq,\otimes})$$

by the property of the maximal groupoid. Thirdly we have

$$Map_{\mathrm{Alg}_{\mathbb{E}_k}(\widehat{\mathcal{S}})}(G^{\otimes}, \mathcal{C}^{\simeq,\otimes}) \simeq Map_{\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})}(G^{\otimes}, \mathcal{P}ic(\mathcal{C})^{\otimes})$$

since an $\mathbb{E}_k$-monoidal functor maps invertible objects into invertible objects. Combining all above, we have a natural equivalence

$$Map_{\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})}(\mathrm{PSh}(G)^{\otimes}, \mathcal{C}^{\otimes}) \simeq Map_{\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})}(G^{\otimes}, \mathcal{P}ic(\mathcal{C})^{\otimes}).$$

□

Generalized Thom spectrum functor

Let $\mathcal{C}^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably $\mathbb{E}_k$-monoidal ∞-category.

Definition

By the adjunction $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \underset{\mathcal{P}ic}{\overset{\mathrm{PSh}}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Pr}^{\mathrm{L}})$, we have the natural counit map $\mathrm{PSh}(\mathcal{P}ic(\mathcal{C}))^{\otimes} \to \mathcal{C}^{\otimes}$, which is a colimit-preserving $\mathbb{E}_k$-monoidal functor.

By unstraightening we get natural equivalences of ∞-categories $\mathrm{PSh}(X) \simeq \mathrm{RFib}_{/X} = \mathrm{KanFib}_{/X} \xrightarrow{\sim} \mathcal{S}_{/X}$ for any space $X \in \mathcal{S}$. This can be promoted as an $\mathbb{E}_k$-monoidal equivalence $\mathrm{PSh}(X)^{\otimes} \simeq \mathcal{S}^{\otimes}_{/X}$ when X is an $\mathbb{E}_k$-space.

Definition (Monoidal enhancement)

By the identification above we have the natural colimit-preserving $\mathbb{E}_k$-monoidal functor $M^{\otimes} : \mathcal{S}^{\otimes}_{/\mathcal{P}ic(\mathcal{C})} \simeq \mathrm{PSh}(\mathcal{P}ic(\mathcal{C}))^{\otimes} \to \mathcal{C}^{\otimes}$, which makes the generalized Thom spectrum functor M monoidal.

Generalized orientation theory

Definition

Let $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. We define the $\mathbb{E}_k$-spaces $BGL_1(\mathbf{1}_{\mathcal{C}})_{\downarrow A}$ and $\mathcal{P}ic(\mathcal{C})_{\downarrow A}$ by requiring the following squares to be pullbacks of $\mathbb{E}_k$-monoidal ∞-categories.

$$\begin{array}{ccccc} BGL_1(\mathbf{1}_{\mathcal{C}})_{\downarrow A} & \longrightarrow & \mathcal{P}ic(\mathcal{C})_{\downarrow A} & \longrightarrow & \mathcal{C}_{/A} \\ \downarrow & & \downarrow & & \downarrow \\ BGL_1(\mathbf{1}_{\mathcal{C}}) & \longrightarrow & \mathcal{P}ic(\mathcal{C}) & \longrightarrow & \mathcal{C} \end{array}$$

Theorem (Camarena-Barthel 2018)

The pair $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})_{/\mathcal{P}ic(\mathcal{C})} \underset{\mathcal{P}ic(\mathcal{C})_{\downarrow(-)}}{\overset{M}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ *is an adjunction. So we have a natural equivalence of spaces*

$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})}(M(X), A) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})_{/\mathcal{P}ic(\mathcal{C})}}(X, \mathcal{P}ic(\mathcal{C})_{\downarrow A})$ *for any* $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$.

Generalized orientation theory

Corollary (Camarena-Barthel 2018)

Let $f : X \to \mathcal{P}ic(\mathcal{C})$ be an $\mathbb{E}_k$-map and $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. The $\mathbb{E}_k$-algebra structure of Mf is characterized by the following universal property: the space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})}(Mf, A)$ is equivalent to the space of $\mathbb{E}_k$-lifts of f indicated below:

$$\begin{array}{ccc} & & \mathcal{P}ic(\mathcal{C})_{\downarrow A} \\ & \nearrow & \downarrow \\ X & \xrightarrow{f} & \mathcal{P}ic(\mathcal{C}) \end{array}$$

Generalized orientation theory

For the remainder of slides let $\mathcal{C}^{\otimes}$ be a **presentably stable symmetric monoidal** ∞-category and $R \to A$ be a morphism in $\mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C})$ where $1 \leq k \leq \infty$.

Definition (Generalized orientation)

1. By HA 5.1.4, the ∞-category $\mathrm{RMod}_R(\mathcal{C})$ of (right) modules over R is an $\mathbb{E}_k$-monoidal ∞-category. We denote $\mathcal{P}ic(R)$ as the $\mathcal{P}ic(\mathrm{RMod}_R(\mathcal{C}))$.
2. Let $B(R, A)$ be the full $\mathbb{E}_k$-subgroupoid of $\mathcal{P}ic(R)_{\downarrow A}$ consisting of morphisms of R-modules $h : M \to A$ such that the adjoint $h^{\dagger} : A \otimes_R M \to A$ is an equivalence in $\mathrm{RMod}_A(\mathcal{C})$.
3. We define the space of $\mathbb{E}_k$ A-orientations of an $\mathbb{E}_k$-map $f : X \to \mathcal{P}ic(R)$ as the space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})_{/\mathcal{P}ic(R)}}(X, B(R, A))$ of $\mathbb{E}_k$-lifts of f indicated below:

$$\begin{array}{ccc} & & B(R, A) \\ & \nearrow & \downarrow \\ X & \xrightarrow{f} & \mathcal{P}ic(R) \end{array}$$

Generalized orientation theory

Proposition (Camarena-Barthel 2018)

If X is a group-like $\mathbb{E}_k$-space, then for any $\mathbb{E}_k$-map $X \to \mathcal{P}ic(R)_{\downarrow A}$ we have the factorization:

$$\begin{array}{ccc} B(R, A) & \longrightarrow & \mathcal{P}ic(R)_{\downarrow A} \\ \uparrow & \nearrow & \\ X & & \end{array}$$

Remark

Note that this proposition does not necessarily hold if we do not assume $\mathcal{C}$ is stable.

So in this group-like X case, any $\mathbb{E}_k$-map $Mf \to A$ is an $\mathbb{E}_k$ A-orientation.

Generalized orientation theory

Proposition (Camarena-Barthel 2018)

The following diagram is a pullback diagram of $\mathbb{E}_k$*-spaces, where* $B(A,A)$ *is a contractible* $\mathbb{E}_k$*-space.*

$$\begin{array}{ccc} B(R,A) & \longrightarrow & B(A,A) \\ \downarrow & & \downarrow \\ \mathcal{P}ic(R) & \xrightarrow{A\otimes_R(-)} & \mathcal{P}ic(A) \end{array}$$

Corollary

Let $f : X \to \mathcal{P}ic(R)$ be an $\mathbb{E}_k$-map. Then the space of $\mathbb{E}_k$ A-orientations for f is either empty or equivalent to the space of $\mathbb{E}_k$ A-orientations of the constant map $c_R : X \to \mathcal{P}ic(R)$, namely $\Omega \operatorname{Map}_{\mathbb{E}_k}(X, \mathcal{P}ic(A))$.

Generalized orientation theory

A quick proof: Let θ be an $\mathbb{E}_k$ A-orientation. We have the following diagram of $\mathbb{E}_k$-spaces.

$$\begin{array}{ccccc} & & B(R,A) & \longrightarrow & B(A,A) \\ & \overset{\theta}{\nearrow} & \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathcal{P}ic(R) & \xrightarrow{A\otimes_R(-)} & \mathcal{P}ic(A) \end{array} \qquad h : X \to \mathcal{P}ic(A)$$

So h is null-homotopy by the contractibility of $B(A,A)$. Therefore we have equivalences of spaces

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})_{/\mathcal{P}ic(R)}}(X, B(R,A)) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})_{/\mathcal{P}ic(A)}}(X, B(A,A)) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})}(X, \Omega\mathcal{P}ic(A))$$

and $\mathrm{Map}_{\mathrm{Alg}^R_{\mathbb{E}_k}}(Mf, A) \simeq \mathrm{Map}_{\mathrm{Alg}^A_{\mathbb{E}_k}}(A \otimes_R Mf, A) \simeq \mathrm{Map}_{\mathrm{Alg}^A_{\mathbb{E}_k}}(A \otimes \Sigma^\infty_+ X, A)$ (Thom isomorphism).

References

In the $\mathbb{E}_\infty$-case, we can go further by combing the infinite loop space machine.

Proposition

If $k = \infty$ *and* X *is group-like, then then by* $\mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S})$ *we have* $\Omega\,\mathrm{Map}_{\mathbb{E}_\infty}(X, \mathcal{P}ic(A)) \simeq \Omega\,\mathrm{Map}_{Sp}(x, \mathrm{pic}(A))$.

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